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SCRIPTORES LOGARITHMICI

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SCRIPTORES LOCARITHMICI

SCRIPTORES LOGARITHMICI;

OR

A COLLECTION

OF

SEVERAL CURIOUS TRACTS

ON THE

NATURE AND CONSTRUCTION

OF

LOGARITHMS,

MENTIONED IN DR. HUTTON'S HISTORICAL INTRODUCTION TO HIS NEW EDITION OF SHERWIN'S MATHEMATICAL TABLES:

TOGETHER WITH

SOME TRACTS ON THE BINOMIAL THEOREM AND OTHER SUBJECTS CONNECTED WITH THE DOCTRINE OF LOGARITHMS.

VOLUME VI.

LONDON:

PRINTED BY R. WILKS, IN CHANCERY-LANE;
AND SOLD BY J. WHITE, IN FLEET-STREET.

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THE

PREFACE.

I NOW have printed a Sixth Volume of this Collection of Mathematical Tracts, to which I have given the title of *Scriptores Logarithmici*, though ſome of the tracts that have been inſerted in it have not related ſtrictly to Logarithms, but to ſome other uſeful ſubjects, which have had but a remote connection with them. And, as the volume is pretty large, extending to 950 pages, I will here proceed to lay before my readers a previous account of the contents of the ſeveral tracts contained in it, to the end that, if they ſhould not have leiſure, or inclination, to read the whole volume regularly through, they may be the better able to judge before-hand which of the tracts contained in it will be beſt ſuited to the courſe of ſtudy they are purſuing, and will be moſt likely to gratify their curioſity.

Of the Contents of the firſt Tract in this Volume.

The firſt tract in this Volume is written by Dr. Andrew Mackay, L.L.D. late of Aberdeen in Scotland, who is a very able Mathematician, and particularly well-ſkilled in the Doctrine of Navigation; and it is intitled *An Examination of Mr. Baron Maseres's Solution of Mr. Bouguer's Example to Dr. Halley's Problem, as given in page 60th of the fourth Volume of the* SCRIPTORES LOGARITHMICI. The occaſion of his writing it was as follows.

Dr. Halley had propoſed to the learned Mathematicians of the latter part of the 17th century a certain Problem concerning Navigation, of which he very much deſired to ſee a direct ſolution. The Problem was this. "Supposing a Ship to ſail from a given place, of which the latitude is known; and to ſail neither upon a Meridian-circle directly *towards*, or directly *from*, the Equator, nor on the Equator itſelf, nor on a Parallel of Latitude, but in an intermediate direction between a Meridian-circle and a Parallel of Latitude, and on a Rhumb-line, or Loxodromick curve, or ſo as to cut all the Meridian-circles, over which it paſſes, in the ſame angle; but that the ſaid angle is not known; and ſuppoſing it to have run-over, in ſuch an oblique direction, a certain known number of leagues, or miles; and alſo, in ſuch run, or voyage, to have increaſed, or diminiſhed, it's Longitude, (meaſured on the circumference of the Equator,) by a given arch, or a given number of leagues, or miles; it is required to find the ſaid angle of the Ship's courſe, and the new Latitude at which the Ship will have arrived at the end of the voyage." This is a very difficult Problem; and no direct ſolution was, as I believe, ever given of it before that which I publiſhed in the 4th Volume of the *Scriptores Logarithmici*, and which had been communicated to me by Mr. George Atwood, who had, with great ſkill and ſagacity, derived it from a certain infinite ſeries invented by Mr. James Gregory. But this ſolution, when explained as fully as possible, is ſtill very ſubtle and difficult, and requires a great quantity of calculation, both Algebräick and Arithmetical, before we can obtain, (by reſolving, by ſome Method of Approximation, an equation conſiſting of an infinite number of terms,) a tolerably accurate near value of x, or the co-ſine of the angle of the courſe, to find which is the object of the Problem. And the Problem admits of three different caſes; in the firſt of which the Ship is ſuppoſed to ſail from a place on one ſide of the Equator to another place on the ſame ſide of the Equator, but which is *leſs* diſtant from it; and in the ſecond it is ſuppoſed to ſail from a place on one ſide of the Equator to another place on the ſame ſide of the Equator, but which is *more* diſtant from it; and in the third it is ſuppoſed to ſail from a place on one ſide of the Equator to a place on the other ſide of it; which variety of caſes increaſes the length and difficulty of the ſolution in a very conſiderable degree. In order, therefore, to make the ſolution as intelligible as I could, I ſubjoined to each of theſe three caſes a ſeparate example with particular quantities, expreſſed in numbers, for the quantities that were

were before ſuppoſed to be given, or known, and I ſolved it by the application of the rules of the General Solution of the caſe under which it fell. And the values of x, (or the co-ſine of the angle of the Ship's courſe,) which were thereby obtained in each of theſe three examples, were tolerably exact; as was proved by deriving, reverſely, from the values of x obtained in thoſe examples, the values of the Differences of the Longitudes of the Ship at the beginning and at the end of the voyage, and finding that the new values of the ſaid Differences of the Longitudes, when ſo derived, were very nearly equal to their former values, when they had been given, or known.

Thus, in the firſt of thoſe examples (in Vol. 4, page 33,) the Ship at the beginning of the voyage is ſuppoſed to be in North-latitude 51°, 18′, and in Weſt-longitude, (reckoned from London,) 22°, 6′; and the diſtance run by the Ship in the voyage in the ſame courſe, (or on the ſame rhumb-line, or loxodromick curve, or ſo as to cut all the meridian lines, over which it paſſes, in the ſame angle,) is ſuppoſed to be 564 miles; and the Difference of the Longitudes of the ſhip in the place of her departure and in the place of her arrival, or the length of the arch of the Equator intercepted between the two meridian-circles that paſs through the ſaid firſt and laſt places of the ſhip, is ſuppoſed to be 786 miles. And, from theſe three things given, it is required to find the angle of the ſhip's courſe.

And by the ſolution of this example (contained in page 33, 34, 35, - - - 44,) it is found that, if the radius of the earth be called 1, the co-ſine x of the angle ſought will be $= 0.4272$, and conſequently the ſaid angle itſelf will be $= 64°, 42'$, and the new Latitude of the ſhip at the end of the voyage will be 47°, 49′.

And then, as a proof that this value of x is pretty near the truth, it is ſhewn (in Art. 47, pages 45, 46, and 47,) that, if the ſaid angle of the courſe be ſuppoſed to be 64°, 42′, and the new Latitude of the ſhip be ſuppoſed to be 47°, 49′, the Difference of the two Longitudes of the ship in it's firſt and it's laſt places will be 790.4 miles; which differs ſo little from 786 miles, or the former, or given, Difference of the Longitudes of the ſhip in the ſaid two

places, that we may juſtly conclude that the value of x, the co-ſine of the angle of the courſe, found by the foregoing computation, to wit, 0.4272, muſt be pretty nearly equal to it's true value.

The example to the ſecond caſe of Dr. Halley's Problem begins in page 60, and ends in page 75, of Vol. 4. In this example the ſhip at the beginning of the voyage is ſuppoſed to be in North-latitude 40°, 45′, and in Weſt longitude, (reckoned from London,) 15°; and it is ſuppoſed to ſail, in a direction partly tending to the North, and partly tending to the Eaſt, along a rhumb-line, or loxodromick curve, or ſo as to cut all the meridian-lines, over which it paſſes, in the ſame angle; and in the courſe of the ſaid voyage to have run over a line of 60 leagues, or 180 nautical miles, in length, and to have varied it's Longitude by an arch of 2°, 15′, meaſured on the Equator, or by a line of 147.375 miles. And, from theſe three quantities ſo given, it is required to find the angle of the ſhip's courſe, of which the co-ſine is called x.

And, by the ſolution of this example (contained in pages 60, 61, 62,&c, - - - 75,) it is found that, if the radius of the earth be called 1, the co-ſine x of the angle ſought will be $=$ 0.7947, and conſequently the ſaid angle itſelf will be $=$ 37°, 22′, and the new Latitude of the ſhip at the end of the voyage will be 42°, 48′, 29″.

And then, as a proof that this value of x is pretty near the truth, it is ſhewn (in Art. 84, pages 76, 77, and 78,) that, if the ſaid angle of the courſe be ſuppoſed to be 37°, 22′, and the new Latitude of the ſhip be ſuppoſed to be 42°, 48′, 29″, the Difference of the two Longitudes of the ſhip in it's firſt and laſt places will be 147.14 miles, or 147 $\frac{14}{100}$ miles; which differs ſo little from 147.375, or 147 $\frac{375}{1000}$, which was the former, or given, Difference of the Longitudes of the ſhip in the ſaid two places, that we may juſtly conclude that the value of x, (the co-ſine of the angle of the courſe,) that was found by the foregoing computation, to wit, 0.7947, muſt be pretty nearly equal to it's true value.

This example has been reſolved alſo by Monſieur Bouguer in his *Traité de Navigation*; and he makes the new Latitude of the ſhip at the end of the voyage to be 43°, 15′. This is greater than the new Latitude found by my ſolution of this example, to wit, 42°, 48′, 29″, by the difference of only 26′, 31″, which is leſs than half a degree, and is therefore not very conſiderable upon an arch of nearly 43 degrees.

The example to the third caſe of Dr. Halley's Problem begins in page 95, and ends in page 111, of Vol. 4. In this example the ſhip is ſuppoſed to be, at the beginning of the voyage, in North latitude 5 degrees, and to ſail, in a direction partly tending to the South and partly to the Weſt, on a rhumb-line, or loxodromick curve, or ſo as to cut all the meridian-lines, over which it paſſes, in the ſame angle, which is ſuppoſed to be unknown, and of which x is ſuppoſed to be the co-ſine in a circle of which r, or 1, or the radius of the earth, is the radius. And the ſaid ſhip is ſuppoſed, in it's voyage, to have run over a line of 564 miles, and to have gone to the South ſide of the Equator, and to have varied it's Longitude by an Equatorial arch of the length of 324.904 miles. And it is required, from theſe three quantities ſo given, to wit, the firſt Latitude of 5 degrees North, the diſtance run of 564 miles, and the Difference of the ſhip's two Longitudes in it's firſt and it's laſt places, or the length of the Equatorial arch intercepted by two meridian-circles paſſing through the ſaid two places of the ſhip, which is 324.904 miles, to find the angle of the ſhip's courſe (of which angle x is the co-ſine,) and the new Latitude at which the ſhip arrives at the end of the voyage.

And by the ſolution of this example (which is contained in pages 95, 96, 97,&c, - - - 111, of Vol. 4,) it is found that the co-ſine x of the angle ſought is $=$ 0.82, and conſequently that the ſaid angle itſelf, or the angle of the courſe, is $=$ 34°, 55′, and that the new Latitude of the ſhip at the end of the voyage will be 1°, 38′, 16″, South.

And then, as a proof that the ſaid value of the co-ſine x obtained by this ſolution, to wit, 0.82, is pretty near the truth, it is ſhewn (in Art. 117, pages 111, 112, and 113,) that, if the ſaid angle of the ſhip's courſe be ſup-

poſed

poſed to be 34°, 55′, and the new Latitude of the ſhip at the end of the voyage be ſuppoſed to be 1°, 38′, 16″, South, the Difference of the two Longitudes of the ſhip in it's firſt and laſt places will be 323.576 miles; which differs ſo little from 324 904 miles, or the Difference of thoſe two Longitudes that was before ſuppoſed, or given, that we may juſtly conclude that 0.82, or the value of *x*, (the co-ſine of the angle of the ſhip's courſe,) which we had found by the foregoing computation, muſt be pretty nearly equal to it's true value.

And by theſe computations of the values of the co-ſine *x* in theſe three examples, and theſe proofs of their being tolerably near the truth by the ſucceſs of theſe reverſe operations, I was inclined to think that the accuracy of this ſolution of Dr. Halley's Problem in a direct method had been ſufficiently eſtabliſhed. But ſome time after I was induced to entertain ſome doubts upon this ſubject, and to fear that I had made ſome ſlip in the computation of the ſecond example in conſequence of Dr. Mackay's having ſolved the ſame example by a different and an indirect Method of Solution, and found the angle ſought, or angle of the ſhip's courſe, (which I had made to be 37°, 22′,) to be only 33°, 52′, which is 3°, 30′ leſs than my value of it. The occaſion of Dr. Mackay's computing this example was as follows.

When I had compleated this difficult and complicated ſolution of Dr. Halley's Nautical Problem in a direct method, and had illuſtrated it by applying it to the computation of the foregoing three examples to the three different caſes of the Problem, (which was a very laborious and fatiguing taſk,) I communicated the whole to Dr. Mackay, who was at that time reſident at Aberdeen in Scotland: and he took occaſion from the peruſal of it to draw-up another ſolution of the ſame problem in an *indirect*, or *tentative*, method, which was incomparably leſs laborious in the application of it to particular examples, than *the direct*, but very long and abſtruſe, ſolution which I had given of it in my tract. And this new and indirect method of ſolving this problem, which he had invented, he thought to be not only ſhorter and eaſier, and therefore fitter for practice, than my direct ſolution of it (in which opinion I intirely agree with him,) but to be preferable alſo to the ſeveral *indirect* methods

of

of ſolving this difficult problem that had been given by other writers on this ſubject. And, to ſhew that it was ſo, he inſerted in the ſaid tract which he drew-up on this occaſion, a number of examples of this Problem that had been ſolved by Mr. *John Robertſon*, (the author of the celebrated treatiſe on Navigation,) and by *Monſieur Bouguer*, (a famous French writer on the ſame ſubject,) and by Mr. *Emerſon*, and other writers, by *indirect* methods of ſolution, and he ſolved them all anew, together with my three examples above-mentioned, by his own new method. And this tract he ſent to me for my peruſal, with a requeſt that I would print it in the ſame volume of the *Scriptores Logarithmici*, in which I had printed my long, direct, ſolution of this problem; which has accordingly been done in pages 275, 276, 277,&c, - - - 299, of Vol. 4. The reſults of his ſolutions of my three examples in this tract are as follows.

In the firſt of my examples, in which I had found the angle of the courſe to be 64°, 42′, and the new Latitude of the ſhip to be 47°, 49′, he makes the angle of the courſe to be 65°, 10′, 40″, and the new Latitude to be 47°, 21′, 14″. Theſe reſults do not differ greatly from mine.

In the ſecond of my examples, in which I had made the angle of the courſe to be 37°, 22′, and the new Latitude of the ſhip to be 42°, 48′, 29″, he makes the angle of the courſe to be only 33°, 52′, and the new Latitude to be 43°, 14′, 30″. Theſe values of the angle of the courſe differ from each other conſiderably, Dr. Mackay's value of it being leſs than mine by 3°, 30′.

In the third of my examples, in which I had made the angle of the courſe to be 34°, 55′, and the new Latitude of the ſhip to be 1°, 38′, 16″ South, Dr. Mackay makes the angle of the courſe to be 35°, 8′, 33″, and the new Latitude to be 2°, 41′, 12″ South.

Theſe values of the angle of the courſe differ from each other by only 13′, 33″, or leſs than 15′, or $\frac{1}{4}$ of a degree, or $\frac{1}{4\times 35}$, or $\frac{1}{140}$, of 35 degrees, which ſeems to be no important difference: but the difference of the two magnitudes of the

the new Latitude, to wit, 1°, 38′, 16″, and 2°, 41′, 2″, ſeems conſiderable and ſomewhat ſurprizing.

When I obſerved theſe differences between the reſults of my ſolutions of theſe examples and the reſults of Dr. Mackay's ſolutions of them, and more particularly in the ſecond example, where I had found the angle of the courſe to be 37°, 22′, and Dr. Mackay had made it to be only 33°, 52′, which is leſs than 37°, 22′ by three degrees and a half, I began to ſuſpect that ſome miſtake had been made in ſome of the numerous arithmetical operations that it had been neceſſary to perform in my ſolution of that example, and therefore I wiſhed to have that ſolution performed over-again by ſome ſkilful calculator who might be able to diſcover and correct any errors that I might have made in it. And for this purpoſe I applied to Dr. Mackay to undertake this buſineſs for me, and he was ſo obliging as to comply with my requeſt. And he has accordingly performed this calculation of my example to the ſecond caſe of Dr. Halley's Problem over-again compleatly, from the beginning to the end, with great care and ſkill, and, by my deſire, in the ſame words that I had made uſe of, and without any other change than a correction of the numbers where they were found to be erroneous; excepting that, for the ſake of obtaining a more exact value of x, or the co-ſine of the angle of the courſe, than I had obtained, he has retained two more terms of James Gregory's infinite ſeries than I had employed. And this new and more exact computation of this example to the ſecond caſe of Dr. Halley's Problem according to the direct method of ſolution explained in Vol. 4, which Dr. Mackay made by my deſire, together with ſome remarks that Dr. Mackay has added, at the end of the computation itſelf, in pages 21, 22, and 23, conſtitutes the firſt tract in the preſent, or 6th, Volume of the *Scriptores Logarithmici*, and extends from page 3 to page 23, or through 20 pages.

The reſult of this new and more accurate computation of this example to the ſecond caſe of Dr. Halley's Problem, by the direct method of ſolution, made by Dr. Mackay, is, "that the co-ſine, x, of the angle of the courſe, (inſtead of being $=$ 0.7947, as I had made it,) is $=$ 0.837,264, and conſequently

ſequently that the angle of the courſe itſelf (inſtead of being 37°, 22′, as I had made it,) is = 33°, 9′, which differs from the value of it which Dr. Mackay had before found by his new, *indirect*, method of ſolution, to wit, 33°, 52′, by only 43′, or leſs than $\frac{3}{4}$ of a degree; which difference of the values of the angle of the courſe, obtained by the direct method of ſolution, when the ſolution has been applied to it with greater exactneſs than before, and obtained by Dr. Mackay's new, indirect, method of ſolution, is much leſs than it was before, when it amounted to three degrees and a half. And by this new and more exact computation of this example made by Dr. Mackay, according to the direct method of ſolution, the new latitude of the ſhip at the end of the voyage, (which I had made to be 42°, 48′, 29″,) appears to be 43°, 15′, 42″, which is much nearer than 42°, 48′, 29″ to 43°, 14′, 30″, which is the new latitude of the ſhip found by Dr. Mackay's new, indirect, or tentative, method of ſolution. So that the difference between the two values of the new latitude of the ſhip obtained by the direct ſolution of this example to the ſecond caſe of the problem when the application of the ſaid ſolution to this example had been made by Dr. Mackay with more exactneſs than it had been made by me before, and when obtained by Dr. Mackay's indirect method of ſolution, as well as the difference of the two values of the angle of the ſhip's courſe obtained by the ſaid direct and indirect methods of ſolution, are much leſs than they were before: and therefore the ſaid *direct* and *indirect* methods of ſolution tend to confirm each other, and there is no reaſon to doubt of the truth of either of them.

Of the Contents of the Second Tract in this Volume.

The ſecond tract in this volume is alſo written by Dr. Mackay, and is another indirect method of ſolving the aforeſaid nautical problem of Dr. Halley, different from that which he had given before, and which had been printed in the 4th volume of the *Scriptores Logarithmici*. This method of ſolution is deduced from the principles of Middle-latitude ſailing, and is recommended by Dr. Mackay on account of it's great ſimplicity, which, he preſumes, will make it acceptable to practical navigators in thoſe caſes in which the *data*, upon

which it is founded, can be obtained with ſufficient accuracy. This tract is very ſhort, and takes-up only three pages, to wit, pages 25, 26, and 27.

Of the Contents of the Third Tract in this Volume.

The third tract in this volume is entitled, *Additional Remarks on Mr. Glenie's Problem which was ſolved and conſtructed in Vol.* 4 *of the* SCRIPTORES LOGARITHMICI. Theſe remarks were ſuggeſted to me by Mr. Frend; and the tendency of them was to ſhew that, when, in attempting to ſolve a Geometrical Problem, we have been led, by the conditions of the problem, to an equation of a higher order than a quadratick equation, and particularly when we have been led to an equation that is many degrees higher than a quadratick, as, for example, to an equation of the tenth order, or that involves the tenth power of the unknown quantity which we wiſh to find, it will, for the moſt part, be much ſafer and eaſier to proceed at once to the reſolution of ſuch high equation by, firſt, forming a few conjectures concerning the value of it's root x, or the unknown quantity required to be found, and ſubſtituting ſuch conjectural values of the root (carried only to two decimal figures,) inſtead of x in the terms of the ſaid equation, untill we have obtained a value of the ſaid root that ſhall be ſufficiently near to it's true value to be made a convenient ground-work, or baſis, of a proceſs of M. Raphſon's Method of Approximation, and then employing one, or more, proceſſes of the ſaid Method of Approximation till we have obtained the value of the root x to the propoſed degree of exactneſs, than to endeavour to reduce the ſaid equation of the tenth order to a quadratick equation by finding a long multinomial diviſor, involving the eighth power of x, by which the original equation (when all it's terms ſhall have been brought to the ſame ſide of the mark of equality, and made equal to o,) may be divided without a remainder; becauſe it will often be impoſſible for any ſuch multinomial diviſor to exiſt, and, when ſuch a diviſor may exiſt, there will be more trouble in finding it than in reſolving the original equation by Mr. Raphſon's Method of Approximation. Theſe remarks of Mr. Frend will be found very curious and ſatisfactory.

This

This third tract extends through 13 pages, beginning in page 31, and ending in page 43.

Of the Fourth Tract in this Volume.

The fourth tract in this volume is entitled, *An Inveſtigation of the Differential Series* $a - \frac{bx}{1+x} - \frac{D^{I}x^{2}}{\overline{1+x}|^{2}} - \frac{D^{II}x^{3}}{\overline{1+x}|^{3}} - \frac{D^{III}x^{4}}{\overline{1+x}|^{4}} - \frac{D^{IV}x^{5}}{\overline{1+x}|^{5}} - \frac{D^{V}x^{6}}{\overline{1+x}|^{6}} - \frac{D^{VI}x^{7}}{\overline{1+x}|^{7}}$ — &c ad infinitum, *by which the Law of Continuation of it's terms is ſhewn to extend to all it's terms, as well as to thoſe which have been actually inveſtigated. By Mr. Thomas Manning.*

This is a very valuable tract, that fully performs what is ſtated in it's title concerning the Law of the Continuation of the terms, which is a moſt important concluſion. It was communicated to me by the author Mr. Thomas Manning, who is a gentleman of great ſkill and learning in the Mathematicks, and who publiſhed in the year 1796 a book intitled, *An Introduction to Arithmetick and Algebra*, in one volume, octavo, and afterwards publiſhed a ſecond volume to it in the year 1798. He was, about 12 years ago, a ſtudent of the Univerſity of Cambridge at Caius College, and he is now on a voyage to China, with a view to obſerve the manners, and the laws, and the attainments in uſeful arts and ſciences, of that remote and ſingular people. This tract contains 16 pages, beginning in page 47, and ending in page 62.

Of the Fifth Tract in this Volume.

The fifth tract in this volume is reprinted from the poſt-humous works of Dr. Robert Simſon, who was Profeſſor of Mathematicks in the Univerſity of Glaſgow, in Scotland, in the middle of the laſt, or eighteenth, century; which poſt-humous works were printed at Glaſgow in the year 1776 at the expence of Philip, the late Earl Stanhope, the father of the preſent Earl. It

is written in Latin, and intitled *Tractatus de Logarithmis.* It contains 22 pages, beginning in page 65, and ending in page 86.

Of the Sixth Tract in this Volume.

The ſixth tract in this volume was alſo written by the ſame celebrated Geometrician, Dr. Robert Simſon, and is reprinted from the aforeſaid edition of his poſt-humous works. It is written in Latin, as well as the foregoing tract, and is intitled, *De Limitibus Quantitatum et Rationum*, and relates to the principles of Sir Iſaac Newton's Method of Fluxions. It contains only 22 pages, like the foregoing tract on Logarithms, beginning in page 89, and ending in page 110.

Of the Seventh Tract in this Volume.

The ſeventh tract in this volume is a paſſage extracted from the works of *Peter Fermat*, an eminent French Mathematician, who was a co-temporary of the celebrated *Des Cartes*, *Paſcal*, and *Roberval*, for about 30 years together, between the years 1630 and 1665. He was a counſellor of the Parliament of Toulouſe, in the reign of Lewis the XIV, and was a reſpectable lawyer and magiſtrate, and a man of general literature as well as knowledge of the mathematicks, though he was moſt celebrated for his attainments in that laſt ſcience. And his mathematical works were collected together after his death, and publiſhed at Toulouſe in a thin folio volume in the year 1679. From this edition of his works the paſſage which makes the ſeventh tract of the preſent volume, is extracted; and it contains little more than a ſingle page, making part of the 111th and 112th pages of the preſent volume. But it relates to a moſt curious and uſeful branch of mathematicks, "the Doctrine of *Maxima* and *Minima*," and is ſaid to have been one of the firſt attempts, if not the very firſt, that were publiſhed by any writer on mathematicks, to lay the foundations of this doctrine, or to teach us how to find the ſaid *Maxima* and *Minima*, where the ſeveral quantities that are ſuppoſed to increaſe, or to decreaſe,

creaſe, are ſubject to ſuch limitations. It is expreſſed with great conciſeneſs, and ſeems to be rather obſcure and difficult to underſtand. But this obſcurity has been intirely removed by the explanation that Mr. Huygens has given of it, and which forms the next tract of the preſent volume.

Of the Eighth Tract in this Volume.

The eighth tract of the preſent volume is the work of that moſt clear and accurate writer on mathematical ſubjects, Mr. Chriſtian Huygens, of Zuylichem in Holland (the author of the excellent treatiſe, *De Motu Penduli, ſeu Horologium oſcillatorium*,) and is called, in the general collection of his works, *Demonſtratio Regulæ de Maximis et Minimis*. But I have, in the preſent volume, given it the more ample title of *Diatribe illuſtriſſimi viri, Chriſtiani Hugenii, continens Demonſtrationem præcedentis Regulæ, à Petro Fermatio inventæ, de determinatione quantitatum Maximarum et Minimarum*. This tract contains a full and clear explanation, and likewiſe a demonſtration, of the foregoing Rule of Fermat for finding the *Maxima* and *Minima* of increaſing and decreaſing quantities, which had been given by it's author in the foregoing paſſage without a demonſtration. And to make the ſubject ſtill eaſier to my readers, I have added a few notes of my own to what Mr. Huygens has ſaid upon it: but theſe I have added at the bottom of the page, to diſtinguiſh them from the text of Mr. Huygens, which is printed faithfully from the Dutch edition of his works, without the ſmalleſt alteration of it. This tract, with theſe additional notes of mine, takes-up nine pages, beginning in page 113, and ending in page 121.

Of the Ninth Tract in this Volume.

The ninth tract of the preſent volume is of my own compoſition, and is an Addition, or Appendix, to the foregoing excellent tract of Mr. Huygens, being an application of it to the determination of the greateſt and leaſt poſſible magnitudes of the following binomial and trinomial quantities, to wit, $px-xx$, $qx-x^3$, px^2-x^3, px^3-x^4, $rx-x^4$, Px^m-x^{m+n}, $qx+px^2-x^3$, $qx-px^2-x^3$, $rx-qx^2-x^4$, and $rx+qx^2-x^4$; from which we may derive the limits

limits of the magnitudes of the two roots of the binomial equations $px-xx = q$, $qx-x^3 = r$, $px^2-x^3 = r$, $px^3-x^4 = s$, $rx-x^4 = s$, $Px^m-x^{m+n} = Q$, and of the two roots of the trinomial quantities $qx+px^2-x^3 = r$, $qx-px^2-x^3 = r$, $rx-qx^2-x^4 = s$, and $rx+qx^2-x^4 = s$; which limits will be of great uſe to us in reſolving any quadratick, cubick, or biquadratick, equations, that are of any of theſe forms, by Mr. Raphſon's Method of Approximation.

This ninth tract I have intitled, *Additamentum ad præcedentem Hugenii Diſtriben de inveniendis Maximis vel Minimis quantitatibus; continens exempla hujus Methodi in quibuſdam quantitatibus binomiis et trinomiis quæ occurrunt in reſolutione quarundam æquationum quadraticarum, cubicarum, et biquadraticarum, et altiorum ordinum.* It takes-up 28 pages of the volume, beginning in page 123, and ending in page 150.

Of the Tenth Tract in this Volume.

This tenth tract relates to the ſame ſubject of *Maxima* and *Minima* as the foregoing, or ninth, tract, and is alſo of my own compoſition. It is intitled, *Alia Methodus inveniendi maximos valores quantitatum quarundam compoſitarum ex duabus, vel pluribus, quantitatibus ſimplicibus, quarum una, vel plures, à reliquis ſubtrahuntur.* And it is grounded upon a very ſimple principle, which appears to me to be ſelf-evident. This principle is laid-down in Art. 4, pages 153 and 154, and may be expreſſed as follows.

If there be two quantities of the ſame kind, (as, for example, lines,) y and z, whereof y is the greater; and the ſaid quantities be, both of them, ſuppoſed to increaſe at the ſame time from their firſt magnitudes to other greater magnitudes by the continual addition to them of very ſmall increments in ſucceſſive very ſmall portions of time; the binomial quantity $y-z$, or the exceſs of the greater quantity y above the leſſer quantity z, will not always neceſſarily increaſe at the ſame time as the two quantities y and z themſelves increaſe; but will increaſe only when the increment of the greater quantity y is greater

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than the contemporary increment of the leſſer quantity z; and will neither increaſe nor decreaſe, but continue of the ſame magnitude, when the increment of y is equal to the contemporary increment of z; and will decreaſe, when the increment of y is leſs than the contemporary increment of z.

This ſeems to me to be the plaineſt and cleareſt principle upon which the Inveſtigation of the *Maxima* and *Minima*, or of the greateſt and leaſt magnitudes of compound, or multinomial, quantities, of which ſome of the terms are ſubtracted from the others, and which do not always continually either increaſe, or decreaſe, can be founded; and indeed it ſeems to be too plain either to need a demonſtration, or to admit of one. And therefore I formerly adopted it in my *Diſſertation on the Uſe of the Negative Sign in Algebra*, publiſhed in November, 1758, and made it the ground of my determination of the *Maxima*, or the greateſt poſſible magnitudes, of the compound quantities $px-xx$, $qx-x^3$, px^2-x^3, $qx+px^2-x^3$, and $qx-px^2-x^3$, in which the quantities px, qx, px^2, $qx+px^2$, and qx, (from which the other quantities xx, x^3, x^3, x^3, and px^2+x^3, are ſubtracted,) anſwer to y, the greater member of the binomial, or, rather, reſidual, quantity $y-z$, and the ſaid ſubtracted quantities xx, x^3, x^3, x^3, and px^2+x^3, anſwer to z, the leſſer, or ſubtracted, member of the ſame reſidual quantity; and by the determination of thoſe *Maxima* of the ſaid compound quantities $px-xx$, $qx-x^3$, px^2-x^3, $qx+px^2-x^3$, and $qx-px^2-x^3$, I obtained the limits of the magnitudes of the roots of the ſeveral equations $px-xx=q$, $qx-x^3=r$, $px^2-x^3=r$, $qx+px^2-x^3=r$, and $qx-px^2-x^3=r$. And, for the ſame reaſon, I have here again brought this principle forward, and have applied it to the ſame uſeful purpoſe of finding the limits of the roots of theſe equations, and likewiſe of ſuch of the trinomial forms of biquadratick equations as admit of more than one real and affirmative root. And I believe this tract will be found to be of great utility to all ſuch readers as are deſirous of becoming perfect maſters of the art of reſolving ſuch cubick and biquadratick equations as admit of ſeveral roots (I mean real and affirmative roots,) by Mr. Raphſon's Method of Approximation. For, by means of the limits of the magnitudes of ſuch roots, (which this method of finding the *Maxima* and *Minima* will diſcover to them,) they will be enabled to find, with but little trouble, ſuch firſt near values of the ſaid roots as will be

be near enough to the true values of them to be fit to be employed as the ground-works, or baſes, of the proceſſes of that excellent method of approximating ſtill further to the ſaid true values.

I am inclined to think that this method of determining the *Maxima* and *Minima* of compound, or multinomial, quantities conſiſting of terms that all increaſe at the ſame time, but of which ſome are ſubtracted from the others, and conſequently, by their increaſing, counter-act the increaſe of the other quantities from which they are ſubtracted, and leſſen the increaſe of the difference, or of the whole compound quantity, is rather clearer and eaſier to underſtand than the method of Monſieur Fermat, even after it has been ſo well explained and illuſtrated by Mr. Huygens in the former tract, though both are equally juſt and true. But of this I leave the reader to judge. But, if both theſe methods ſhould be thought equally clear and eaſy to underſtand, it will ſtill be uſeful to the ſtudents of theſe ſubjects to ſee them placed in different points of view, and demonſtrated by more than one train of reaſoning.

This tenth tract extends through no fewer than 60 pages, beginning in page 151, and ending in page 210.

Of the Eleventh Tract in this Volume.

The eleventh tract in this volume is an inveſtigation of the *Maximum*, or greateſt poſſible value, of the binomial quantity $qx - x^3$ while the variable quantity x increaſes from 0 to $\sqrt{q}$, which was communicated to me by Mr. William Frend, M. A. and Fellow of Jeſus College, Cambridge. It differs both from Fermat's Method of Inveſtigation, which Mr. Huygens has ſo well explained, and from my Method of Inveſtigation, ſet forth in the tenth tract; and I found it ſomewhat ſubtle and difficult to underſtand. But I have taken great pains to deſcribe it as fully and clearly as I could, after I had, by Mr. Frend's repeated explanations of it, been made to underſtand it myſelf. It takes up five pages, beginning in page 211, and ending in page 215.

Of

Of the Twelfth Tract in this Volume.

The twelfth tract in this volume is entitled *Tractatus de Limitibus Æquationum.* It was written by Mr. *Florimond De Beaune,* a very learned and ingenious French mathematician, who was a co-temporary of the celebrated Des Cartes, and reckoned to be almoſt his equal in mathematical ſkill and knowledge, and had written ſome ſhort, but very inſtructive, notes upon his Geometry, which, from the very conciſe manner in which it is written, ſtood in great need of them.

For, when that celebrated tract of Des Cartes on Geometry was firſt publiſhed, (which was in the year 1637,) it was underſtood by very few perſons in all Europe, but was much applauded by thoſe few readers who were capable of comprehending it. And it was, during the ten, or twelve, years immediately following it's publication, the aim and glory of ſome of the ableſt mathematicians on the Continent to penetrate into the true meaning of it's profound contents, and make them known to other lovers of theſe ſciences either by comments, or preparatory tracts, that would facilitate the peruſal of the book itſelf. Among theſe explanatory writers the following three were the moſt diſtinguiſhed, namely, the above-mentioned *Mr. Florimond De Beaune, Mr. Francis Schooten,* or *Van Schooten,* a very learned and moſt induſtrious Profeſſor of Mathematicks at Leyden in Holland, and *Eraſmus Bartholinus,* who was the King of Denmark's Profeſſor both of Medicine and Mathematicks at Copenhagen in the year 1657, and who had, about ten years before, or about the years 1647 and 1648, been a ſtudent of the Mathematicks at Leyden in Holland, under the ſaid Profeſſor Schooten, and had thereby contracted an intimate friendſhip with him, which continued till Schooten's death in the year 1659. The firſt edition of Des Cartes's Geometry, was in the year 1637, and in French, and, I believe, without any notes, or comment, to facilitate the reading it; and accordingly it was but little underſtood. But in the year 1649 Schooten tranſlated it into Latin, and publiſhed his tranſlation of it, with Mr. De Beaune's ſhort, and few, but very inſtructive, notes above-mentioned. And in the year

1651, Eraſmus Bartholinus (who had, a few years before, attended the mathematical lectures of Profeſſor Schooten at Leyden, in which he had explained the principles of Des Cartes's Geometry in ſo full and clear a manner as to enable Bartholinus to underſtand it in a degree that gave him great ſatisfaction,) publiſhed an excellent tract in Latin on the Elementary parts of Algebra, which he intended for an introductory, or preparatory, treatiſe, to facilitate to the ſtudents of Algebra the ſubſequent peruſal of Des Cartes's Geometry itſelf, and accordingly gave it the title of *Principia Matheſeos univerſalis, ſeu Introductio ad Geometriæ Methodum Renati Des Cartes.*

This piece of Bartholinus was publiſhed at firſt in the year 1651 as a ſeparate book, two years after the firſt publication of Schooten's Latin tranſlation of Des Cartes's Geometry in the year 1649; but afterwards it was publiſhed a ſecond time in the year 1659 in conjunction with ſeveral other explanatory tracts on Des Cartes's Geometry, in Schooten's ſecond edition of his Latin tranſlation of that work, with his learned comment on it in two ſmall quarto volumes, in which edition it is the firſt tract of the ſecond volume. This edition of Des Cartes's Geometry of the year 1659 is the edition that I am poſſeſſed-of; and is the only one, I believe, that is now to be eaſily met-with.

Though this piece of *Bartholinus* is not re-printed in the preſent volume of the *Scriptores Logarithmici*, and therefore it's contents are not properly an object of this Preface, yet I will not loſe this opportunity of declaring that it appears to me to be an excellent elementary treatiſe of Algebra, and is much clearer and eaſier to underſtand than moſt of the elementary treatiſes of the ſame ſubject that have been publiſhed ſince: and, amongſt other merits, it has the very great one, in my eſtimation, of omitting all mention of *Negative* quantities, or (as Sir Iſaac Newton defines them in the third page of his *Arithmetica univerſalis*,) quantities that are *leſs than nothing*, or (as Mr. Mac Laurin defines them in the beginning of his treatiſe on Algebra,) quantities obtained *by ſubtracting a greater quantity from a leſſer.* For in page 3 of this uſeful work of *Bartholinus*, where he treats of the ſubtraction of ſimple quantities, he makes the following remark, *Notandum, in ejuſmodi quantitatum ſubtractione, oportere quantitatem*

titatem illam quæ ex aliâ ſubtrahi debet, eſſe minorem : hoc eſt, ad ſubtrahendum b *ex* a, *(ut in ſuperiori exemplo,) opus eſſe ut* b *ſit minor quàm* a; which ſeems to prove, that, in the year 1651, when this uſeful tract was firſt publiſhed, the abſurd and pernicious doctrine of *negative* quantities (which has ſince introduced ſo much obſcurity and confuſion into the, otherwiſe, clear and ſimple ſcience of Algebra, or Univerſal Arithmetick,) had not yet been generally adopted by all the eminent writers on this ſubject.

The above-mentioned Mr. Florimond De Beaune was ſo zealous in his endeavours to promote the ſtudy of Des Cartes's Geometry that, beſides the notes he wrote to it in the firſt Latin edition of it publiſhed by Schooten in the year 1649, he begun to compoſe two ſeparate tracts relating to Algebräick equations, which were calculated to explain and illuſtrate what Des Cartes had delivered upon that ſubject in his Geometry, but died of the gout at *Blois*, (which was the place of his uſual reſidence, and in which he was a Counſellor, or one of the Judges, of the Preſidial Court there eſtabliſhed,) before he had finiſhed them. It happened, however, that at the time of this melancholy event, (which took place in Auguſt, 1652,) and for a year or two before, the learned Eraſmus Bartholinus (who had publiſhed that uſeful piece of Algebra, called *Principia Matheſeos univerſalis*,) had come to reſide for a conſiderable time at *Blois*, in order to perfect himſelf in the knowledge of the French language: and there he had become intimately acquainted with Mr. De Beaune, and had frequently converſed with him upon the ſubjects treated-of in thoſe two tracts which Mr. De Beaune had not had time to finiſh. The manuſcripts of them therefore (which were left by Mr. De Beaune in a very imperfect ſtate,) and all Mr. De Beaune's looſe papers relating to them, were put into the hands of Bartholinus, to be examined and corrected by him, and to have the defects in them ſupplied by him, agreeably to the meaning and intention of Mr. De Beaune, which his frequent converſations with Mr. De Beaune concerning them had enabled him to diſcover. And he accordingly did ſo examine and compleat them for publication, and delivered them to Profeſſor Schooten to be publiſhed in the next edition he ſhould publiſh of Des Cartes's Geometry; and they were publiſhed accordingly in the next edition of that learned work, which was that of the year 1659.

The firſt of theſe poſt-humous tracts of Mr. De Beaune, which were prepared for the preſs by Bartholinus, is intitled, *De Naturâ et Conſtitutione æquationum*, and the ſecond is intitled, *De Limitibus æquationum; ſeu quo pacto, ex formâ æquationum affectarum, definiri poſſint Limites intrà quos radices veræ debent offendi.*

The former of theſe tracts relates to the doctrine of the Generation of quadratick, and cubick, and biquadratick, equations, and all higher equations whatſoever, from ſimple equations by Multiplication, which was invented by *Thomas Harriot*, and was publiſhed in the year 1631, ten years after Harriot's death, by his friend *Walter Warner*, and was adopted by Des Cartes in his Geometry, which was written by him in French, and publiſhed for the firſt time in that language in the year 1637, and afterwards was tranſlated by *Schooten* into Latin, and publiſhed in that language in the year 1649, with the notes of Mr. De Beaune, and afterwards again in the year 1659 with the ſame notes of Mr. De Beaune, and likewiſe with a number of other concomitant and explanatory tracts. I have ſaid "that this Doctrine of the Generation of affected, or multinomial, equations from ſimple equations by multiplication, was *adopted* by Des Cartes," becauſe Harriot's works had been publiſhed in the year 1631, which was ſix years before the firſt edition of Des Cartes's Geometry in the French language in the year 1637; in which interval of time it ſeems highly probable that Des Cartes had ſeen Harriot's works in which this doctrine is ſet-forth at large. But, as, in his book of Geometry, Des Cartes makes no mention of Harriot, as being the inventor of this doctrine; ſome French writers have ſuppoſed that he had never ſeen Harriot's book, and had invented this doctrine himſelf, by his own ſagacity. This, however, is a controverſy in which I take no concern, becauſe I ſincerely wiſh, that neither Harriot, nor Des Cartes, or any other perſon whatſoever, had ever invented this much-boaſted doctrine, and, (inſtead of joining in the general admiration of it which is frequently expreſſed by the modern writers of Algebra, as being a moſt uſeful improvement of that ſcience,) I conſider it is a *great corruption* of Algebra, that has deſtroyed it's ſimplicity and perſpicuity, and added greatly to the obſcurity and confuſion that had been already introduced into it by the doctrine of *negative* quantities. And I think I have given ſufficient

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reaſons for this opinion in pages 93, 94, 95, 96, 97, 98, and 99 of the octavo volume I publiſhed in the year 1803, intitled, *Tracts on the Reſolution of Cubick and Biquadratick Equations*, and more particularly in the tract, called, *Remarks on the Doctrine of the Generation of Algebräick Equations one from another by Multiplication*, in pages 483, 484, 485,&c, - - - 555 of the ſame volume, where the ſubject is examined to the bottom.

The other poſt-humous tract of Mr. De Beaune, which was reviſed and prepared for the preſs by Eraſmus Bartholinus, is that intitled, *De Limitibus æquationum; ſeu quo pacto, ex formâ æquationum affectarum, definiri poſſint Limites inter quos radices veræ debent offendi.* This tract is re-publiſhed in the preſent volume, of which it is the twelfth tract. It is re-printed from the edition of Des Cartes's Geometry in Latin, with Schooten's ample comment and other concomitant explanatory tracts, that was publiſhed in the year 1659, as has been already mentioned; and I have added a few notes to it, where they ſeemed to be neceſſary to the explanation of the text. It is preceeded by a very friendly and reſpectful dedication of it by Eraſmus Bartholinus to his old inſtructor Schooten. And in this dedication he declares that the propoſitions contained in this tract, will be of great uſe in reſolving Algebräick equations by finding the values of their roots in numbers; *Eximium habent uſum ea quæ ſequenti tractatu exponentur ad numeroſam Æquationum reſolutionem; ut reliquas utilitates pertranſeam, quia tu eas ignorare non potes.*

The method of finding the limits of the magnitudes of the roots of affected, or multinomial, equations, deſcribed in this tract, is founded on a compariſon of the ſeveral ſeparate terms that involve x, or the unknown root of the equation, with each other and with the abſolute, or known, term of the equation; whereby we ſhall find whether it is greater, or leſs, than ſome of the known co-efficients, or the ſquare-roots, or the cube-roots, or other roots, of the ſaid co-efficients of the terms, or of the abſolute term, or known quantity. Thus, in the biquadratick equation $x^4+mmxx-n^3x+p^4 = 0$, in Prop. 6, page 237, we proceed as follows. Add n^3x to both ſides of the equation; and we ſhall have $x^4+mmxx+p^4 = n^3x$. Subtract p^4 from both ſides; and we ſhall have $x^4+mmxx = n^3x-p^4$. Therefore n^3x is greater than p^4, and conſequently x is greater

greater than $\frac{p^4}{n^3}$; and thus we obtain the known quantity, or, rather, knowable quantity, $\frac{p^4}{n^3}$, or the quotient of the diviſion of p^4 by n^3, for one limit of the magnitude of x, than which it muſt always be greater. Secondly, ſince $x^4 + mmxx + p^4$ is $= n^3x$, it follows that x^4 alone muſt be leſs than n^3x; and therefore $\frac{x^4}{x}$ will be leſs than $\frac{n^3x}{x}$, that is, x^3 will be leſs than n^3, and conſequently x will be leſs than n, or the cube root of the known co-efficient n^3; and thus we obtain the known quantity n for another limit of the magnitude of x, than which it muſt always be leſs. Therefore x, or the root of the biquadratick equation $x^4 + mmxx - n^3x + p^4 = 0$, or (as I chuſe rather to deſcribe this equation,) the root of the biquadratick equation $n^3x - mmxx - x^4 = p^4$, is greater than $\frac{p^4}{n^3}$, but leſs than n. But this equation has two roots, and the foregoing reaſonings will apply equally to both roots. Therefore both the roots of this equation will be greater than $\frac{p^4}{n^3}$, but leſs than n.

Theſe reaſonings, employed by Mr. De Beaune for finding certain limits of the magnitudes of the roots of equations, are very juſt and clear; and *Mr. Kerſey* in his excellent treatiſe of Algebra, book 2nd, chapter 10, (where he deſcribes *Stevinus's* Method of reſolving affected equations by conjectures and trials,) recommends this method of finding the limits of the roots in the following words; "Mr. Florimond De Beaune, in the latter of two ſmall treatiſes printed in 1659, ſhews how to find-out limits, within which the roots of all compound equations, not aſcending above the biquadratick kind, are confined; which limits, (when they may be diſcovered without much trouble, and are not very wide aſunder,) will help to leſſen the [number of] trials in the general method before delivered." This tenth chapter of the ſecond book of Kerſey's Algebra is reprinted in the octavo volume of *Tracts on the Reſolution of Affected Algebräick Equations by the Methods of Approximation*, which I publiſhed in the year 1800.

In

In obtaining theſe limits in the manner recommended by Mr. De Beaune, we ſhould compute them only to two decimal places of figures. Thus, for example, in computing the limits $\frac{p^4}{n^3}$ and n, or $\sqrt[3]{n^3}$, in the equation $n^3x - mmxx - x^4 = p^4$, we ought to find only the two firſt figures of the quotient of the diviſion of p^4 by n^3, and only the two firſt figures of the cube root of the co-efficient n^3: for otherwiſe the trouble of finding theſe limits of the values of x would be too great, as is intimated in the foregoing paſſage of Mr. Kerſey.

When theſe limits have been obtained to two decimal figures, one of them muſt be ſubſtituted inſtead of x in the firſt, or left-hand, ſide of the equation: and, if either of them, being ſo ſubſtituted, makes that ſide of the equation be tolerably near to the abſolute term, or known quantity of the equation, (as, for example, ſo near to the ſaid abſolute term as to differ from it by leſs than a tenth part of it,) it will be expedient to make ſuch limit the ground-work, or baſis, of a proceſs of Mr. Raphſon's Method of Approximation to the true value of the root, by calling the ſaid limit a, and denoting it's unknown difference from x, or the true value of the root, by the letter z, and ſubſtituting the binomial quantity $a+z$, or $a-z$, inſtead of x in the propoſed equation, with an omiſſion of all the terms that involve any higher power of z than it's ſimple power, or z itſelf, and then reſolving the ſimple equation reſulting from ſuch omiſſion; which will give us the value of z, and conſequently that of $a+z$, or $a-z$, or x, to twice as many figures as were true in a, the preceeding near value of it; and, by repeating theſe proceſſes of Mr. Raphſon's Method of Approximation, we ſhall ſoon obtain the value of the root x to a very great degree of exactneſs.

But it will ſometimes happen that the two limits of the unknown quantity, obtained by this method of Mr. De Beaune, will be very wide aſunder, as is likewiſe intimated in the foregoing paſſage of Mr. Kerſey. And in this caſe we muſt ſubſtitute one of the limits, to wit, that one which we conjecture to be nearer than the other to the true value of the root x in the firſt, or left-hand, ſide of the equation, (which contains all the terms involving x,) in order

der to diſcover whether the quantity reſulting from ſuch ſubſtitution will be greater, or will be leſs, than the abſolute term of the equation, and alſo how great their difference will be with reſpect to the ſaid abſolute term; and, if the ſaid difference is leſs than a tenth part of the abſolute term, we may proceed, as before, to make uſe of the limit ſo ſubſtituted, as the ground-work, or baſis, of a proceſs of Mr. Raphſon's Method of Approximation: but, if this difference is much greater than a tenth part of the abſolute term, it will be expedient to make the like ſubſtitution of the other limit obtained by Mr. De Beaune's method, inſtead of x, in the ſaid firſt, or left-hand, ſide of the equation, in order to diſcover whether the quantity reſulting from ſuch ſubſtitution will be greater, or will be leſs, than the abſolute term of the equation, and alſo how great their difference will be with reſpect to the ſaid abſolute term; and, if their difference is leſs than a tenth part of the ſaid abſolute term, we may proceed to make uſe of this ſecond limit, as the ground-work, or baſis, of a proceſs of Mr. Raphſon's Method of Approximation: but, if their difference is much greater than a tenth part of the abſolute term, as well as the former difference of the quantity reſulting from the ſubſtitution of the former limit, from the ſaid abſolute term, it will be expedient to take, by conjecture, ſome quantity of an intermediate magnitude between the ſaid two limits, (of which neither is found to be near enough to the true value of the root x to be made the ground-work, or baſis, of a proceſs of Mr. Raphſon's Method of Approximation to the ſaid true value,) and to ſubſtitute ſuch intermediate quantity inſtead of x in the firſt, or left-hand, ſide of the equation, and compare the reſult of ſuch ſubſtitution with the ſaid abſolute term, in order to diſcover whether ſuch reſult will be greater, or will be leſs, than the ſaid abſolute term, and whether it will be near enough to the ſaid abſolute term to make it adviſeable to employ the ſaid new intermediate quantity as the ground-work, or baſis, of a proceſs of Mr. Raphſon's Method of Approximation to obtain another value of x that ſhall be much nearer than the ſaid intermediate value to it's true value.

This twelfth tract, with the notes I have added to it to clear-up ſome of the more difficult parts of it, extends through 43 pages, beginning in page 221, and ending in page 263.

Of

Of the Thirteenth Tract in this Volume.

The thirteenth tract in this volume is a ſort of appendix to the preceeding tract of Mr. De Beaune, being a ſummary of all the concluſions obtained in the propoſitions of that tract, without the reaſonings there employed to obtain them. It is therefore a mere table of the ſeveral quadratick, cubick, and biquadratick, equations which are ſet-down in the former tract, with the limits of the roots of each equation, that had been found in the former tract, ſet-down immediately under the equation to which they belong; and is therefore, on account of it's brevity, more eaſy to be conſulted by a calculator who is deſirous of employing theſe limits in the beginning of his reſolutions of any of theſe equations, than the former tract itſelf. I have given it a title that expreſſes it's nature, to wit, that of *Recenſio Limitum, intrà quos radices veræ æquationum quadraticarum, cubicarum, et biquadraticarum, debent offendi, in præcedente tractatu Florimondi De Beaune inventorum.* It extends through 23 pages, beginning in page 265, and ending in page 287.

Of the Fourteenth Tract in this Volume.

The fourteenth tract in this volume is intended to illuſtrate Mr. De Beaune's tract on the limits of the roots of equations by applying thoſe limits to the reſolution of ſome numeral cubick and biquadratick equations, in order to obtain good firſt near values of the roots ſought, or ſuch values of them as are near enough to their true values to be fit to be made the ground-works, or baſes, of proceſſes of Mr. Raphſon's Method of Approximation to their true values. And for this purpoſe I have choſen eight equations from the examples given in Mr. Raphſon's own excellent treatiſe, intitled *Analyſis æquationum univerſalis*, to be ſo reſolved, and have reſolved them with great care and exactneſs, and at great length, that my readers may be fully able to judge how far theſe limits of the roots given us in the foregoing tract of Mr. De Beaune will be of uſe to us in reſolving theſe difficult equations. Theſe eight equations are as follows.

The cubick equation $x^3+24x = 587,914$.
The cubick equation $300x-x^3 = 1000$.
The cubick equation $9xx-x^3 = 100$.
The cubick equation $x^3+74xx+8729x = 560,783$.
The biquadratick equation $x^4+6808xx+672,792x = 43,507,216$.
The biquadratick equation $17515.969,772,051x - 1221.125xx + 62.145,069,375x^3-x^4 = 135,869.138,875,123,885$.
The biquadratick equation $4,228,931.085,087,852 \times x + 323,609.663,689 \times xx-x^4 = 22,540,483,202.613,561,987,516$.
And the biquadratick equation $x^4+4x^3+751xx-9000x = 90,000$.

In the firſt of theſe equations, to wit, the cubick equation $x^3+24x = 587,914$, which has but one root, the limits of the root x obtained by Mr. De Beaune's method are 83 and 84. Theſe limits are very near each other, and either of them is fit to be made the ground-work, or baſis, of a proceſs of Mr. Raphſon's Method of Approximation: and, by ſubſtituting $83+z$ inſtead of x in the equation $x^3+24x = 587,914$, and reſolving the new, or transformed, equation thence ariſing, as if it were a ſimple equation, we find z to be $= 0.68$, and conſequently x, or $83+z$, to be $(= 83+0.68) = 83.68$; and by a ſecond proceſs of Mr. Raphſon's Method of Approximation we find x to be $= 83.677,607$. In this example therefore the limits obtained by Mr. De Beaune's method are found to be very uſeful. This equation is reſolved in pages 288, 289, 290.

In the ſecond of theſe equations, to wit, the cubick equation $300x-x^3 = 1000$, (which has two roots,) it appears from Mr. De Beaune's limits that both the roots will be leſs than 17.32 &c, and greater than $\frac{10}{3}$, or 3.333,333. Therefore it is poſſible that the leſſer root may be very little greater than 3.333,333,&c. To try therefore how near this number 3.333,333,&c approaches to the true value of the leſſer root, I ſubſtitute 3.3 inſtead of x in the binomial quantity $300x-x^3$, and find the value of the ſaid quantity thence reſulting to be 954.063, which differs but little from the abſolute term 1000, the difference being leſs than a 20th part of 1000. I therefore conclude that

that 3.3 is near enough to the true value of x to be fit to be made the ground-work, or baſis, of a proceſs of Mr. Raphſon's Method of Approximation. And one ſuch proceſs gives me 3.47 for a ſecond near value of x; and a ſecond proceſs gives me 3.472,963 for a third near value of it; and a third proceſs gives me 3.472,963,553,338 for a fourth near value of it; of which all the figures are true. It appears therefore that in this equation the leſſer limit 3.333,333&c of the roots that was obtained by Mr. De Beaune's method, was near enough to the true value of the leſſer root to be ſucceſsfully employed as a ground-work for a proceſs of Mr. Raphſon's Method of Approximation. This ſecond equation is reſolved in pages 290, 291, 292, 293, and 294.

In the third of theſe equations, to wit, the cubick equation $9xx - x^3 = 100$, (which has two roots,) it appears from Mr. De Beaune's limits, that both the roots will be leſs than 9, and greater than $\frac{10}{3}$, or 3.333,333. Therefore it ſeems reaſonable to ſuppoſe that the leſſer of two roots will be a little greater than 3.333,333,&c, as, for example, equal to 3.4. But, upon trial of this number 3.4 in the binomial quantity $9xx - x^3$, it appears to be too ſmall to be a good ground-work, or baſis, of a further approach to the true value of this leſſer root by Mr. Raphſon's Method of Approximation. For its true value is the whole number 5. In this equation therefore theſe limits obtained by Mr. De Beaune's are not of much aſſiſtance to us in reſolving it by Mr. Raphſon's Method of Approximation; but we might have found the leſſer root of it more eaſily by ſubſtituting ſucceſſively the ſmall numbers 3, 4, and 5, inſtead of x in the binomial quantity $9xx - x^3$, the laſt of which ſubſtitutions would have made that quantity be $= (9 \times 25 - 125 = 225 - 125 =) 100$, which is the abſolute term of the equation, and therefore would have ſhewn us that 5 is the true value of the ſaid leſſer root.

But, in reſolving this equation in pages 295, 296, 297, and 298, I have proceeded by Mr. Raphſon's Method of Approximation, beginning from the number 4.5; and by the firſt proceſs I have found x, or the ſaid leſſer root, to be $= 4.938 +$ &c, and by the ſecond proceſs I have found it to be $=$ 4.998,564

4.998,564+&c; which naturally creates a ſuſpicion that it's true value is the whole number 5, as, upon trial, it appears to be.

In the fourth of theſe equations, to wit, the cubick equation $x^3+74xx+8729x = 560,783$ (which has but one root,) the limits of the root x obtained by Mr. De Beaune's method are 26 and 64. Theſe limits required a good deal of troubleſome calculation to obtain them; and, when we have found them, they are very wide of each other, and each of them differs very widely from the true value of the root x, which is 41.480,514,263. In this caſe therefore theſe limits are of no uſe to us in obtaining the value of the root x: I have, however, reſolved this equation at great length, and with great exactneſs, by Mr. Raphſon's Method of Approximation, in order to illuſtrate that excellent method of reſolution. And I have taken 41 (which is ſomewhat leſs than 45, or the arithmetical mean between 26 and 64, the two limits of the value of x found by Mr. De Beaune's method,) for the ground-work, or baſis, of the firſt proceſs of that method of approximation, and thus I have obtained ſucceſſively the numbers 41.48, and 41.480,514,263 for more accurate values of the root x of that equation. The reſolution of this equation extends from page 298 to page 303.

In the fifth of theſe equations, to wit, the biquadratick equation $x^4+6808xx+672,792x = 43,507,216$, (which has but one root,) the limits of the root x obtained by Mr. De Beaune's method, are 25 and 63. Theſe limits required a good deal of troubleſome calculation to obtain them; and, when we have found them, they are very wide of each other, and each of them differs very widely from the true value of the root x, which is 42.083,595,737. In this caſe therefore, as well as in the laſt, theſe limits are of no uſe in obtaining the value of the root x. I have, however, reſolved this equation at great length, and with great exactneſs, by Mr. Raphſon's Method of Approximation, taking the number 42 (which is ſomewhat leſs than 44, or the arithmetical mean between 25 and 63, the two limits of the value of x found by Mr. De Beaune's method,) for the ground-work, or baſis, of the firſt proceſs of that method of approximation. And thus I have obtained ſucceſſively the numbers 42.083

and

and 42.083,595,737 for more accurate values of the root x of that equation. The reſolution of this equation extends from page 303 to the end of page 308.

In the ſixth of theſe equations, to wit, the biquadratick equation $17575.969,772,051x - 1221.125xx + 62.145,069,375x^3 - x^4 = 135,869.138,875,123,885$ (which, by the changes of the ſigns + and — prefixed to it's terms, ſeems capable of having four roots, that is, real and affirmative roots,) the limits of the roots obtained by Mr. De Beaune's method are indeed very wide of each other, being the whole number 88 and the decimal fraction 0.47, which is leſs than 0.5, or $\frac{1}{2}$, ſo that all the roots are leſs than 88, and greater than 0.47. The roots of this equation are only two, to wit, 40.7583 and 13.666, which are, both of them, very diſtant from the foregoing limits 88 and 0.47. Yet we may obſerve that the greater of theſe roots, to wit, 40.7583, is not a great deal leſs than 44.23, or the arithmetical mean between the ſaid limits 88 and 0.47, as was the caſe with the roots of the two preceeding equations. However, it muſt be acknowledged that in this equation the inveſtigation of theſe limits given by Mr. De Beaune's methods (the obtaining of which required a good deal of troubleſome calculation in page 312,) has not been of any uſe to us in finding the roots 40.7583 and 13.666, and that it would have been eaſier and better to ſeek for the firſt tolerably near values of them, to wit, 41 and 13.6, (which are made the ground-works, or baſes, of proceſſes of Mr. Raphſon's Method of Approximation, by which their more accurate values 40.7583 and 13.666 have been obtained,) by mere conjectures and trials with ſmall and eaſy numbers, ſuch as 1, 10, 20, 30, 40, and 1, 10, 12, 13, according to the method recommended by Stevinus and Kerſey.

This equation I have examined and reſolved at great length, and ſhewn that it can have only the two roots 40.7583 and 13.666, notwithſtanding the number of changes of the ſigns + and — prefixed to it's terms. The examination and reſolution of it extend through 19 pages, beginning in page 309, and ending in page 327.

In

In the ſeventh of theſe equations, to wit, the biquadratick equation 4,228,931.085,087,852×x + 323,609.663,689xx — x^4 = 22,540,483,202.613,561,987,516, (which has two roots,) the limits of the two roots obtained by Mr. De Beaune's method are the numbers 160 and 591, which are very wide of each other, and very diſtant from the true values of either of the two roots, which, after a long inveſtigation, are found to be 302.750,106 and 487.140,484,17. In this equation theſe limits obtained by Mr. De Beaune's method are of no uſe to us in finding a firſt near value of either of the two roots that is near enough to it's true value to be employed ſucceſsfully as the ground-work, or baſis, of a proceſs of Mr. Raphſon's Method of Approximation.

This equation is reſolved at great length, and the reſolution of it extends through 16 pages, beginning in page 327, and ending in page 342.

In the 8th and laſt of theſe equations, to wit, the biquadratick equation x^4 + $4x^3$ + 751xx — 9000x = 90,000, (which has but one root,) the limits of the root x obtained by Mr. De Beaune's method, are 17.32 and 4.8, which differ conſiderably from each other, and likewiſe from the true value of the root x, which is = 12. Therefore theſe limits are of no uſe to us in finding a firſt near value of x that is near enough to it's true value to be employed as the ground-work, or baſis, of a proceſs of Mr. Raphſon's Method of Approximation. However, we may obſerve that in this eighth equation, (as well as in ſome of the preceeding equations,) the true value of the root x, to wit, the number 12, though conſiderably diſtant from either of the two limits 17.32 and 4.8, yet is but little different from the arithmetical mean between them, or from half their ſum, (or $\frac{17.32 + 4.80}{2}$, or $\frac{22.12}{2}$,) or 11.6.

The reſolution of this eighth equation extends through ſix pages, beginning in page 342, and ending in page 347.

I recommend to all ſuch of my readers as are deſirous of being fully and familiarly acquainted with Mr. Raphſon's excellent method of reſolving equations by approximation, to peruſe all theſe eight examples of equations reſolved by that method, with great care and attention, and to go through all the arithmetical operations that occur in them with patience; the reſolutions of theſe

equations

equations having been here ſet-down very much at length, and every difficulty that occurred in performing them having been ſtated and explained. Theſe examples are all taken from Mr. Raphſon's own original work called *Analyſis æquationum univerſalis.*

The reſult of theſe reſolutions of the foregoing eight examples by the application of the limits of their roots obtained by Mr. De Beaune's method, is not ſo much in favour of the utility of thoſe limits as I had expected. However, it is prudent to have recourſe to a variety of different methods of proceeding, in order to obtain a good firſt near value of the root x of the propoſed equation, or one that is ſufficiently near to it's true value to be fit to be employed as the ground-work, or baſis, of a proceſs of Mr. Raphſon's Method of Approximation, by which we may obtain another value of it that ſhall approach much nearer to it's true value.

Of the Fifteenth Tract in this Volume.

The fifteenth tract in this volume is a very uſeful and ingenious tract of Mr. James Ivory, of the Royal Military College at Marlow in Buckinghamſhire, and relates to the reſolution of Algebräick equations by the methods of approximation. It is intitled, *A Method of aſcertaining the number of figures that are exact in the value of a root of an Algebräick equation that has been obtained by Mr. Raphſon's Method of Approximation.* This method Mr. Ivory deſcribes in the following manner by an example.

In reſolving the biquadratick equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, (which Dr. Wallis has derived from the difficult Arithmetical Problem propoſed by Colonel *Silas Titus*,) it is found in Art. 28 of my octavo volume of Tracts on the reſolution of Algebräick equations by Approximation, that it's ſecond root, or it's leaſt root but one, (for it has four real and affirmative roots,) is greater than 12, but leſs than 13. Therefore, if e be put for the exceſs of this root above 12, we are ſure that e muſt be leſs than 1, or, in other words, that the two limits of the magnitude of e are 0 and 1. Now let the letter a be (for the ſake of brevity,) ſubſtituted for the known number 12, which is the leſſer limit

limit of this ſecond value of x, or ſecond root of the equation $14{,}937x - 1{,}998x^2 + 80x^3 - x^4 = 5000$. And we ſhall then have $x\ (= 12+e) = a+e$, and $x^2 = \overline{a+e}^2 = a^2+2ae+e^2$, and $x^3\ (= \overline{a+e}^3) = a^3+3a^2e+3ae^2+e^3$, and $x^4\ (= \overline{a+e}^4) = a^4+4a^3e+6a^2e^2+4ae^3+e^4$, and conſequently $14{,}937x\ (= 14{,}937 \times \overline{a+e}) = 14{,}937a+14{,}937e$, and $1998x^2\ (= 1998 \times \overline{a^2+2ae+e^2}) = 1998a^2+3996ae+1998e^2$, and $80x^3\ (= 80 \times \overline{a^3+3a^2e+3ae^2+e^3}) = 80a^3 + 240a^2e+240ae^2+80e^3$. Therefore the whole quadrinomial quantity $14{,}937x - 1998x^2+80x^3-x^4$ will be equal to the multinomial quantity

$$\left\{\begin{array}{rrrrrrrr} & 14{,}937a & + & 14{,}937e & & & & \\ - & 1{,}998a^2 & - & 3{,}996ae & - & 1998e^2 & & \\ + & 80a^3 & + & 240a^2e & + & 240ae^2 & + & 80e^3 \\ - & a^4 & - & 4a^3e & - & 6a^2e^2 & - & 4ae^3 \quad - \quad e^4 \end{array}\right\}$$

or (by inſerting 12 inſtead of a,) to the multinomial quantity

$$\left\{\begin{array}{rrrrrrrr} & 14{,}937 \times 12 & + & 14{,}937e & & & & \\ - & 1998 \times 144 & - & 3{,}996 \times 12 \times e & - & 1998e^2 & & \\ + & 80 \times 1728 & + & 240 \times 144e & + & 240 \times 12e^2 & + & 80e^3 \\ - & 20{,}736 & - & 6912 \times e & - & 864e^2 & - & 48e^3 \quad - \quad e^4 \end{array}\right\}$$

or the multinomial quantity

$$\left\{\begin{array}{rrrrrrrr} & 179{,}244 & + & 14{,}937e & & & & \\ - & 287{,}712 & - & 47{,}952e & - & 1998e^2 & & \\ + & 138{,}240 & + & 34{,}560e & + & 2880e^2 & + & 80e^3 \\ - & 20{,}736 & - & 6{,}912e & - & 864e^2 & - & 48e^3 \quad - \quad e^4 \end{array}\right\}$$

or the multinomial quantity

$$\left\{\begin{array}{rrrrrrrr} & 317{,}484 & + & 49{,}497e & - & 2862e^2 & + & 80e^3 \\ - & 308{,}448 & - & 54{,}864e & + & 2880e^2 & - & 48e^3 \quad - \quad e^4 \end{array}\right\}$$

or the quinquinomial quantity $9036 - 5367e + 18e^2 + 32e^3 - e^4$.

But the quadrinomial quantity $14{,}937x - 1998x^2 + 80x^3 - x^4$ is $= 5000$.

Therefore the quinquinomial quantity $9036 - 5367e + 18e^2 + 32e^3 - e^4$ will alſo be $= 5000$. And conſequently (adding $5367e + e^4$ to both ſides,) we ſhall have $9036 + 18e^2 + 32e^3 = 5000 + 5367e + e^4$; and (ſubtracting 5000 from both ſides,) we ſhall have $4036 + 18e^2 + 32e^3 = 5367e + e^4$; and (ſubtracting $13e^2 + 32e^3$ from both ſides,) we ſhall have $4036 = 5367e - 18e^2 - 32e^3 + e^4$; and,

and, laſtly, (dividing both ſides by the quadrinomial quantity $5367-18e-32e^2+e^3$,) we ſhall have $e =$ the fraction $\frac{4036}{5367-18e-32e^2+e^3}$.

Now we know that e is leſs than 1, or that the two limits of the magnitude of e are 0 and 1. Therefore the true value of e will be of an intermediate magnitude between the value of the fraction $\frac{4036}{5367-18e-32e^2+e^3}$ that would ariſe by ſubſtituting 0 inſtead of e in it's denominator, and the value of it that would ariſe by ſubſtituting 1 inſtead of e in it's denominator. But, if we were to ſubſtitute 0 inſtead of e in the fraction $\frac{4036}{5367-18e-32e^2+e^3}$, that fraction would be $(= \frac{4036}{5367-0-0+0} = \frac{4036}{5367}) = 0.750$, which is the near value of e obtained by a proceſs of Mr. Raphſon's Method of Approximation; and, if we were to ſubſtitute 1 inſtead of e in the denominator of that fraction, the ſaid fraction would be $(= \frac{4036}{5367-18\times1-32\times\overline{1}^2+\overline{1}^3} = \frac{4036}{5367-18-32+1} = \frac{4036}{5368-50} = \frac{4036}{5318}) = 0.758$. Therefore the true value of e muſt be of an intermediate magnitude between 0.750 and 0.758; and conſequently the two firſt figures of it muſt be 0.75, which are found in both it's limits. Therefore the firſt four figures of $a+e$, or $12+e$, which is equal to x, or the ſecond root of the equation $14{,}937x-1998x^2+80x^3-x^4 = 5000$, muſt be 12.75.

Q. E. D.

Having thus found with certainty that x is greater than 12.75, or 12.750, but leſs than 12.758, Mr. Ivory puts e for the exceſs of x above 12.75, and ſubſtitutes $12.75+e$ inſtead of x in the equation $14{,}937x-1998x^2+80x^3-x^4 = 5000$, but with an omiſſion of the terms that involve the cube and fourth power of e, on account of their extreme ſmallneſs. And, by reaſonings ſimilar to thoſe juſt now deſcribed, he finds e to be equal to the fraction $\frac{34.058{,}593{,}75}{5287.6875-86.725e}$. And then he ſubſtitutes 0, or the leſſer limit of the magnitude of e, inſtead of e, in the denominator of the ſaid fraction, and finds the value of the ſaid fraction reſulting from ſuch ſubſtitution to be $(= \frac{34.058{,}593{,}75}{5287.6875-0}$

$\frac{34.058,593,75}{5287.6875-0} = \frac{34.058,593,75}{5287.6875}$) = 0.006,441,7; and he afterwards ſubſtitutes 0.008, or the greater limit of the magnitude of e, inſtead of e, in the denominator of the ſaid fraction, and finds the value of the ſaid fraction reſulting from ſuch ſubſtitution to be ($= \frac{34.058,593,75}{5287.6875-86.725\times 0.008} = \frac{34.058,593,75}{5287.6875-0.6938}$ $= \frac{34.058,593,75}{5286.9037}$) = 0.006,441,9, which agrees with the former value of the ſaid fraction, to wit, 0.006,441,7, in the firſt four figures 0.006,441. And thence he concludes with certainty that the true value of e, (which is of an intermediate magnitude between theſe two limiting values 0.006,441,7 and 0.006,441,9,) muſt have the ſame four figures 0.006,441, for it's four first figures; and conſequently that the firſt eight figures of the value of $12.75+e$, or of x, the ſecond root of the equation $14,937x-1998x^2+80x^3-x^4 = 5000$, muſt be 12.756,441.

Mr. Ivory further illuſtrates this method of aſcertaining the number of figures that are exact in the values of the roots of equations obtained by Mr. Raphſon's Method of Approximation, by applying it to the firſt, or leaſt, of the four roots of the foregoing equation $14,937x-1998x^2+80x^3-x^4 = 5000$, which is = 0.350,987,045,866,03, in the ſame manner as he had before applied it to the ſecond root, 12.756,461, of the ſame equation.

This method ſeems to me to be both very ingenious and very ſatisfactory, ſince nothing can be more evident than that ſo many of the figures of the numeral values of the two limits of the magnitude of the correction e as are found to be the ſame with each other, muſt likewiſe be found in the value of the ſaid correction e itſelf, which lies between the ſaid limits.

Of the Sixteenth Tract in this Volume.

The ſixteenth tract in this volume is a New Solution of the difficult Arithmetical Problem propoſed by Colonel Silas Titus to Dr. Wallis, which was communicated to me by Mr. Ivory. It is very different both from Dr. Wallis's Solution, and from Mr. Frend's Solution, of the ſame Problem, which I have inſerted in the octavo volume of tracts on the Reſolution of Algebräick Equations by the Methods of Approximation, that was publiſhed in

in the year 1800. And, in one reſpect at leaſt, it ſeems preferable to both thoſe ſolutions, to wit, in producing a final equation which has only the ſame number of roots as there are ſolutions to the Problem itſelf, which are two; whereas the equation obtained by Dr. Wallis's Solution of the Problem, to wit, the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, has four roots, of which only the firſt and ſecond, or the leaſt and leaſt but one, to wit, 0.350,987,&c and 12.756,441,&c, are of uſe for the ſolution of the problem, and the other two roots, to wit, 32.060,290,&c and 34.832,280,&c, relate to two other problems that bear a near reſemblance to the ſaid problem, but differ from it in ſome of it's conditions; and the equation obtained by Mr. Frend's ſolution of the ſame problem, to wit, the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, has three roots, of which only the middle root, and the third, or greateſt, root are fitted to ſolve the problem from which it was derived, and the firſt, or leaſt, root has no relation to that problem, but is fitted to ſolve another problem a little different from it, but which would produce the ſame final equation. Now, when the final equation, derived from the conditions of a problem that is required to be ſolved, is exactly commenſurate to the problem that produces it, and has *no* ſuperfluous roots, or roots that are of *no* uſe in ſolving the problem that has been propoſed, but relate to ſome other problems differing a little in their conditions from the problem under conſideration, ſuch a ſolution ſeems to be ſomewhat preferable, in point of perſpicuity and elegance, to another ſolution that produces a final equation that admits of more roots than are wanted for the ſolution of the propoſed problem. And in that reſpect Mr. Ivory's ſolution of this problem of Colonel Titus ſeems to be preferable to either of the two former ſolutions of it by Dr. Wallis and Mr. Frend. The ſolution of this problem is reduced by Mr. Ivory to the reſolution of the biquadratick equation $34z + 5z^2 - 34z^3 - z^4 = 8$, which, Mr. Ivory tells us, has two roots, of which the firſt, or leſſer, is nearly equal to $\frac{8}{34}$, or $\frac{4}{17}$, and the ſecond, or greater, is ſomewhat greater than $\frac{9}{10}$. This ſolution extends through ten pages, beginning in page 361, and ending in page 370. And in the 24 following pages, to wit, pages 371, 372, 373, &c, - - - - 394, I have given a very full reſolution of this equation by Mr. Raphſon's Method of Approximation, by which it appears that the leſſer root of this equation is =

0.240,831,917,697, and the greater root of it is = 0.926,828,970,505,7; and I have applied theſe roots to the ſolution of Colonel Titus's problem, and ſhewn that they enable us to perform that ſolution with great exactneſs. And in the three following pages 395, 396, and 397, I have inſerted another letter of Mr. Ivory, (which I received ſoon after that which accompanied the aforeſaid ſolution of this famous problem,) in which he explains the true foundation of all the different methods of approximating to the roots of Algebräick equations, and the reaſon and degree of their approximation, and alſo makes ſome curious and valuable remarks on Dr. Halley's Nautical Problem, of which a direct Solution has been given in the fourth volume of this Collection of Tracts, called *Scriptores Logarithmici.*

Of the Seventeenth Tract in this Volume.

The next tract in this volume is of my own compoſition, and is intitled, *A convenient Method of reſolving Biquadratick Equations of all kinds, without taking away their ſecond terms; which may alſo be applied to the reſolution of all kinds of Cubick Equations, and of all Equations of the fifth, ſixth, ſeventh, and all other higher powers.*

This tract was drawn-up in the year 1777, and it exhibits examples of the Reſolution of Biquadratick Equations by Mr. Raphſon's excellent Method of Approximation, performed at great length and with great exactneſs, and alſo of the reduction of a biquadratick equation, that has more than one root, to a cubick equation by diviſion, in order to find the remaining roots of the former equation by reſolving the latter, or cubick, equation, of which the roots will be the ſame with the remaining roots of the biquadratick. But it will often-times be found eaſier to inveſtigate the remaining roots of the firſt, or biquadratick, equation without reducing it to a cubick equation by diviſion than by means of ſuch a reduction; on account of the quantity of calculation that is required for ſuch reductions. And of the expediency of reſolving thoſe equations in the one way rather than in the other, the calculator muſt judge for himſelf in each particular biquadratick equation that he may have occaſion to reſolve, as no general rules can be ſafely laid-down for his direction. But, that he may be able to reſolve ſuch an equation by the method which he ſhall think it expedient

expedient to adopt, it is neceſſary that he ſhould be acquainted with both theſe methods of proceeding; and that knowledge is what a careful peruſal of this tract, with a diligent performance of all the arithmetical operations that occur in it, will contribute to render familiar to him. This tract extends through no fewer than 47 pages, beginning in page 402, and ending in page 448.

Of the Eighteenth Tract in this Volume.

The eighteenth tract in this volume is a Computation of the Length of the Sine of an Arch of 1 Minute of a Degree of a Circle, by continual ſections of an arch of 60 Degrees, the chord of which is known to be equal to the radius. The finding the length of this ſine is a problem of great importance in Trigonometry, as it is neceſſary to the computation of the Trigonometrical canon, or of a table of the values of the ſines, tangents, and ſecants of circular arches to every Minute of a Degree throughout the whole quadrant. And the uſual method of ſolving this problem, or of finding the length of a ſine of 1 minute, in the middle of the 17th century, (or before Sir Iſaac Newton had found out and publiſhed to the world the infinite ſeries $a - \frac{a^3}{2.3.rr} + \frac{a^5}{2.3.4.5.r^4} - \frac{a^7}{2.3.4.5.6.7.r^6} + \frac{a^9}{2.3.4.5.6.7.8.9.r^8} - \frac{a^{11}}{2.3.4.5.6.7.8.9.10.11.r^{10}}$ + &c for expreſſing the ſine s of any arch in a quadrant of a circle called a, in powers of the ſaid arch and the radius r,) was to biſect the arch of 30 degrees, (of which the ſine is $= \frac{r}{2}$,) continually, or to inveſtigate the ſines of the halves of the ſeveral arches, of which the ſines ſucceſſively become known, till the calculator had obtained the value of the ſine of an arch that was leſs than 1 minute. This required no fewer than eleven ſucceſſive biſections of the arch of 30 degrees, by which the calculator ſucceſſively obtained the ſines of an arch of 15 degrees, and of an arch of 7 degrees and 30 minutes, and of arches of 3°, $45'$, and of 1°, $52'$, $30''$, and of $56'$, $15''$, and of $28'$, $7''$, $30'''$, and of $14'$, $3''$, $45'''$, and of $7'$, $1''$, $52'''$, $30''''$, and of $3'$, $30''$, $56'''$, $15''''$, and of $1'$, $45''$, $28'''$, $7''''$, $30'''''$, and, laſtly, of $52''$, $44'''$, $3''''$, $45'''''$. Theſe numerous biſections require a great deal of laborious calculation; and, when they have been performed with due care, they do not give

give us the value of the ſine of an arch of 1 Minute, which was the thing ſought, but the ſine of an arch of $52''$, $44'''$, $3''''$, $45'''''$, that is leſs than 1 Minute; from which we are afterwards obliged to derive the value of the ſine of an arch of 1 Minute by means of the rule of proportion; which rule of proportion between the ſines of ſmall arches of a circle and the arches themſelves is not accurately, but only nearly, true. This made me conjecture that it might be more convenient, and would, at leaſt, be more ſatisfactory, to employ quinquiſections and triſections of an arch, as well as biſections, in this Inveſtigation; becauſe we ſhould then arrive at laſt with certainty at the very quantity which we were ſeeking, to wit, the length of the ſine of an arch of 1 Minute. And, when I mentioned this matter to Mr. Hutton in the year 1777, he agreed with me in this opinion, and, at my deſire, undertook and performed, with great ſkill and ſucceſs, the following computation of the length of the ſine of an arch of 1 Minute in the manner that had been ſuggeſted. He ſuppoſes the radius of the circle to be called 1, and conſequently the chord of an arch of 60 Degrees (which is equal to the radius of the circle,) to be equal to 1 likewiſe. And he then begins his calculation by quinquiſecting the arch of 60 Degrees, or finding the chord of it's fifth part, or of an arch of 12 Degrees, by reſolving, by Mr. Raphſon's Method of Approximation, the equation $a^5 - 5a^3 + 5a = b$, which expreſſes the relation between thoſe two chords upon a ſuppoſition that b denotes the chord of the greater arch, of which the chord is given, or known, (to wit, in the preſent caſe, the chord of 60 Degrees, which is $= 1$,) and that the leſſer of the two values of a denotes the chord of it's fifth part, or of an arch of 12 Degrees. And he finds this latter chord to be $= 0.209,056,926,535,307$; and he afterwards proves this number to be true in all it's fifteen figures.

He then triſects the arch of 12 Degrees, or finds the chord of it's third part, or of an arch of 4 Degrees, by reſolving, by Mr. Raphſon's Method of Approximation, the equation that expreſſes the relation between the ſaid two chords, to wit, the equation $3a - a^3 = c$, in which c denotes the chord of the greater arch, to wit, 12 Degrees, and is therefore $= 0.209,056,926,535,307$, and the leſſer of it's two roots, or the leſſer value of a, denotes the chord of the third part of the ſaid greater arch, or the chord of an arch of 4 Degrees. And he finds this latter chord to be $= 0.069,798,993,405,002$; and he afterwards proves this number to be true in all it's fifteen figures.

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The next operation is to biſect an arch of 4 Degrees, or to find the chord of it's half, or of an arch of 2 Degrees by reſolving the equation that expreſſes the relation between the chords of thoſe arches, to wit, the equation $4a^2 - a^4 = c^2$, in which c denotes the chord of the greater arch, to wit, 4 Degrees, and is therefore = 0.069,798,993,405,002, and the leſſer of the two values of a denotes the chord of it's half, or of an arch of 2 Degrees. And he finds this latter chord to be = 0.034,904,812,874,569; and he afterwards proves this number to be true in the firſt 14 places of figures, to wit, 0.034,904,812, 874,56, and very nearly true alſo in the laſt figure 9, which muſt be either an 8, or a 9.

The next operation is to biſect an arch of 2 Degrees, or to find the chord of it's half, or of an arch of 1 Degree, by a calculation ſimilar to that employed in the biſection of an arch of 4 Degrees. And Dr. Hutton finds the chord of the latter arch, or of an arch of 1 Degree, to be = 0.017,453,070,996,753; and he afterwards proves this number to be true in all it's fifteen figures.

Having thus found the chord of an arch of 1 Degree, or 60 Minutes, to be = 0.017,453,070,996,753, Dr. Hutton proceeds to quinquiſect this arc, or to find the chord of it's fifth part, or of an arch of 12 Minutes, by reſolving, by Mr. Raphſon's Method of Approximation, the equation $a^5 - 5a^3 + 5a = b$, which expreſſes the relation between thoſe two chords, upon a ſuppoſition that b denotes the chord of the greater arch, of which the chord is given, or known, to wit, in the preſent caſe, the chord of an arch of 1 Degree, or 60 Minutes, (which is = 0.017,453,070,996,753,) and that the leſſer of the two values of a denotes the chord of it's fifth part, or of an arch of 12 Minutes. And he finds this latter chord to be = 0.003,490,656,731,798; and he afterwards proves this number to be true in the firſt 14 places of figures, to wit, in the figures 0.003,490,656,731,79, and very nearly true in the 15th figure 8.

He then triſects the arch of 12 Minutes, or finds the chord of it's third part, or of an arch of 4 Minutes, by reſolving, by Mr. Raphſon's Method of Approximation, the equation that expreſſes the relation between thoſe two chords, to wit, the equation $3a - a^3 = c$, in which c denotes the chord of the greater arch, to wit, 12 Minutes, and is therefore = 0.003,490,656,731,798, and the

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leſſer of the two values of a denotes the chord of it's third part, or of an arch of 4 Minutes. And he finds this latter chord to be = 0.001,163,552,769,027; and he afterwards proves this number to be true in the firſt 14 places of figures, to wit, 0.001,163,552,769,02, and very nearly true in the laſt figure 7.

He then biſects the arch of 4 Minutes, or finds the chord of an arch of 2 Minutes, by reſolving the equation that expreſſes the relation between the chords of thoſe two arches, to wit, the equation $4a^2 - a^4 = c^2$, in which c denotes the chord of the greater arch, to wit, 4 Minutes, and is therefore = 0.001,163,552,769,027, and the leſſer of the two values of a denotes the chord of it's half, or of an arch of 2 Minutes. And he finds this latter chord to be = 0.000,581,776,409,127; and he afterwards proves that this number is true in the firſt 14 places of figures, to wit, 0.000,581,776,409,12, and that it is very nearly ſo in the laſt figure 7.

He then divides this number 0.000,581,776,409,127, (which is equal to the chord of an arch of 2 minutes,) by 2; and the quotient 0.000,290,888,204,563,5 is equal to the ſine of half the ſaid arch of 2 Minutes, or to the ſine of an arch of 1 Minute; which is the quantity that was ultimately to be found, as the reſult of the whole computation. And he afterwards proves that this value of the ſaid ſine, to wit, the number 0.000,290,888,204,563,5, is true in the firſt 15 places of figures, to wit, in the figures 0.000,290,888,204,563, and that it's laſt figure 5, in the 16th place of decimal fractions, differs from the truth by only an unit. And thus the length of the ſine of an arch of only 1 Minute of a Degree is, by this valuable and well-conducted computation, found exactly to 15 places of decimal fractions, without the help of Sir Iſaac Newton's ſeries for expreſſing the ſine of a given arch of a circle in terms involving the powers of the arch and the radius; which ſeries is obtained by firſt inveſtigating, by means of Sir Iſaac's Method of Fluxions, the infinite ſeries for expreſſing the arch of a circle in terms involving the powers of it's ſine and the radius, and afterwards reverting the ſaid ſeries according to ſome of the known methods of reverting ſerieſes; which methods are very abſtruſe and difficult to underſtand, and likewiſe often attended with very great labour in the practical application of them to particular ſerieſes.

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This computation of the ſine of an arch of 1 Minute, with the preliminary obſervations by which it is introduced, extends through 23 pages, beginning in page 451, and ending in page 473.

Of the Nineteenth Tract in this Volume.

The nineteenth tract of this 6th volume of the *Scriptores Logarithmici*, is the original tract of Lord Napier on Logarithms, publiſhed at Edinburgh in the year 1614, under the title of *Mirifici Logarithmorum Canonis Deſcriptio; ejúſ-que uſûs in utrâque Trigonometriâ, ut etiàm in omni Logiſticâ Mathematicâ, ampliſſimi, facillimi, et expeditiſſimi, explicatio. Auctore et Inventore Joanne Nepero, Barone Merchiſtonii, &c, Scoto. Edinburgi, ex Officinâ Andreæ Hart, Bibliopolæ.* CIↃ.DC.XIV. Of this tract there has never, as I believe, been any ſecond edition in the Latin language, in which it was written by it's learned author: but a very good Engliſh tranſlation of it, in a ſmall duodecimo volume, was publiſhed at London in the year 1615, (the very next year after that in which the Latin original had been publiſhed at Edinburgh,) which had been made by Mr. Edward Wright, (the celebrated Engliſh writer on Navigation,) immediately after the publication of the original, and had been ſent to Lord Napier for his peruſal and examination, and had been intirely approved-of by him, "*as being moſt exact, and preciſely conformable to his mind and the original.*" It was not, however, publiſhed by Mr. Edward Wright himſelf, (who died very ſoon after he had compleated it, and before he had had time to publiſh it,) but by his ſon, Mr. Samuel Wright, as is related more fully in page 704 of the preſent volume. But this tranſlation I have not thought it neceſſary to reprint.

This original tract of Lord Napier is divided into two books; of which the firſt contains, in the two firſt chapters, his definition of Logarithms, and his geometrical manner of generating them by the motion of a point over a right line, and a deſcription of their uſeful property in enabling Arithmeticians to find third proportionals to two given quantities, and fourth proportionals to three given quantities, without performing the laborious operations of multiplication and diviſion which were formerly neceſſary for that purpoſe, but in the room of which he ſubſtituted the much eaſier operations of the addition and ſubtraction of the Logarithms correſponding to

the ſaid quantities; and, (in the following, or 3d, chapter,) the ſaid 1ſt book contains a deſcription of the table of Logarithms which Lord Napier had compoſed and publiſhed for that purpoſe, and which conſiſts of ſeven different columns of numbers placed one under another in the ſame page; and, (in the 4th and 5th chapters,) it contains a ſet of directions as to the manner of making uſe of the ſaid table, or of finding in it the Logarithms correſponding to given Numbers, and the Numbers correſponding to given Logarithms, together with ſome examples of their great uſe in finding continual proportionals to two given quantities, or in raiſing the powers of given quantities, and in finding intermediate proportionals between two given quantities, or in extracting the ſquare-roots and cube-roots, and other higher roots, of given quantities. And the ſecond book is an application of this new and uſeful invention of Logarithms to the ſolution of all the caſes of the reſolution of triangles, that can occur either in Plane, or in Sphærical, Trigonometry, with numerical examples of the reſolution of each of the ſaid caſes by means of the ſaid table of Logarithms.

The firſt of theſe two books contains only 18 pages, beginning in page 483, and ending in page 500; and the ſecond contains 33 pages, beginning in page 501, and ending in page 533. After which follows the table of Logarithms itſelf, which begins in page 534, and ends in page 623, containing 90 pages.

I have taken great pains in printing this tract of Lord Napier as correctly as poſſible, and have rectified the punctuation of it in many places: ſo that I believe it will be found to be more eaſy to read and underſtand in this edition of it than in the original edition from which this has been copied. And I have reprinted all the introductory pieces of it, ſuch as the dedication of the work to Charles, Prince of Wales, (who was afterwards king of England by the title of Charles the firſt, but who then had been very recently made Prince of Wales, upon the death of his elder brother, Prince Henry,) and the Latin verſes written, in honour of the work and it's learned author, by ſeveral of his friends at Edinburgh.

Of the Twentieth Tract in this Volume.

In perusing Lord Napier's Tract on Logarithms I observed several passages which appeared to me to be somewhat subtle and difficult to understand, and to require some explanation. And I have therefore drawn up and printed, immediately after this Tract of Lord Napier, another Tract that I have intitled "*Observations on the foregoing Tract of Lord Napier*," and which, I hope, will be found sufficient to remove all the difficulties which my readers may have found in the perusal of it. In this tract I, in the first place, set-forth (in article 1, pages 625, 626,) the object which Lord Napier had in view in contriving this new and useful invention and computing a Table of Logarithms; which was to facilitate the resolution of triangles in both Plane and Spherical Trigonometry by substituting, instead of the laborious operations of multiplying and dividing the lengths of sines and tangents of circular arches, expressed in numbers of seven places of decimal figures, by each other, (which operations were continually occurring in the study of practical astronomy, to which Lord Napier was much devoted,) the much easier operations of adding, or subtracting, the Logarithms of the said numbers by which it was formerly necessary to make those multiplications and divisions. And then (in art. 2, 3, and 4,) I describe Lord Napier's method of generating the sines of the circular arches contained in a quadrant of a circle, by supposing the radius to decrease proportionally from it's first magnitude, or that of the sine of an arch of 90 degrees, to that of the sine of one Minute of a Degree, or of 1 second Minute of a Degree, or of such other very small arch as shall be chosen to be the last, or least, arch that will be the object of contemplation in any Trigonometrical calculation; which decrease he conceives to be effected by the motion of a point moving along the said radius with a velocity that decreases continually at such a rate that the whole line, or radius, and the remainders of it at the ends of several successive equal portions of time shall be to the first, and second, and third, and other following decrements corresponding to them respectively, or generated in the said successive equal portions of time, always in the same proportion. This decreasing motion is a matter of some nicety and difficulty, and therefore I have described it at considerable

length in art. 2, and 3, (pages 627, 628, 629, and 630); and have afterwards demonſtrated in art. 4, (pages 630, 631, and 632), that the radius and it's ſeveral remainders at the ends of the ſaid ſeveral ſucceſſive equal portions of time will form a decreaſing Geometrical Progreſſion. Then, in art. 5, and 6, (pages 632, 633, 634, and 635) I have deſcribed Lord Napier's Method of generating the Logarithms of the Sines of the ſeveral Arches in the Quadrant of a Circle, or the Logarithms of the ratios of the ſaid ſines to the radius of the circle. Then in art. 7, (pages 635 and 636,) it is obſerved that the lengths of all the ſines of all the arches in the quadrant of a circle, and the lengths of all their Logarithms, are expreſſed by Lord Napier in whole numbers, and not in mixt numbers conſiſting of whole numbers with decimal fractions added to them, as they are often repreſented by ſubſequent writers on this ſubject. And in art. 8, (pages 636, 637, and 638,) it is ſhewn that the Decreaſing Geometrical Progreſſion which Lord Napier has choſen for the foundation of his Logarithms, is that of which the high number 10,000,000, or ten millions, (which is the number of equal parts into which he has thought fit to ſuppoſe the radius of the circle to be divided,) is the firſt, or greateſt, term, and 10,000,000—1, or 9,999,999, is the ſecond term. And in art. 9 (page 639,) it is ſhewn that the natural numbers 1,2,3,4,5,6,7, &c. in their order, are proportional to, or meaſures or Logarithms of, the ſeveral ratios of minority of the ſecond, third, fourth, fifth, ſixth, ſeventh, and eighth and other following terms of this decreaſing geometrical ſeries, reſpectively, to wit, of the terms $\overline{9,999,999}^{1}$, $\frac{\overline{9,999,999}^{2}}{10,000,000}$, $\frac{\overline{9,999,999}^{3}}{\overline{10,000,000}^{2}}$, $\frac{\overline{9,999,999}^{4}}{\overline{10,000,000}^{3}}$, $\frac{\overline{9,999,999}^{5}}{\overline{10,000,000}^{4}}$, $\frac{\overline{9,999,990}^{6}}{\overline{10,000,000}^{5}}$, $\frac{\overline{9,999,999}^{7}}{\overline{10,000,000}^{6}}$, $\frac{\overline{9,999,999}^{8}}{\overline{10,000,000}^{7}}$, &c, to 10,000,000, the firſt and greateſt term of the ſaid ſeries: which is as clear a definition, or deſcription, of Logarithms as can well be given.

In art. 10, 11, 12, 13, &c.---21, pages 640, 641, 642, &c.---653, it is ſhewn how the Logarithms of the Sines in the Trigonometrical Canon may be derived from the Logarithms of the terms of the foregoing Geometrical Progreſſion, to ſome of which terms every ſine in the Quadrant muſt be equal, or ſo little different from it that it may be conſidered as equal, their difference being leſs than 1, or one ten-millionth part of the radius, or 10,000,000. And

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it is ſhewn alſo how the Secants of circular arches, and the Logarithms of the ſaid Secants, may be generated by the motion of two moveable points, in a manner analogous to that of generating the ſines of arches and their Logarithms; which generation of the ſaid Secants and their Logarithms ſeems to be ſuppoſed by Lord Napier in the foregoing tract, but is not expreſſly deſcribed by him. And the manner of finding the Logarithms of the tangents of circular arches is alſo explained in theſe articles.

In art. 22, pages 653, 654, and 655, there is a Scholium that I think particularly curious, inaſmuch as it gives us an account of the diſſatisfaction expreſſed by ſome elderly German Mathematicians in the year 1621, or 7 years after the publication of Lord Napier's Tract on Logarithms, at the ſubtle and abſtruſe manner in which he had conceived Logarithms to be generated, by the motion of two points along two different right lines, the one with a velocity continually decreaſing according to a certain law, and the other with an uniform velocity. This we are told by the celebrated John Kepler in the following paſſage. *Cum anno* 1621 *veniſſem in Germaniam ſuperiorem, paſsímque cum peritis rerum mathematicarum de Logarithmis Neperianis contuliſſem; deprehendi eos quibus ætas prudentiam addebat, promptitudinem minuebat, ſupèr hoc genere numerorum, loco Canonis Sinuum, in uſum recipiendo, cunctari: Quòd dicerent turpe eſſe Profeſſori Mathematico ſupèr compendio aliquo calculi puerilitèr exultare, interímque ſine demonſtratione legitimâ formam calculi in uſum recipere, quæ olìm, cum minimè metueres, in erroris inſidias te pertrahere poſſet. Querebantur Neperi demonſtrationem niti figmento motûs cujuſdam Geometrici, cujus lubricitas et fluxibilitas inepta eſſet in quâ ſolidus ille ſtylus rationis demonſtrationúmque firmum poneret veſtigium. Hæc mihi cauſa fuit ſtatìm tunc concipiendi rudimentum aliquod demonſtrationis legitimæ*, &c. as in pages 95 and 96 of the firſt volume of the *Scriptores Logarithmici*, in which Kepler's two excellent tracts upon Logarithms (which are the cleareſt and beſt that have ever been publiſhed on the ſubject,) are re-printed. The words *figmento motûs cujuſdam Geometrici, cujus lubricitas et fluxibilitas inepta eſſet*, &c., relate to the motions by which Lord Napier ſuppoſed the ſeveral ſines in the arch of a quadrant of a circle, and the Logarithms of the ſaid ſines, or of the ratios of the ſaid ſines to the radius of the circle, to be generated; and they ſeem to be very happily choſen to expreſs the objection of theſe German

German Mathematicians to theſe motions, and particularly to the motion with a velocity that continually decreaſes, by which Lord Napier ſuppoſes the ſines to be generated, " as being a motion of a very ſubtle and ſlippery kind, of which it is not eaſy to form a clear conception, and which differs widely from the plain ideas of fixed and determinate magnitudes that occur in the elementary parts of Geometry, and more eſpecially in the works of Euclid." And for this objection, I muſt needs confeſs, I think there is ſome foundation. But Lord Napier might have deſcribed his noble invention of Logarithms, and given his readers a clear idea of the table of them which he had computed, (and which is reprinted in the preſent volume, at the end of his Lordship's tract,) without having recourſe to the ſuppoſition of this ſubtle generation of the ſines of circular arches by motion, or departing in any degree from the ſimple conceptions of antient Geometry and the known properties of the terms of a Geometrical Progreſſion. And therefore, in the two following articles, to wit, art. 23 and 24, (pages 655, 656, - - - 659,) I have ſhewn how he might have done this; and in art. 25, (pages 660, 661, and 662,) I have actually computed the 3d, 4th, 5th, 6th, 7th, and 8th terms of the decreaſing geometrical ſeries of numbers of which the two firſt terms are 10,000,000 and 9,999,999, and which is the grand foundation of Lord Napier's Table of Logarithms; theſe Logarithms being nothing more than the exponents of the diſtances of the ſecond, and third, and other following terms of this decreaſing progreſſion from its firſt, or greateſt, term 10,000,000, which exponents are proportional to, or meaſures of, the ſeveral ratios which the terms to which they belong, bear to the ſaid firſt term, and expreſs the numbers of equal *ratiunculæ*, or ſmall ratios of minority, equal to that of 9,999,999, to 10,000,000, that are contained in the ratios of the ſecond, third, fourth, and other following terms, to which they reſpectively belong, to its firſt and greateſt term 10,000,000, and from this circumſtance have obtained the name of *Logarithms*, as being ἀριθμοὶ τῶν λογῶν.

Though the method in which the 3d and 4th and ſix following terms of this decreaſing geometrical progreſſion are computed in art. 25 from the two firſt terms 10,000,000 and 9,999,999, is exceedingly clear and eaſy, every new term being produced by the multiplication of the preceeding term into the fraction

fraction $\frac{9,999,999}{10,000,000}$, or $\frac{10,000,000-1}{10,000,000}$, yet, from the prodigious number of the terms of which this decreasing progression, if carried to a term that was less than 1, or even if carried only to a term that was less than 2909, or the sine of an arch of 1 Minute of a Degree, would consist,it would, perhaps,be impracticable to compute them all in this manner. And therefore it would be highly convenient to have a method of finding the Logarithm of any proposed sine contained in the Trigonometrical Canon without having such a table of the values of the terms of this Geometrical Progression ready-computed to our hands. And this may be done by resolving the equation $\frac{a^x}{r^{x-1}} = S$; in which r stands for the radius of the circle, which, in Lord Napier's Table of Sines and their Logarithms, is = 10,000,000; and a stands for 9,999,999; and S is the known, or given, sine, of which the Logarithm is to be found; and x is the Logarithm of the said Sine, or the Logarithm sought. Now this equation has been resolved in a very ingenious manner by the help of the Binomial Theorem by the learned Mr. James Ivory, M. A. of the Royal Military College at Great Marlow in Buckinghamshire; and the result of his Solution is, "That, if v be put $= r-S$, or 10,000,000 — the numeral value of the given Sine S, the Logarithm x will be $= v \times$ the infinite Series $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \frac{v^5}{6r^5} + \frac{v^6}{7r^6} +$ &c." This Solution Mr. Ivory has communicated to me with permission to publish it; and I have therefore printed it at length in this volume in pages 665, 666, 667, - - - 671, and have afterwards applied it to four different examples, in which I have, by means of it, computed Napier's Logarithms of four given sines. These examples are contained in pages 671, 672, 673, &c. - - - 681.

And, as it may sometimes be also convenient to be able to derive from a given Logarithm in Lord Napier's System the value of the Sine to which such Logarithm belongs, or of the ratio of which to the radius r, or 10,000,000, it is the measure, Mr. Ivory has likewise solved this second Problem in a very clear and satisfactory manner by means of the same useful binomial Theorem

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which he had employed in the ſolution of the former Problem, and has communicated his Solution to me with permiſſion to publiſh it in this collection: which I have accordingly done in pages 681, 682, 683, &c.---688, and have afterwards illuſtrated it by applying it to four examples, or computing by means of it the Sines that correſpond to four different given Logarithms in Lord Napier's Table. Theſe examples begin in page 688, and end in page 693. The reſult of Mr. Ivory's Solution of this Second Problem is, "That, "if x be put for the given Logarithm in Lord Napier's Table, of ſome unknown Sine S of an arch in a Quadrant of a Circle of which the radius is "called r, the ſaid ſine S will be equal to the infinite ſeries $r-x+\frac{x^2}{2r}-$ "$\frac{x^3}{2.3.r}+\frac{x^4}{2.3.4.r^3}-\frac{x^5}{2.3.4.5.r^4}+\frac{x^6}{2.3.4.5.6.r^5}-\frac{x^7}{2.3.4.5.6.7.r^6}+$ &c; or, if "we put L for the ſaid given Logarithm inſtead of x, the ſaid Sine S will be "$=$ the infinite Series $r-L+\frac{L^2}{2r}-\frac{L^3}{2.3.r^2}+\frac{L^4}{2.3.4.r^3}-\frac{L^5}{2.3.4.5.r^4}+$ $\frac{L^6}{2.3.4.5.6.r^5}-\frac{L^7}{2.3.4.5.6.7.r^6}+$ &c."

In the remaining pages of this 20th Tract, intitled *Obſervations on the foregoing Tract of Lord Napier*, from page 693 to page 710, I have endeavoured to give a diſtinct account of the changes made, or intended to be made, afterwards by Lord Napier in the Syſtem of Logarithms deſcribed in the foregoing Latin Tract, and ſet-down in the Table at the end of it, and of the ſhare which Mr. Henry Briggs had in making thoſe changes, and of the further changes made in them by Mr. Briggs after Lord Napier's death; and for that purpoſe I have cited at full length a paſſage or two from Mr. Briggs's *Arithmetica Logarithmica*, and from Mr. Edward Wright's Engliſh tranſlation of Lord Napier's Tract, that relate to this ſubject, but which require to be carefully examined and compared, in order to our forming a clear conception of theſe changes, there being ſome expreſſions in theſe paſſages which are not eaſily to be made conſiſtent with other parts of them. But I have laid them all fairly before the reader, that he may form his own judgment of their meaning, and determine whether, or not, he finds reaſon to concur with me in the interpretation I have given of them, and in the opinion I have formed of the ſeveral gradual changes and improvements that have been made in the ſyſtem of Logarithms, from the firſt publication of Lord Napier's Tract to their being publiſhed in their preſent

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ſtate in the Tables now in general uſe. This opinion is ſet-forth in art. 45, pages 700, 701, 702, and 703.

This 20th tract extends through 86 pages, beginning in page 625, and ending in page 710.

Of the Twenty-first Tract in this Volume.

The twenty-firſt tract in this volume is, *An Account of John Speidell's New Logarithms, derived by Subtraction from the Logarithms of Lord Napier's firſt Syſtem, and publiſhed, firſt, in the year* 1619, *and afterwards in the year* 1628. This tract has been drawn-up by my learned friend Mr. James Ivory, without whoſe aſſiſtance I was not able to underſtand the nature of theſe New Logarithms, as Mr. Speidell himſelf had not explained them ſufficiently. But Mr. Ivory has ſhewn us in this tract that they are the ſeveral remainders that ariſe from the Subtraction of the ſeveral Logarithms ſet-down in Lord Napier's Table from the great number 100,000,000, or an hundred millions, which is greater than the greateſt of thoſe Logarithms, to wit, the Logarithm 81,425,681, which is that of 2909, or the Sine of an arch of 1 Minute, or of the ratio of the ſaid Sine of 1 Minute to the radius, or 10,000,000.

Mr. Speidell's object in publiſhing theſe new Logarithms, inſtead of thoſe ſet-down in Lord Napier's Table, was to baniſh from the Trigonometrical Canon of Sines, Tangents, and Secants of circular arches with their Logarithms annexed to them, the diſtinction of Logarithms into two kinds, or claſſes, to wit, *affirmative* Logarithms, or ſuch as were to be added to others in the reſolution of Plane and Spherical Triangles, and *negative* Logarithms, or ſuch as were to be ſubtracted from others in performing thoſe reſolutions, which diſtinction aroſe from the two contrary kinds of ratios, (to wit, ratios of majority, or of greater quantities to leſſer quantities, and ratios of minority, or of leſſer quantities to greater quantities,) of which they were the repreſentatives, or meaſures. And this he effects by chuſing the number 100,000,000 (which is greater than 81,425,681, or the Logarithm of 2909, or the Sine of 1 Minute, or of the ratio of the ſaid Sine of 1 Minute to the radius, or 10,000,000, which is the greateſt Logarithm that occurs in Lord Napier's Table of Logarithms,) for

the number from which he ſubtracts all the Logarithms in the ſaid table, and of which he conſiders them as parts. But, as it is not eaſy to explain this matter in fewer words than are employed for this purpoſe by Mr. Ivory in this 21ſt tract itſelf, I refer the reader to the ſaid tract for a fuller illuſtration of it.

As this Table of *New Logarithms*, publiſhed by John Speidell in the year 1619, and again in the year 1628, ſeems to be intirely ſuperſeded by the Tables of Briggs's Logarithms that are now in general uſe, I have not thought fit to reprint it.

But in the edition of John Speidell's Logarithms that was publiſhed in the year 1628, there is, beſides the above-mentioned Table of *New Logarithms*, another Table of Logarithms relating to numbers in general, without any reference to Trigonometry, in which are contained the Logarithms of all the natural numbers 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, as far as the number 1000, by which he underſtands (as Mathematicians in general do at this day,) the Logarithms of the ratios of thoſe numbers to unity, or 1. Theſe Logarithms are the ſame as Lord Napier's Logarithms of the ſame ratios, except that they are carried to one figure leſs than his Logarithms; the Logarithm of the ratio of 10 to 1 in this ſecond Table of Speidell being the whole number 2,302,584, which with the additional figure 2, makes 23,025,842, which is Napier's Logarithm of that ratio, or of the equal and contrary ratio of 1 to 10. The deſcription of this ſecond Table is given in the laſt, or 12th, article of Mr. Ivory's account of Speidell's Logarithms, in page 727. And, as it is ſuppoſed that this ſecond table of *John Speidell*, which was publiſhed in the edition of the year 1628, may be of conſiderable uſe to Mathematicians in cultivating ſome branches of Sir Iſaac Newton's Method of Fluxions, I have here reprinted it in pages 728, 729, 730, &c, - - - - 759.

Of the Twenty-second, and last, Tract in this Volume.

The laſt tract in this volume is a very long and laborious Computation of the Length of the Sine of an arch of 1 Minute of a Degree in a Circle of which the Radius is called 1, derived from the Tangent of an arch of 45 Degrees,

Degrees, which is known to be equal to the Radius. This computation is performed by various ſections, or diviſions, of an arch of 45 Degrees into leſſer arches by reſolving the equations that expreſs the relations of the tangent of any given arch of a quadrant to the tangent of a leſſer arch that is a known aliquot part of the former arch; by which we may obtain, firſt, by the Quinquiſection of an arch of 45 Degrees, the Tangent of an arch of 9 Degrees; and 2dly, by the Triſection of an arch of 9 Degrees, the Tangent of an arch of 3 Degrees; and 3dly, by the Triſection of an arch of 3 Degrees, the Tangent of an arch of 1 Degree; and 4thly, by the Quinquiſection of an arch of 1 Degree, or 60 Minutes, the Tangent of an arch of 12 Minutes; and 5thly, by the Triſection of an arch of 12 Minutes, the Tangent of an arch of 4 Minutes; and 6thly, by the Biſection of an arch of 4 Minutes, the Tangent of an arch of 2 Minutes; and 7thly, by the Biſection of an arch of 2 Minutes, the Tangent of an arch of 1 Minute. And, when we have thus found the length of the Tangent of an arch of 1 Minute exact to the intended number of decimal figures, we muſt multiply the ſaid Tangent into itſelf, in order to obtain it's ſquare, and muſt then add the ſaid ſquare to the ſquare of the radius 1, that is, to (1×1, or) 1; and the ſum thence ariſing will be equal to the ſquare of the Secant of the ſaid arch of 1 Minute, and the ſquare-root of the ſaid ſum will be equal to the ſaid Secant itſelf. And, having thus found the value of the ſaid Secant of an arch of 1 Minute, we muſt make the following proportion, to wit, "As the Secant of an arch of 1 Minute of a Degree is to the Tangent of the ſame arch, ſo is the radius of the circle, to wit, 1, to the Sine of the ſame arch;" and the fourth quantity obtained by this proportion will be the length of the ſaid Sine of an arch of 1 Minute; which was the quantity to be found. And thus we may, by common Geometry, and without the help of the infinite Series $a - \frac{a^3}{2.3.r^2} + \frac{a^5}{2.3.4.5.r^4} - \frac{a^7}{2.3.4.5.6.7.r^6}$ + &c, which has been given us by Sir Iſaac Newton for expreſſing the length of the Sine of the arch a in a circle of which the radius is called r, (and which is obtained by means both of the Doctrine of Fluxions and of that of the Reverſion of infinite Serieſes,) obtain the value of this important Sine of 1 Minute, which is the foundation of the Trigonometrical Canon, or the Table of Sines, Tangents, and Secants.

This Computation of the Sine of an arch of 1 Minute by means of the Tangent of the ſame arch, after having computed the ſaid Tangent by repeated ſections of an arch of 45 Degrees, is of the ſame degree of utility with reſpect to the compoſition of the Trigonometrical Canon, as the Computation of the Sine of the ſame arch of 1 Minute made above in the 18th Tract of this volume by Dr. Hutton by repeated ſections of an arch of 60 Degrees; the chord of which is known to be equal to the radius; and the agreement of the numeral value of the ſaid Sine of 1 Minute found by means of the Tangents with the value of it found before by means of the Chords, is a ſtrong confirmation of the accuracy of Dr. Hutton's Computation, and is ſufficient to remove every doubt concerning the exactneſs of the value of the ſaid Sine of 1 Minute which reſults from both Computations.

This grand Computation of the length of the Sine of 1 Minute by means of repeated Sections of an arch of 45 Degrees, is preceeded by a Geometrical Problem that has been ſolved by Mr. John Machin, (a very eminent Mathematician in the beginning of the eighteenth century,) "for finding the tangent of the ſum of two circular arches, (which together are leſs than 90 Degrees, or the arch of a Quadrant of the circle,) from the tangents of the ſaid two arches." This Problem and it's Solution are given in pages 766, 767, and 768; and in the Corollaries to it, (in pages 768, 769, 770,&c, - - - 773,) there are derived from it the four following equations expreſſing the relations between the Tangent of any given circular arch, (leſs than 90 Degrees, or the arch of a Quadrant,) and the tangents of it's half, and of it's third part, and of it's fourth part, and of it's fifth part; to wit, that, if T be put for the tangent of the greater arch, and t for the tangent of it's half, and r for the radius of the circle, the relation between T and t will be expreſſed by the Quadratick equation $t^2 + \frac{2r^2}{T} \times t = r^2$; and, if t be put for the tangent of it's third part, the relation between T and t will be expreſſed by the cubick equation $3rrt + 3T \times t^2 - t^3 = rrT$; and, if t be put for the tangent of it's fourth part, the relation between T and t will be expreſſed by the Biquadratick equation $\frac{4r^4}{T} \times t + 6rr \times t^2 - \frac{4rr}{T} \times t^3 - t^4 = r^4$; and, if t be put for the tangent of it's fifth part, the relation between T and t will be expreſſed by the following equation of the fifth order, to wit, $5r^4t +$

$10rrT \times t^2$

$10rrT \times t^2 - 10rrt^3 - 5Tt^4 + t^5 = r^4T$. Therefore, if the greater tangent T be known, or given, the tangent t of half the arch of which T is the tangent, may be found by reſolving the Quadratick equation $t^2 + \frac{2rr}{T} \times t = r^2$; and the tangent of the third part of the arch of which T is the tangent, may be found by reſolving the Cubick equation $3rrt + 3Tt^2 - t^3 = rrT$; and the tangent of the fourth part of the arch of which T is the tangent, may be found by reſolving the Biquadratick equation $\frac{4r^4}{T} \times t + 6rr \times t^2 - \frac{4rr}{T} \times t^3 - t^4 = r^4$; and the tangent of the fifth part of the arch of which T is the tangent, may be found by reſolving the following equation of the fifth order, to wit, $5r^4t + 10rrT \times t^2 - 10rrt^3 - 5Tt^4 + t^5 = r^4T$.

And, if the radius r of the Circle be called 1, theſe four equations for the Biſection, Triſection, Quadriſection, and Quinquiſection, of the arch of which T is the tangent, will be thereby much ſimplified, and reduced to the four following equations, to wit, for the Biſection of the arch of which the tangent T is known, or given, to the Quadratick equation $t^2 + \frac{2}{T} \times t = 1$; and for the Triſection of the ſaid arch, to the Cubick equation $3t + 3T \times t^2 - t^3 = T$; and for the Quadriſection of the ſaid arch, to the Biquadratick equation $\frac{4}{T} \times t + 6t^2 - \frac{4}{T} \times t^3 - t^4 = 1$; and for the Quinquiſection of the ſaid arch, to the equation, of the fifth order, $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$. And this is the caſe in the preſent Computation, the radius of the circle being therein ſuppoſed to be equal to 1.

Further, in the preſent Computation no uſe is made of the equation for Quadriſecting an arch of which the tangent T is known, but only of the equations for Biſecting, Triſecting, and Quinquiſecting it, that is, of the equations $t^2 + \frac{2}{T} \times t = 1$, $3t + 3T \times t^2 - t^3 = T$, and $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$.

This Computation has been made with great care and ſkill by Dr. Andrew Mackay, L. L. D. late of Aberdeen in Scotland, and a fellow of the Royal Society

Society of Edinburgh, and has been carried to more than 20 places of decimal figures; and the ſeveral ſtages, or branches, of it may be briefly deſcribed as follows.

The firſt Stage, or branch, of this Computation is to Quinquiſect an arch of 45 Degrees, of which the tangent is known to be equal to the radius of the circle, which in this Computation is 1. We muſt therefore ſubſtitute 1 inſtead of T in the equation $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$, by which it will be converted into the equation $(5t + 10 \times 1 \times t^2 - 10t^3 - 5 \times 1 \times t^4 + t^5 = 1$, or) $5t + 10t^2 - 10t^3 - 5t^4 + t^5 = 1$; and we muſt then reſolve this laſt equation. And this Dr. Mackay has done in the following manner.

He, firſt, by means of the number 3.141,592,653,589,793,238,462,6 &c (which has been found by the induſtrious Dutch Calculator *Van Ceulen*, by the principles of Common Geometry, to expreſs the length of the whole circumference of a circle of which the Diameter is called 1, and which therefore will expreſs the length of the ſemi-circumference of a circle of which the radius is called 1,) computes the length of the arch of which we are ſeeking the tangent, to wit, the arch which is a fifth part of an arch of 45 Degrees, or the arch of 9 Degrees, and finds it to be $= \frac{3.141,592,653,\&c}{20}$, or 0.157,079, 632,&c. And then, as the tangent of a circular arch is always ſomewhat greater than the arch itſelf, he conjectures the tangent of this arch, or of 9 Degrees, to be nearly equal to 0.1571; and, in order to try the truth of this conjecture, he ſubſtitutes 0.1571 inſtead of t in the quinquinomial quantity $5t + 10t^2 - 10t^3 - 5t^4 + t^5$, and finds the value of the ſaid quinquinomial quantity reſulting from this ſubſtitution to be = 0.990,581,255,867,552, 868,51; which is leſs than 1, or the abſolute term of the equation $5t + 10t^2 - 10t^3 - 5t^4 + t^5 = 1$. And hence he concludes that 0.1571 is leſs than the true value of t, or the leaſt root of that equation.

Having thus diſcovered that 0.1571 is leſs than the true value of t, Dr. Mackay puts z for their unknown difference, and ſubſtitutes the binomial quantity $0.1571 + z$ inſtead of t in the equation $5t + 10t^2 - 10t^3 - 5t^4 + t^5 =$ 1, and

1, and reſolves the new equation reſulting from ſuch ſubſtitution according to Mr. Raphſon's Method of Approximation, and finds z to be $= 0.001,28$, and conſequently t, or $0.1571+z$, to be $(= 0.1571 + 0.001,28) = 0.15838$. And thus he obtains the number 0.15838 for a ſecond near value of t, or the leaſt root of the equation $5t+10t^2-10t^3-5t^4+t^5 = 1$.

He then ſubſtitutes this number 0.15838 inſtead of t in the quinquinomial quantity $5t+10t^2-10t^3-5t^4+t^5$, and finds the value of the ſaid quantity reſulting from the ſaid ſubſtitution to be $= 0.999,967,413,423,063,877,778$; which is leſs than 1, or the abſolute term of the equation $5t+10t^2-10t^3-5t^4+t^5 = 1$. And conſequently 0.158,38 muſt be leſs than the true value of t, or the leaſt root of that equation.

Then, by another Proceſs of Mr. Raphſon's Method of Approximation, Dr. Mackay obtains the number 0.158,384,440,3 for a third near value of t; and, by a third Proceſs of the ſame Method, he obtains the number 0.158,384,440,324,536,293,83 for a fourth near value of t; which is exact in all it's twenty figures. And he afterwards, by further calculations, proves that the number 0.158,384,440,324,536,293,838,883 will be a fifth near value of t, or the tangent of an arch of 9 Degrees, that will be ſtill nearer to it's true value, and will be true in all it's twenty-four figures.

This reſolution of the quinquinomial equation $5t+10t^2-10t^3-5t^4+t^5 = 1$ is very long and laborious, and takes-up more than 22 pages, beginning at the bottom of page 781, and ending in page 803. But all the Arithmetical operations in it are ſet-down at length, that the induſtrious reader, that has a taſte for calculations of this nature, may be enabled to go-over them carefully, line by line, with his eye, and perceive that they are rightly performed; which he may do in this manner with much leſs labour than would be neceſſary for the performance of the ſame operations by his own endeavours, and without having them thus preſented before him for his inſpection and examination.

The next, or ſecond, branch of this Computation is to Triſect an arch of 9 Degrees, of which the tangent has been found to be $= 0.158,384,440,324,536,293,$

536,293,838,883, by ſubſtituting this number inſtead of T in the Cubick equation $3t+3Tt^2-t^3 = T$, and reſolving the equation ($3t+3\times0.158,384,440,324,536,293,838,883\times t^2-t^3 = 0.158,384,440,324,536,293,838,883$, or) $3t+0.475,153,320,973,608,881,516,649\times t^2-t^3 = 0.158,384,440,324,536,293,838,883$, reſulting from ſuch ſubſtitution; whereby we ſhall obtain the value of t, or the leſſer root of this Cubick equation, which will be the value of the tangent of the third part of an arch of 9 Degrees, or of the tangent of an arch of 3 Degrees.

To find this value of t, or the tangent of an arch of 3 Degrees, Dr. Mackay, in the firſt place, obſerves that an arch of 3 Degrees is equal to the third part of an arch of 9 Degrees, or to $\frac{0.15707}{3}$, or to 0.05236, and that the tangent of an arch of 3 Degrees muſt therefore be ſomewhat greater than 0.05236; and he therefore conjectures that it will be $= 0.0524$, and ſubſtitutes this number inſtead of t in the trinomial quantity $3t+0.475,153,320,973,608,881,516,649\times t^2-t^3$, in order to diſcover whether the value of this trinomial quantity reſulting from ſuch ſubſtitution will be greater, or will be leſs, than 0.158,384,440,324,536,293,838,883, or the abſolute term of the equation that is to be reſolved. And he finds that the ſaid value reſulting from this ſubſtitution is 0.158,360,779,158,596,496,322,514, which is leſs than the ſaid abſolute term of the equation; whence it follows that 0.0524 is leſs than the true value of t, or the leſſer root of the ſaid equation, or than the tangent of an arch of 3 Degrees.

He therefore puts z for the difference between 0.0524 and t, and ſubſtitutes the binomial quantity $0.0524+z$ inſtead of t in the propoſed equation, and reſolves the new equation thence reſulting by Mr. Raphſon's Method of Approximation, and thereby finds z to be $= 0.000,007,779,289,3$, and conſequently t, (or $0.0524+z$,) to be $= 0.052,407,779,289,3$; of which number it is probable that the firſt nine figures 0.052,407,779 are correct.

Dr. Mackay then ſubſtitutes 0.052,407,779, inſtead of t, in the trinomial quantity $3t+0.475,153,320,973,608,881,516,649\times t^2-t^3$, and finds the value

value of the ſaid quantity reſulting from ſuch ſubſtitution to be = 0.158, 384,439,463,648,432,218,748, which is leſs than the abſolute term 0.158,384, 440,324,536,293,838,883, of the propoſed equation. Therefore 0.052,407, 779 is leſs than the true value of t, or the leſſer root of the ſaid equation.

Dr. Mackay then puts z for the difference between 0.052,407,779 and t, and ſubſtitutes the binomial quantity $0.052,407,779+z$, inſtead of t, in the propoſed equation, and reſolves the new equation thence reſulting by Mr. Raphſon's Method of Approximation, and thereby finds z to be = 0.000,000,000,283, 041,204, and conſequently t, or $0.052,407,779+z$, to be = 0.052,407,779, 283,041,204. Therefore t, or the tangent of an arch of 3 Degrees, is equal to 0.052,407,779,283,041,204.

This number 0.052,407,779,283,041,204 is exact in all it's figures. But Dr. Mackay has carried his inveſtigation of this Tangent ſtill further by means of a third Proceſs of Mr. Raphſon's Method of Approximation, and has found t, or the tangent of an arch of 3 Degrees, to be equal to 0.052,407, 779,283,041,204,038,805, and afterwards proves, by the neceſſary trials and ſubſtitutions, that this laſt number is exact in all it's 24 figures.

This Triſection of an arch of 9 Degrees extends through 16 pages, beginning in page 804, and ending in page 819.

The third branch of this Computation is to triſect an arch of 3 Degrees by reſolving the Cubick equation $3t+3T\times t^2-t^3 = T$, in which T is the Tangent of 3 Degrees, and is = 0.052,407,779,283,041,204,038,805, and t, or the leſſer root of the ſaid equation, is the tangent of an arch of 1 Degree.

By reaſonings ſimilar to thoſe employed in the foregoing triſection of an arch of 9 Degrees, Dr. Mackay conjectures that the tangent of an arch of 1 Degree will be = 0.017,455, and ſubſtitutes this number inſtead of t in the trinomial quantity $3t+3T\times t^2-t^3$, in order to diſcover whether the reſult of this ſubſtitution will be greater, or will be leſs, than the abſolute term T, or 0.052,407,779,283,041,204,038,805; and he finds that the ſaid reſult will

be = 0.052,407,584,201,365,065,880,997; which is leſs than 0.052,407,779,283,041,204,038,805, or T. Therefore 0.017,455 will be leſs than the true value of t in the ſaid equation $3t+3T\times t^2-t^3 = T$.

He therefore puts z for the difference between 0.017,455, and t, and ſubſtitutes the binomial quantity 0.017,455 + z, inſtead of t, in the equation $3t+3T\times t^2-t^3 = T$, (or $3t+3\times 0.052,407,779,283,041,204,038,805\times t^2-t^3 = 0.052,407,779,2\ 3,041,204,038,8\ 5,$) or $3t+0.157,223,337,849,123,612,116,415\times t^2-t^3 = 0.052,407,779,283,041,204,038,805$, and then reſolves the equation reſulting from this ſubſtitution by Mr. Raphſon's Method of Approximation, and finds z to be = 0.000,000,064,928,217,733, and conſequently 0.017,455+z, or t, to be = 0.017,455,064,928,217,733; that is, the tangent of an arch of 1 Degree will be nearly equal to 0.017,455,064,928,217,733, of which number it is probable that at leaſt the firſt 12 figures 0.017,455,064,928, are exact.

Dr. Mackay then ſubſtitutes the number 0.017,455,064,928 inſtead of t in the trinomial quantity $3t+0.157,223,337,849,123,612,116,415\times t^2-t^3$, in order to diſcover whether the reſult of this ſubſtitution will be greater, or will be leſs, than the abſolute term 0.052,407,779,283,041,204,038,805; and he finds the ſaid reſult to be = 0.052,407,779,282,387,451,364,858, which is leſs than the ſaid abſolute term. Therefore 0.017,455,064,928 is leſs than the true value of t.

Dr. Mackay then puts z for the exceſs of t above 0.017,455,064,928, and ſubſtitutes the binomial quantity 0.017,455,064,928 + z inſtead of t in the propoſed equation $3t+3T\times t^2-t^3 = T$, and reſolves the equation thence ariſing by Mr. Raphſon's Method of Approximation, and finds z to be = 0.000,000,000,000,217,585,765,126, and conſequently 0.017,455,064,928 + z, or t, to be = 0.017,455,064,928,217,585,765,126; that is, the Tangent of 1 Degree will be = 0.017,455,064,928,217,585,765,126.

This number is true in all but the two loweſt figures 26, as Dr. Mackay proves afterwards by making the proper ſubſtitutions; and he further ſhews that, if 4 be added to the laſt figure 6, the reſulting number 0.017,455,064,928,

217,585,

217,585,765,130 will agree with the more exact value of t, or the Tangent of 1 Degree, in all it's 24 figures. And thus he compleats this ſecond Triſection of a circular arch by means of the Tangential equation $3t + 3T \times t^2 - t^3 = T$, to a very great degree of exactneſs.

This Triſection of an arch of 3 Degrees extends through 12 pages, beginning in page 820, and ending in page 831.

The fourth branch of this Computation is to Quinquiſect an arch of 1 Degree, or 60 Minutes, of which the Tangent has been found to be = 0.017, 455,064,928,217,585,765,130, by reſolving the equation $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$, in which T is equal to the tangent of 1 Degree, or 60 Minutes, and the leaſt of the three values of t is equal to the tangent of an arch of 12 Minutes.

By ſubſtituting, inſtead of T, in this equation the number 0.017,455,064, 928,217,585,765,130, to which T is equal, the ſaid equation will become $5t + 0.174,550,649,282,175,857,651,30 \times t^2 - 10t^3 - 0.087,275,324,641,087,928,825,65 \times t^4 + t^5 = 0.017,455,064,928,217,585,765,130$; which Dr. Mackay reſolves in the following manner.

By reaſonings ſimilar to thoſe employed in the reſolutions of the foregoing tangential equations Dr. Mackay conjectures that the tangent of an arch of 12 Minutes, or the leaſt value of t in this equation, will be very nearly equal to 0.003,490,67: and then, by ſubſtituting this number inſtead of t in the Quinquinomial quantity $5t + 0.174,550,649,282,175,857,651,30 \times t^2 - 10t^3 - 0.087,275,324,641,087,928,825,65 \times t^4 + t^5$, he finds that the value of the ſaid quantity reſulting from ſuch ſubſtitution is = 0.017,455,051,517,948, 824,771,82; which is leſs than 0.017,455,064,928,217,585,765,130, or the abſolute term of the equation $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$; whence it follows that the number 0.003,490,67 is leſs than the true value of t, or the leaſt root of that equation.

Dr. Mackay then puts z for the exceſs of t above 0.003,490,67, and ſubſtitutes the binomial quantity $0.003,490,67 + z$ inſtead of t in the propoſed

equation $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$, and then reſolves the equation reſulting from this ſubſtitution by Mr. Raphſon's Method of Approximation, and finds z to be $=$ 0.000,000,002,681,596,250,401, and conſequently 0.003,490,67 $+ z$, or t, to be $=$ 0.003,490,672,681,596,250,401; of which number it is highly probable that the firſt fifteen figures, (reckoned from the place of units,) to wit, 0.003,490,672,681,596, are true.

Dr. Mackay then ſubſtitutes this number 0.003,490,672,681,596, inſtead of t, in the Quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$, and finds the reſult to be $=$ 0.017,455,064,928,216,333,959,018; which is leſs than 0.017,455,064,928,217,585,765,130, or the abſolute term of the propoſed equation; whence it follows that the number 0.003,490,672,681,596 is leſs than the true value of t, or the leaſt root of that equation, or the tangent of an arch of 12 Minutes.

Dr. Mackay then puts z for the exceſs of t above 0.003,490,672,681,596, and ſubſtitutes the binomial quantity 0.003,490,672,681,596 $+ z$, inſtead of t, in the propoſed equation $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$, or $5t +$ 0.174,550,649,282,175,857,651,30 $\times t^2 - 10t^3 -$ 0.087,275,324,641,087,928,825,65 $\times t^4 + t^5 =$ 0.017,455,064,928,217,585,765,130, and then reſolves the new equation reſulting from ſuch ſubſtitution by Mr. Raphſon's Method of Approximation, and thereby finds z to be $=$ 0.000,000,000,000,000,250,318,516, and conſequently t, or 0.003,490,672,681,596 $+ z$, to be $=$ 0.003,490,672,681,596,250,318,516; that is, the Tangent of an arch of 12 Minutes will be equal to 0.003,490,672,681,596,250,318,516. And this number Dr. Mackay afterwards proves, by making the proper ſubſtitutions and trials, to be exact in all it's 24 decimal figures.

This Quinquiſection of an arch of 1 Degree, or 60 Minutes, extends through 18 pages, beginning in page 832, and ending in page 849.

The fifth branch of this Computation is to Triſect an arch of 12 Minutes, (of which the tangent is now found to be $=$ 0.003,490,672,681,596,250,318,516,) by reſolving the Cubick equation $3t + 3T \times t^2 - t^3 = T$, in which T is

T is equal to the Tangent of 12 Minutes, or to the number 0.003,490,672,681,596,250,318,516.

By ſubſtituting 0.003,490,672,681,596,250,318,516 inſtead of T in this equation $3t+3T\times t^2-t^3 = T$, it will be converted into the equation $(3t+3\times 0.003,490,672,681,596,250,318,516\times t^2-t^3 = 0.003,490,672,681,596,250,318,516$, or) $3t + 0.010,472,018,044,788,750,955,548 \times t^2 - t^3 = 0.003,490,672,681,596,250,318,516$; which Dr. Mackay reſolves in the following manner.

By reaſonings ſimilar to thoſe employed in the reſolutions of the preceeding Tangential equations, he conjectures that t, or the tangent of an arch of 4 Minutes, is nearly equal to 0.001,163,553, and ſubſtitutes that number, inſtead of t, in the trinomial quantity $3t+0.010,472,018,044,788,750,955,548\times t^2-t^3$, and finds the reſult of ſuch ſubſtitution to be $=$ 0.003,490,671,602,317,377,578,15; which is leſs than the abſolute term 0.003,490,672,681,596,250,318,516 of the propoſed equation $3t+3T\times t^2-t^3 = T$, and therefore the number 0.001,163,553 muſt be leſs than t, or the leaſt root of the ſaid equation.

He therefore puts z for the exceſs of t above 0.001,163,553, and ſubſtitutes the binomial quantity $0.001,163,553+z$, inſtead of t, in the propoſed equation $3t+3T\times t^2-t^3 = T$, and reſolves the equation thence reſulting by Mr. Raphſon's Method of Approximation, and thereby finds z to be $=$ 0.000,000,000,359,757,188,938,9, and conſequently $0.001,163,553+z$, or t, to be $=$ 0.001,163,553,359,757,188,938,9; of which it is probable that the firſt 16 figures, to wit, 0.001,163,553,359,757,1, are correct.

He then ſubſtitutes this number 0.001,163,553,359,757,1 inſtead of t in the Trinomial quantity $3t+3T\times t^2-t^3$, and finds the value of the ſaid quantity reſulting from ſuch ſubſtitution to be $=$ 0.003,490,672,681,595,983,500,79; which is leſs than the abſolute term T, or 0.003,490,672,681,596,250,318,516; and therefore the ſaid number 0.001,163,553,359,757,1 muſt be leſs than the true value of t, or the leſſer root of the propoſed equation.

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Dr. Mackay then puts z for the exceſs of t above the number 0.000,163,553,359,757,1, and ſubſtitutes the binomial quantity 0.001,163,553,359,757,1 + z, inſtead of t, in the equation $3t + 3T \times t^2 - t^3 = T$, and reſolves the equation reſulting from ſuch ſubſtitution by Mr. Raphſon's Method of Approximation, and thereby finds z to be = 0.000,000,000,000,000,088,938,65, and conſequently 0.001,163,553,359,757,1 + z, or t, to be = 0.001,163,553,359,757,188,938,65. Therefore the Tangent of an arch of 4 Minutes will be = 0.001,163,553,359,757,188,938,65. And this number he afterwards proves, by making the proper ſubſtitutions and trials, to be true in all it's figures.

This Triſection of an arch of 12 Minutes extends through 11 pages, beginning in page 849, and ending in page 859.

The ſixth branch of this Computation is to Biſect an arch of 4 Minutes, of which the Tangent is now found to be = 0.001,163,553,359,757,188,938,65, by reſolving the Quadratick equation $t^2 + \frac{2}{T} \times t = 1$, in which T is the Tangent of an arch of 4 Minutes, or is = 0.001,163,553,359,757,188,938,65.

If we divide by 0.001,163,553,359,757,188,938,65 the Quotient will be = 1718.872,609,690,509,838,486,820,225,3, which is therefore the coefficient of t in the Quadratick equation $t^2 + \frac{2}{T} \times t = 1$. Therefore the Quadratick equation $t^2 + \frac{2}{T} \times t = 1$ will become $t^2 + 1718.872,609,690,509,838,486,820,225,3 \times t = 1$, which Dr. Mackay reſolves in the following manner.

By reaſonings ſimilar to thoſe employed by him in the reſolutions of the preceeding Tangential equations, he conjectures that the root of this equation, or the tangent of an arch of 2 Minutes, will be very nearly equal to 0.000,581,776,42, and he ſubſtitutes that number inſtead of t in the binomial quantity $t^2 + 1718.872,609,690,509,838,486,820,225,3 \times t$, and finds the value of the ſaid quantity reſulting from ſuch ſubſtitution to be = 0.999,999,891,765,604,989,826,040,5; which is leſs than 1, or the abſolute

term

term of the equation $t^2 + \frac{2}{T} \times t = 1$; and consequently 0.000,581,776,42 is less than the true value of t in that equation.

He therefore puts z for the excess of t above 0.000,581,776,42, and substitutes the binomial quantity 0.000,581,776,42+z, instead of t, in the equation $t^2 + \frac{2}{T} \times t = 1$, or t^2+1718.872,609,690,509,838,486,820,225,3 $\times t = 1$, and resolves the equation resulting from such substitution by Mr. Raphson's Method of Approximation, and finds z to be = 0.000,000,000,062,968,204,352, and consequently 0.000,581,776,42+z, or t, to be = 0.000,581,776,482,968,204,352. Therefore the Tangent of an arch of 2 Minutes will be nearly equal to 0.000,581,776,482,968,204,352.

He then substitutes this number 0.000,581,776,482,968,204,352, instead of t, in the binomial quantity t^2+1718.872,609,690,509,838,486,820,225,3 $\times t$, and finds the value of the said quantity resulting from such substitution to be = 0.999,999,999,999,999,998,710,568,2; which is less than 1, or the absolute term of the equation $t^2 + \frac{2}{T} \times t = 1$; and consequently 0.000,581,776,482,968,204,352 is less than the true value of t in that equation, or than the Tangent of an arch of 2 Minutes.

Dr. Mackay then puts z for the excess of t above 0.000,581,776,482,968,204,352, and substitutes the binomial quantity 0.000,581,776,482,968,204,352+z, instead of t, in the equation $t^2 + \frac{2}{T} \times t = 1$, or t^2+1718.872,609,690,509,838,486,820,225,3 $\times t = 1$, and resolves this equation by Mr. Raphson's Method of Approximation, and finds z to be = 0.000,000,000,000,000,000,000,750,16, and consequently 0.000,581,776,482,968,204,352+z, or t, to be = 0.000,581,776,482,968,204,352,750,16; of which, if no mistakes have been made in the arithmetical operations, all the 26 figures must be true. Therefore the Tangent of an arch of 2 Minutes will be = 0.000,581,776,482,968,204,352,750,16.

But this equation $t^2 + \frac{2}{T} \times t = 1$, or t^2+1718.872,609,690,509,838,

486,

$486,820,225 \times t = 1$, being a Quadratick equation, may be reſolved in a direct manner to as great a degree of exactneſs as we pleaſe, by ſquaring the half of the co-efficient of t, or the half of the long number 1718.872,609,690,509,838,486,820,225, and adding the ſquare to both ſides of the equation, and extracting the ſquare-roots of the ſums. And Dr. Mackay has given us alſo this reſolution of this equation in the manner following.

He divides the co-efficient 1718.872,609,690,509,838,486,820,225, (omitting only the three laſt figures 225,) by 2, and finds the quotient, or the ſaid co-efficient, to be = 859.436,304,845,254,919,243,410; and he then multiplies this number into itſelf, and finds the product, or ſquare, to be = 738,630.762,086,065,944,179,205,412,018; and, adding this ſquare to both ſides of the propoſed equation $t^2 + \frac{2}{T} \times t = 1$, he converts it into the equation $t^2 + 1718.872,609,690,509,838,486,820 \times t + \overline{859.436,304,845,254,919,243,410}^2 = 738,631.762,086,065,944,179,205,412,018$; and then, extracting the ſquare-roots of both ſides, he has $t + 859.436,304,845,254,919,243,410 = 859.436,886,621,737,887,447,762,7$, and conſequently

$$t = \left\{ \begin{array}{l} 859.436,886,621,737,887,447,762,7 \\ - \ 859.436,304,845,254,919,243,410,0 \end{array} \right\}$$

$$= \quad 0.000,581,776,482,968,204,352,7;$$

that is, the root of the equation $t^2 + \frac{2}{T} \times t = 1$, or the Tangent of an arch of 2 Minutes, will be = 0.000,581,776,482,968,204,352,7. This number agrees with the number 0.000,581,776,482,968,204,352,750,16, found before by Mr. Raphſon's Method of Approximation, in all it's twenty-two figures 0.000,581,776,482,968,204,352,7.

This Biſection of an arch of 4 Minutes extends through 14 pages, beginning in page 860, and ending in page 873.

The ſeventh, and laſt, branch of this Computation for finding the length of the Tangent of an arch of 1 Minute, is to Biſect an arch of 2 Minutes, (of which

which we now know the tangent to be equal to 0.000,581,776,482,968,204,352,75,) by reſolving the Quadratick equation $t^2 + \frac{2}{T} \times t = 1$, in which T is $=$ 0.000,581,776,482,968,204,352,75. This reſolution Dr. Mackay performs, firſt, by Mr. Raphſon's Method of Approximation, and afterwards by the direct Method, by adding the ſquare of half the numeral co-efficient, $\frac{2}{T}$, of t to both ſides of the equation $t^2 + \frac{2}{T} \times t = 1$, and then extracting the ſquare-roots of the ſums. Theſe reſolutions are as follows.

Dr. Mackay, in the firſt place, divides the numerator, 2, of the ſaid co-efficient $\frac{2}{T}$, by it's denominator T, or 0.000,581,776,482,968,204,352,75, and finds the Quotient to be $=$ 3437.746,382,933,985,613,383,297,287, and thereby converts the equation $t^2 + \frac{2}{T} \times t = 1$ into the equation $t^2 +$ 3437.746,382,933,985,613,383,297,287$\times t = 1$. And then, by reaſonings ſimilar to thoſe which he employed in the reſolutions of the preceeding Tangential equations, he conjectures that t, or the tangent of an arch of 1 Minute, will be $=$ 0.000,290,888,21, and ſubſtitutes this number 0.000,290,888,21, inſtead of t, in the binomial quantity $t^2 + \frac{2}{T} \times t$, or $t^2 +$ 3437.746,382,933,985,613,383,297,287$\times t$, and finds the value of the ſaid quantity reſulting from ſuch ſubſtitution to be $=$ 0.999,999,976,381,592,340,246,920; which is leſs than 1, or the abſolute term of the equation $t^2 + \frac{2}{T} \times t = 1$. Therefore the number 0.000,290,888,21 is leſs than the true value of t in that equation. He then puts z for the exceſs of t above 0.000,290,888,21, and ſubſtitutes the binomial quantity 0.000,290,888,21$+z$, inſtead of t, in the equation $t^2 + \frac{2}{T} \times t = 1$, or $t^2 +$ 3437.746,382,933,985,613,383,297,287$\times t = 1$, and reſolves the equation reſulting from this ſubſtitution by Mr. Raphſon's Method of Approximation, and thereby finds z to be $=$ 0.000,000,000,006,870,315,908,123, and conſequently t, (or 0.000,290,888,21$+z$,) to be $=$ 0.000,290,888,216,870,315,908,123; of which number the firſt 21 figures, to wit, 0.000,290,888,216,870,315,908, will be exact, if no miſtake has been made in the calculation. And Dr. Mackay

afterwards proves that these 21 figures are exact, by substituting them, instead of t, in the binomial quantity $t^2 + \frac{2}{T} \times t$, or $t^2 + 3437\cdot746,382,933,985,613,383,297,287 \times t$, and finding that the result is $= 0.999,999,999,999,999,999,574$, which is less than 1, or the absolute term of the equation $t^2 + \frac{2}{T} = 1$, and by then adding 0.000,000,000,000,000,000,001 to the said number 0.000,290,888,216,870,315,908, and substituting the new number thence arising, to wit, 0.000,290,888,216,870,315,909, instead of t, in the same binomial quantity $t^2 + \frac{2}{T} \times t$, and finding that the result is $= 1.000,000,000,000,000,003,011$, which is greater than 1, or the absolute term of the equation $t^2 + \frac{2}{T} = 1$. For it follows from these results that the true value of the root t must be greater than 0.000,290,888,216,870,315,908, but less than 0.000,290,888,216,870,315,909, and consequently that all the 21 figures in the number 0.000,290,888,216,870,315,908 are exact.

Dr. Mackay then resolves the same equation $t^2 + \frac{2}{T} \times t = 1$, or $t^2 + 3437\cdot746,382,933,985,613,383,297,287 \times t = 1$, by the direct Method, as follows. He first divides the co-efficient $3437\cdot746,382,933,985,613,383,297,287$ by 2, and finds the quotient, or half of it, to be $= 1718.873,191,466,992,806,691,648,643,5$; and he then squares this last number, and finds it's square to be $= 2,954,525\cdot048,343,925,312,842,322,795,310$. He then adds this square to both sides of the equation, and thereby obtains the equation $t^2 + 3437\cdot746,382,933,985,613,383,297,287 \times t + \overline{1718.873,191,466,992,806,691,648,643,5}^2 = (1 + 2,954,525\cdot048,343,925,312,842,822,795,310) = 2,954,526.048,343,925,312,842,822,795,310$; and then (by extracting the square-roots on both sides,) obtains the equation $t + 1718.873,191,466,992,806,691,648,643,5 = 1718.873,482,355,209,677,007,556,766,9$. Therefore t will be $=$

$$\left\{\begin{array}{l} 1718.873,482,355,209,677,007,556,766,9 \\ -\ 1718.873,191,466,992,806,691,648,643,5 \end{array}\right\}$$

$$= \quad 0.000,290,888,216,870,315,908,123,4.$$

This value of t, or the tangent of an arch of 1 Minute, agrees with the value of

of it found before by Mr. Raphſon's Method of Approximation in the twenty-four figures 0.000,290,888,216,870,315,908,123.

And thus, by finding by two different methods of reſolving the equation $t^2 + \frac{2}{T} \times t = 1$, or $t^2 + 3437.746,382,933,985,613,383,297,287 \times t = 1$, that it's root t (which is equal to the tangent of an arch of 1 Minute,) is $=$ 0.000,290,888,216,870,315,908,123, the long and difficult Computation undertaken by Dr. Mackay for deriving the length of the Tangent of an arch of 1 Minute from the Tangent of an arch of 45 Degrees, by the Principles of Plane Geometry, and without the help of the Infinite Serieſes invented by Mr. James Gregory for expreſſing the length of a circular arch in powers of the tangent and the radius, and the length of the tangent in powers of the arch and radius (which ſerieſes are obtained by the Method of Fluxions and the Doctrine of the Reverſion of Serieſes,) is, at laſt, ſucceſsfully concluded.

This Biſection of an arch of 2 Minutes extends through 12 pages, beginning in page 874, and ending in page 886. And the whole Computation extends through 106 pages, beginning in page 781, and ending in page 886.

The ſecond Computation, by which, after we have found, by the reſolution of the foregoing tangential equations, that the tangent of an arch of 1 Minute of a Degree is $=$ 0.000,290,888,216,870,315,908, we may derive from it the length of the Sine of the ſame arch, is performed by Dr. Mackay as follows.

He, in the firſt place, multiplies the number 0.000,290,888,216,870,315,908, (which is equal to the tangent of an arch of 1 Minute,) into itſelf, and finds the product, or the ſquare of the ſaid tangent, to be equal to 0.000,000,084,615,954,713,990. To this ſquare he adds the ſquare of the radius 1, that is, 1×1, or 1, and thereby obtains the number 1.000,000,084,615,954,713,990 for the ſquare of the Secant of the ſame arch of 1 Minute. And then, by extracting the ſquare-root of this number 1.000,000,084,615,954,713,990,

he obtains the number 1.000,000,042,307,976,462,012, for the value of the ſaid Secant itſelf. And he then makes the following proportion, "As 1.000, 000,042,307,976,462,012, the Secant of an arch of 1 Minute, is to 0.000, 290,888,216,870,315,908, the tangent of the ſame arch, ſo is 1, the radius of the circle, to the Sine of the ſaid arch of 1 Minute; and therefore the ſaid Sine will be $= \frac{1 \times 0.000,290,888,216,870,315,908}{1.000,000,042,307,976,462,012} = \frac{0.000,290,888,216,870,315,908}{1.000,000,042,307,976,462,012}$ $=$ 0.000,290,888,204,563,424,596. Therefore the Sine of an arch of 1 Minute of a Degree is $=$ 0.000,290,888,204,563,424,596.

This value of the Sine of an arch of 1 Minute of a Degree agrees with the value of it found before in Dr. Hutton's Computation of it in page 473, to wit, the number 0.000,290,888,204,563,5, in the firſt 15 places of figures, reckoned from the place of units, to wit, the figures 0.000,290,888,204,563, and therefore confirms the truth of the ſaid value of the Sine of 1 Minute, obtained by that Computation, in all the ſaid 15 figures: which was the object of the two Computations performed by Dr. Mackay.

When Dr. Mackay had, in the manner here deſcribed, obtained, by means of theſe two Computations founded on the principles of Common Geometry, the lengths of the tangent and the ſine of an arch of 1 Minute, to twenty-one places of decimal figures, (which he had done at my requeſt,) he was led to compute them likewiſe by means of the two infinite ſerieſes which had been given by Mr. James Gregory and Sir Iſaac Newton for expreſſing them in powers of the arch of 1 Minute and the radius, in order to ſhew the perfect agreement of the values of them that would be thereby obtained with the values of them that he had before obtained by Common Geometry. This he has done in the manner following.

If the radius of a circle be called r, and any arch in it, leſs than an arch of 45 Degrees, be called a, and it's tangent be called t, the tangent t will be equal to the infinite Series $a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} + \frac{1382a^{11}}{155,925r^{10}}$ + &c, or, if r is $= 1$, (as it is in the foregoing Computations,)

to

to the infinite Series $a + \frac{a^3}{3} + \frac{2a^5}{15} + \frac{17a^7}{315} + \frac{62a^9}{2835} + \frac{1382a^{11}}{155,925} + \&c.$ And, if the ſine of the ſaid arch be called s, the ſaid ſine s will be equal to the infinite ſeries $a - \frac{a^3}{2.3.r^2} + \frac{a^5}{2.3.4.5.r^4} - \frac{a^7}{2.3.4.5.6.7.r^6} + \frac{a^9}{2.3.4.5.6.7.8.9.r^8} - \frac{a^{11}}{2.3.4.5.6.7.8.9.10.11.r^{10}} + \&c$, or, $a - \frac{a^3}{6r^2} + \frac{a^5}{120r^4} - \frac{a^7}{5040r^6} + \frac{a^9}{362,880r^8} - \frac{a^{11}}{39,916,800r^{10}} + \&c$, or, if r is $= 1$, to the infinite Series $a - \frac{a^3}{6} + \frac{a^5}{120} - \frac{a^7}{5040} + \frac{a^9}{362,880} - \frac{a^{11}}{39,916,800} + \&c.$ The former of theſe ſerieſes was invented by Mr. James Gregory, about A. D. 1670, and the latter by Sir Iſaac Newton.

The arch of 1 Minute is the 60th part of a Degree, or of the 180th part of the ſemi-circumference, which, when the radius is called 1, is $= 3.141,592,653,589,793,238,462,6$ &c. Therefore the arch of 1 Minute is $\left(= \frac{3.141,592,653,589,793,238,462,6\ \&c}{60 \times 180} = \frac{3.141,592,653,589,793,238,462,6\ \&c}{10800}\right) = 0.000,290,888,208,665,721,596,153,\&c$; or, if we carry the figures only to 21 places below the place of units, it will be $= 0.000,290,888,208,665,721,596$. Dr. Mackay therefore takes $a = 0.000,290,888,208,665,721,596$, and then computes the four following powers of a, to wit, a^2, a^3, a^4, and a^5, and finds a^2 to be $= 0.000,000,084,615,949,940,752,2$, and a^3 to be $= 0.000,000,000,024,613,782,102,8$, and a^4 to be $= 0.000,000,000,000,007,159,859,0$, and a^5 to be $= 0.000,000,000,000,000,002,082,6$; and then computes $\frac{a^3}{3}$, or $\frac{0.000,000,000,024,613,782,102,8}{3}$, and finds it to be $= 0.000,000,000,008,204,594,034,3$, and computes $\frac{2a^5}{15}$ and finds it to be (equal $\frac{0.000,000,000,000,004,165,2}{15}$) $= 0.000,000,000,000,000,000,277,7$, and conſequently the Series $a + \frac{a^3}{3} + \frac{2a^5}{15} + \&c$ to be $=$

$$\left\{\begin{array}{l} 0.000,290,888,208,665,721,596 \\ + \ 0.000,000,000,008,204,594,034,3 \\ + \ 0.000,000,000,000,000,000,277,7 \\ + \ \&c \end{array}\right\}$$

$= 0.000,290,888,216,870,315,908,0$; that is, the tangent

of

of an arch of 1 Minute will be equal to 0.000,290,888,216,870,315,908; which agrees with the value of the said tangent found above by the foregoing Computation in all it's 21 figures.

And he then computes $\frac{a^3}{6}$, or $\frac{0.000,000,000,024,613,782,102,8}{6}$, and finds it to be equal to 0.000,000,000,004,102,297,017,1, and computes $\frac{a^5}{120}$, or $\frac{0.000,000,000,000,000,002,082,6}{120}$, and finds it to be = 0.000,000,000,000,000, 000,017,3, and consequently finds the Series $a - \frac{a^3}{6} + \frac{a^5}{120} -$ &c to be

$$= \left\{\begin{array}{ll} & 0.000,290,888,208,665,721,596 \\ - & 0.000,000,000,004,102,297,017,1 \\ + & 0.000,000,000,000,000,000,017,3 \\ - & \&c \end{array}\right\}$$

$$= \left\{\begin{array}{ll} & 0.000,290,888,208,665,721,613,3 \\ - & 0.000,000,000,004,102,297,017,1 \end{array}\right\}$$

$$= 0.000,290,888,204,563,424,596,2.$$

Therefore s, or the Sine of 1 Minute of a Degree, in a circle of which the radius is called 1, will be = 0.000,290,888,204,563,424,596,2; which agrees in the first twenty-one figures, reckoned from the place of units, with the value of the said Sine found before, from the principles of common Geometry, in art. 63, page 890, to wit, 0.000,290,888,204,563,424,596.

These Computations of the lengths of the Tangent and the Sine of an arch of 1 Minute of a Degree, by the infinite Series of James Gregory and Sir Isaac Newton, take-up only 5 pages, beginning in page 891, and ending in page 895.

Dr. Mackay, after having thus computed the values of the Tangent and the Sine of an arch of 1 Minute of a Degree, by the two infinite Series invented by Mr. James Gregory and Sir Isaac Newton for expressing them in powers of the arch and the radius of the circle, and having shewn the perfect agreement of

of theſe new values of the ſaid Tangent and Sine with the values of them that had been found before by the principles of common Geometry, proceeds to make ſome very judicious Obſervations on the long Computation by which the ſaid Tangent of 1 Minute had been derived from the Tangent of an arch of 45 Degrees by repeated ſections of the ſaid arch; in which he ſhews that the firſt and ſecond branches of that long Computation, (which conſiſt of the Quinquiſection of the ſaid arch of 45 Degrees, in order to obtain the Tangent of an arch of 9 Degrees, and of the Triſection of an arch of 9 Degrees in order to obtain the tangent of an arch of 3 Degrees, and which, together, extend through no fewer than 39 pages of the preſent volume, to wit, from page 780 to page 820,) might have been avoided by taking a much leſs laborious method of finding the length of the tangent of an arch of 3 Degrees, and which yet was equally derived from the principles of common Geometry with thoſe former methods of obtaining it. This Method is as follows.

An arch of 3 Degrees is the Difference of an arch of 15 Degrees and an arch of 18 Degrees. Now it is ſhewn in books of Trigonometry that, if there be taken any two unequal arches in the Quadrant of a Circle, each of which is leſs than the arch of the whole Quadrant, and the leſſer of theſe two arches be ſubtracted from the greater, and we draw the tangents and the ſecants of the ſaid two arches, and likewiſe the tangent and ſecant of the arch which is the Difference of the two former arches, the tangent of this third arch (which is equal to the Difference of the two former arches,) will be to the difference of the tangents of the two former arches in the ſame proportion as the ſquare of the radius of the circle to the ſum of the ſaid ſquare of the radius and the rectangle contained under the tangents of the two former arches; or, if r be put for the radius of the circle, and a for the tangent of the greater of the two former arches, and b for the tangent of the leſſer of them, and c for the tangent of their difference, the tangent c will be to $a-b$, (the difference of the two former tangents,) as rr is to $rr+ab$. And conſequently, when r is $=1$, we ſhall have c to $a-b$ as 1×1 is to $1\times1+ab$, or c to $a-b$ as 1 is to $1+ab$; and therefore c will be $=\frac{a-b}{1+ab}$. Therefore, if the two tangents a and b of the two former arches are known, the

tangent

tangent c of the third arch (which is equal to the difference of the two former arches,) may be derived from them by computing the expreſſion $\frac{a-b}{1+ab}$. And conſequently, if the two former arches were arches of 18 Degrees and 15 Degrees, and we knew the lengths of their tangents a and b, the tangent c of their difference, which is an arch of 3 Degrees, would be $(= \frac{a-b}{1+ab}) = \frac{\text{tang. } 18^\circ - \text{tang. } 15^\circ}{1+\text{tang. } 18^\circ \times \text{tang. } 15^\circ}$, and might be obtained by computing the value of the ſaid fraction. Now the values of the tangents of 18 Degrees and of 15 Degrees may be obtained with very little labour of calculation by proceeding in the following manner.

It has been ſhewn above in page 773 of the preſent volume that, if T be put for the tangent of any arch in the Quadrant of a Circle that is leſs than the arch of the whole quadrant, and t for the tangent of the fifth part of the ſaid arch, and the radius of the circle be called 1, the leſſer tangent t will be the leaſt root of the quinquinomial equation $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$. Therefore t will alſo be equal to the leaſt root of the equation $\frac{5t}{T} + \frac{10T \times t^2}{T} - \frac{10t^3}{T} - \frac{5T \times t^4}{T} + \frac{t^5}{T} = \frac{T}{T}$, or of the equation $\frac{5t}{T} + 10t^3 - \frac{10t^3}{T} - 5t^4 + \frac{t^5}{T} = 1$. And this equation will always be true ſo long as the greater arch, of which T is the tangent, is of any magnitude leſs than 90 Degrees, or the arch of a Quadrant, that is, ſo long as the Tangent T is of any finite magnitude, how great ſoever. Therefore it will alſo be true when the greater arch is equal to 90 Degrees, which is the limit of all the former caſes; as might be proved, (if it ſhould be thought neceſſary,) by a demonſtration *ex abſurdo*. But, when the greater arch is equal to 90 Degrees, the tangent T becomes infinitely great, and conſequently the three terms $\frac{5t}{T}$, $\frac{10t^3}{T}$, and $\frac{t^5}{T}$, become infinitely ſmall in compariſon of the other three terms $10t^2$, $5t^4$, and 1, and therefore the equation $\frac{5t}{T} + 10t^2 - \frac{10t^3}{T} - 5t^4 + \frac{t^5}{T} = 1$ becomes in this caſe $0 + 10t^2 - 0 - 5t^4 + 0 = 1$, or $10t^2 - 5t^4 = 1$. Therefore $\frac{10t^2}{5} - t^4$ will be $= \frac{1}{5}$, or $2t^2 - t^4$ will be $= \frac{1}{5}$;

$\frac{1}{5}$; and (ſubtracting both ſides from $\frac{2}{2}\Big|^2$, or $\overline{1}\Big|^2$, or 1,) we ſhall have $1-2t^2+t^4 = 1-\frac{1}{5} = \frac{4}{5}$; and (extracting the ſquare-roots of both ſides,) $1-t^2 = \frac{2}{\sqrt{5}}$, and conſequently $1 = t^2 + \frac{2}{\sqrt{5}}$, and $t^2 = 1 - \frac{2}{\sqrt{5}}$, and (again extracting the ſquare-roots of both ſides,) $t = \sqrt{1-\frac{2}{\sqrt{5}}} = \sqrt{1-\frac{2\sqrt{5}}{5}} = \sqrt{1-\frac{4\sqrt{5}}{10}} = \sqrt{1-0.4\times\sqrt{5}}$. Therefore the tangent of an arch of 18 Degrees is $= \sqrt{1-0.4\times\sqrt{5}}$.

And another demonſtration that the tangent of an arch of 18 Degrees is equal to $\sqrt{1-\frac{2}{\sqrt{5}}}$, or $\sqrt{1-\frac{2\sqrt{5}}{5}}$, or $\sqrt{1-\frac{4\sqrt{5}}{10}}$, or $\sqrt{1-0.4\times\sqrt{5}}$, has been given in the 3d volume of this Collection of Mathematical Tracts called *Scriptores Logarithmici*, in pages 148 and 149; to which I refer the reader.

The computation of this expreſſion $\sqrt{1-0.4\times\sqrt{5}}$ requires four arithmetical operations, to wit, 1ſt, the extraction of the ſquare-root of the number 5, 2dly, the multiplication of this ſquare-root into the decimal fraction 0.4, 3dly, the ſubtraction of the product of this multiplication from 1, and 4thly, the extraction of the ſquare-root of the remainder of this ſubtraction. Neither of theſe four operations is very difficult, and the ſecond and third are remarkably eaſy. The extraction of the ſquare-root of 5 has been made to a great number of figures by Mr. Abraham Sharp, the great Engliſh Calculator of the latter part of the 17th Century; and he has found the ſaid ſquare-root to be = 2.236,067,977,499,789,696,409,173,668,731,276,235,440,618,357,375,457,7 &c, as may be ſeen in page 149 of the third volume of this Collection of Tracts called *Scriptores Logarithmici*, lines 7th and 8th from the bottom. Therefore $0.4\times\sqrt{5}$ is (= 0.4 × 2.236,067,977,499,789,696,409,173,668,731,276,) = 0.894,427,190,999,915,878,563,669,467,492,710,4, and $1-0.4\times\sqrt{5}$ is =

$$\left\{\begin{array}{l} 1.000,000,000,000,000,000,000,000,000,000,000,000,0 \\ -\ 0.894,427,190,999,915,878,563,669,467,492,710,4 \end{array}\right\}$$
$$= 0.105,572,809,000,084,121,436,330,532,507,289,6.$$

And the ſquare-root of this laſt number, which is equal to $1-0.4\times\sqrt{5}$, has been found by Mr. Abraham Sharp to be = 0.324,919,696,232,906,326, 155,871,412,215,134,4 &c; as may be ſeen in the third volume of theſe *Scriptores Logarithmici*, page 146.

Therefore $\sqrt{1-0.4\times\sqrt{5}}$, or the tangent of an arch of 18 Degrees, in a circle of which the radius is called 1, is = 0.324,919,696,232,906,326,155, 871,412,215,134,4 &c. Q. E. I.

The length of the tangent of an arch of 15 Degrees may be found as follows.

It is ſhewn in books of Trigonometry that the Difference between the Secant and the Tangent of any arch in the Quadrant of a Circle is equal to the tangent of half the complement of the ſaid arch to an arch of 90 Degrees, or the arch of the whole Quadrant. See my Elements of Plane Trigonometry, Prop. 23, Coroll. in page 65. Therefore the Difference between the Secant and the Tangent of an arch of 60 Degrees is equal to the Tangent of an arch of 15 Degrees, which is the half of 30 Degrees, or of the complement of an arch of 60 Degrees to an arch of 90 Degrees, or the arch of the whole Quadrant. But the Secant of an arch of 60 Degrees is double of the radius of the circle, and therefore, when the radius is called 1, is = 2. Therefore it's ſquare is = 4, and the ſquare of the tangent of the ſame arch is = 4 — 1 = 3, and the tangent itſelf is = $\sqrt{3}$. Therefore the difference of the tangent and ſecant of 60 Degrees is = $2-\sqrt{3}$. And conſequently the tangent of an arch of 15 Degrees will alſo be = $2-\sqrt{3}$. But the ſquare-root of 3 has been computed by Mr. Abraham Sharp, and found to be = 1.732,050,807,568,877,293,527,446,341,505,872, &c. Therefore $2-\sqrt{3}$ will be =

$$\begin{array}{l}\{\ 2.000,000,000,000,000,000,000,000,000,000,000\ \}\\ \{-\ 1.732,050,807,568,877,293,527,446,341,505,872\}\\ \hline\end{array}$$

= 0 267,949,192,431,122,706,472,553,658,494,128; that is, the tangent of an arch of 15 Degrees will be equal to the decimal fraction 0.267,949,192,431,122,706,472,553,658,494,128. Q. E. I.

Having

Having thus found the tangent of an arch of 18 Degrees to be = 0.324,919,696,232,906,326,155,871,412,215,134, and the tangent of an arch of 15 Degrees to be = 0.267,949,192,431,122,706,472,553,658,494,128, we muſt ſubtract the latter of theſe tangents from the former, and their difference will be =

$$\left\{\begin{array}{l} 0.324,919,696,232,906,326,155,871,412,215,134 \\ -\ 0.267,949,192,431,122,706,472,553,658,494,128 \end{array}\right\}$$

$$= 0.056,970,503,801,783,619,683,317,753,721,006$$

; or, if the tangent of 18 Degrees be denoted by a, and the tangent of 15 Degrees be denoted by b, we ſhall have $a-b$ (or the numerator of the fraction $\frac{a-b}{1+ab}$,) = 0.056,970,503,801,783,619,683,317,753,721,006. We muſt then multiply the tangent a, or 0.324,919,696,232,906,326,155,871,412, &c into the tangent b, or 0.267,949,192,431,122,706,472,553,658, &c, or (increaſing the laſt figure 8 by an unit, on account of the following figures 494,128, &c, which are omitted,) 0.267,949,192,431,122,706,472,553,659; and we ſhall find the product ab to be = 0.087,061,970,210,572,952,731,324,783, according to the multiplication of them performed by Dr. Mackay, which is ſet-forth in page 904. And, if to this number we add 1, we ſhall have $1+ab$, (or the denominator of the fraction $\frac{a-b}{1+ab}$,) = 1.087,061,970,210,572,952,731,324,783. And, laſtly, we muſt divide the number 0.056,970,503,801,783,619,683,317,753, or the numerator $a-b$ of the ſaid fraction, by the number 1.087,061,970,210,572,952,731,324,783, or it's denominator; and the quotient of this Diviſion, (which has been performed by Dr. Mackay, and is ſet-forth in page 904,) will be = 0.052,407,779,283,041,204,038,805, and will be equal to the fraction $\frac{a-b}{1+ab}$, or $\frac{\text{tangent of } 18^\circ - \text{tangent of } 15^\circ}{\text{square of radius } 1 + \text{tang. } 18^\circ \times \text{tang. } 15^\circ}$, and conſequently will be equal to the tangent of an arch of 3 Degrees.

Q. E. I.

This value of the tangent of an arch of 3 Degrees, found by this method ſuggeſted by Dr. Mackay, to wit, the number 0.052,407,779,283,041,204,038,805, agrees with the value of the ſame tangent found before in art. 32, page 817, in all it's twenty-four figures. And therefore it appears that, by

theſe calculations, which have been ſuggeſted and performed by Dr. Mackay, (and which have taken up only 9 pages, beginning in page 897, and ending in page 905,) we have obtained the length of the tangent of an arch of 3 Degrees to the ſame degree of exactneſs as it was obtained before, by the quinquiſection of an arch of 45 Degrees and the ſubſequent triſection of an arch of 9 Degrees in the 39 pages from page 780 to page 820.

But there is ſtill another method of proceeding to facilitate the Computation of the Tangent of an arch of 1 Minute of a Degree, in the former part of ſuch Computation, and until we have obtained the value of the Tangent of an arch of 1 Degree; which was communicated to me by my very learned and ingenious friend Mr. James Ivory of the Royal Military College at Great Marlow in Buckinghamſhire, in conſequence of my having mentioned to him the foregoing laborious Computation of it by the Quinquiſection of an arch of 45 Degrees, and the following two Triſections, the ſecond Quinquiſection, the third Triſection, and the two following Biſections, of arches, which had been performed by Dr. Mackay. This method of Mr. Ivory is ſo very ingenious, and contributes ſo greatly to leſſen the Labour of computing the length of a Tangent of 1 Degree, that I have inſerted it in this Volume immediately after Dr. Mackay's obſervations on the foregoing laborious Computation, under the title of *A Sketch of a Method of computing the length of a tangent of an arch of* 1 *Degree in a circle of which the radius is called* 1, *by Mr. James Ivory*, being confident that the peruſal of it will give great ſatisfaction to the Lovers of theſe ſubjects. The ſeveral ſteps, or branches, of this Method may be ſhortly deſcribed as follows.

Mr. Ivory, in the firſt place, puts t for the tangent of an arch of 18 Degrees, which is the fourth part of an arch of 72 Degrees. Therefore the tangent of an arch of 72 Degrees will be $= \frac{4t-4t^3}{1-6t^2+t^4}$, as is ſhewn before in pages 770, 771, 772, and 773. Further, an arch of 18 Degrees is the complement of 72 Degrees to an arch of 90 Degrees. Therefore the rectangle under the tangent of 72 Degrees and the tangent of 18 Degrees is equal to the ſquare of the radius 1; that is, $\frac{4t-4t^3}{1-6t^2+t^4} \times t$ is $= 1$, or $\frac{4t^2-4t^4}{1-6t^2+t^4}$ is $=$ 1, and

1, and consequently $4t^2-4t^4$ will be $= 1-6t^2+t^4$. Therefore (adding $6t^2$ to both sides,) we shall have $10t^2-4t^4 = 1+t^4$, and (subtracting t^4 from both sides,) we shall have $10t^2-5t^4 = 1$, and (dividing all the terms by 5,) we shall have $2t^2-t^4 = \frac{1}{5}$, which is a quadratick equation of which tt is the root. Therefore, if we subtract both sides from 1, we shall have $1-2t^2+t^4$ $(= 1-\frac{1}{5} = \frac{5}{5}-\frac{1}{5}) = \frac{4}{5}$, or $\overline{1-t^2}^2 = \frac{4}{5}$. Therefore $1-t^2$ will be $(= \sqrt{\frac{4}{5}}) = \frac{2}{\sqrt{5}}$, and 1 will be $= \frac{2}{\sqrt{5}}+t^2$, and t^2 will be $= 1-\frac{2}{\sqrt{5}}$, and consequently t will be $(= \sqrt{1-\frac{2}{\sqrt{5}}} = \sqrt{1-\frac{2\sqrt{5}}{5}}) = \sqrt{\frac{5-2\sqrt{5}}{5}}$; that is, the tangent of an arch of 18 Degrees will be $= \sqrt{\frac{5-2\sqrt{5}}{5}}$. And this I conceive to be the easiest Method that can be taken to find the length of the Tangent of an arch of 18 Degrees.

Having thus found the tangent of an arch of 18 Degrees to be equal to $\sqrt{\frac{5-2\sqrt{5}}{5}}$, Mr. Ivory puts T for this value, and t for the value of the tangent of an arch of 9 Degrees, and finds the value of t by resolving the Quadratick equation $t^2+\frac{2}{T}\times t = 1$, in which T is equal to $\sqrt{\frac{5-2\sqrt{5}}{5}}$. The value of t in this equation is $\sqrt{1+\frac{1}{T^2}}-\frac{1}{T}$, which, (being properly reduced into numbers by substituting for T it's numeral value $\sqrt{\frac{5-2\sqrt{5}}{5}}$,) is $= 1+\sqrt{5}-\sqrt{5+2\sqrt{5}}$; as is shewn in art. 79, page 909. Therefore the Tangent of an arch of 9 Degrees is $= 1+\sqrt{5}-\sqrt{5+2\sqrt{5}}$; the computation of which expression requires little more labour than that of extracting the two square-roots of the numbers 5 and $5+2\sqrt{5}$, which is trifling in comparison of the labour employed in the foregoing Computation in Quinquisecting an arch of 45 Degrees in order to obtain the same quantity, or the tangent of an arch of 9 Degrees, which took-up no fewer than 23 pages, beginning in page 781 and ending in page 803.

Mr. Ivory,

Mr. Ivory, in the 3d place, computes the length of an arch of 30 Degrees, which is eaſily ſhewn to be equal to $\frac{1}{\sqrt{3}}$, or to $\frac{\sqrt{3}}{3}$, or $\frac{1.732,050,807,568,877,293,527,\&c,}{3}$ or 0.577,350,269,189,625,764,509 &c. He then triſects the arch of 30 Degrees by reſolving the Cubick equation $3t+3T\times t^2-t^3=T$, in which T is equal to $\frac{1}{\sqrt{3}}$, or the tangent of an arch of 30 Degrees, and the leſſer of the two roots of this equation, or the leſſer value of t, is equal to the tangent of an arch of 10 Degrees. And thus he obtains the values of the tangents of the two arches of 9 Degrees and of 10 Degrees, the difference of which is only 1 Degree.

And then, in the 4th place, he puts a for the tangent of an arch of 10 Degrees, and b for the tangent of an arch of 9 Degrees, and multiplies the one into the other, and thereby obtains the value of ab; and, by adding ab to 1, obtains the value of $1+ab$. He then ſubtracts the tangent b from the tangent a, and thereby obtains the value of $a-b$; and then he divides the value of $a-b$ by the value of $1+ab$, and thereby obtains the value of the fraction $\frac{a-b}{1+ab}$, which, (by the Trigonometrical Propoſition employed by Dr. Mackay in his abridgement of the firſt Computation) is equal to the tangent of the arch which is the difference of the two arches of which a and b are the tangents, that is, to the tangent of an arch of 1 Degree.

And thus the whole Labour of computing the length of an arch of 1 Degree is reduced to that of computing the numeral expreſſion $1+\sqrt{5}-\sqrt{5+2\sqrt{5}}$, and triſecting an arch of 30 Degrees, of which the tangent is $=\frac{1}{\sqrt{3}}$, or $\frac{\sqrt{3}}{3}$, and thereby finding the value of the tangent of an arch of 10 Degrees, and then computing the value of the fraction $\frac{a-b}{1+ab}$, in which a denotes the tangent of an arch of 10 Degrees, and b denotes the tangent of an arch of 9 Degrees.

The Deſcription of this method of computing the tangents of an arch of 1 Degree communicated by Mr. Ivory, extends through 7 pages, beginning in

in page 907 and ending in page 913. And in the following 22 pages, to wit, in pages 914, 915, 916, &c. - - - - 935, I have inſerted the whole computation itſelf, with all it's arithmetical operations ſet-forth at length, as they have been performed by Dr. Mackay, by which the ſaid tangent of an arch of 1 Degree is found to be = 0.017,455,064,928,217,585,765; which agrees in all it's twenty-one figures with the firſt twenty-one figures of the number 0.017,455,064,928,217,585,765,130,659, which had been found before in art. 39, page 831, for the value of the ſame tangent, by the former computation of it, which had extended through no fewer than 51 pages, to wit, the pages from page 780 to page 832. So that Mr. Ivory's Method of computing this tangent muſt be acknowledged to be much ſhorter and leſs laborious than that former method of computing it, and yet equally clear and ſatisfactory.

The remaining 15 pages of this Volume are employed in the deſcription of another method of quinquiſecting an arch of 45 Degrees, or of reſolving the equation $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5 = r^5$, and finding it's leaſt root, (which is the tangent of an arch of 9 Degrees,) by, firſt, obſerving that one of it's other roots (for it has three in all,) will be the radius r, or the tangent of the arch of 45 Degrees itſelf, and then making uſe of that known root r to reduce the quinquinomial equation $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5 = r^5$ to a quadrinomial, or biquadratick, equation, and then reſolving the ſaid biquadratick equation by Mr. Raphſon's Method of Approximation.

Now, that the radius r is one of the roots of the quinquinomial equation $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5 = r^5$, will eaſily appear by ſubſtituting r, inſtead of t, in the quinquinomial quantity $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5$, which forms the firſt, or left-hand, ſide of that equation. For, if that ſubſtitution of r for t be made in the ſaid quantity, it will be thereby converted into the quantity $5r^4\times r+10r^3\times r^2-10r^2\times r^3-5r\times r^4+r^5$, or $5r^5+10r^5-10r^5-5r^5+r^5$, which is equal to r^5, or the abſolute term of the propoſed quinquinomial equation $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5 = r^5$. And conſequently r is one of the roots of the ſaid equation. Q. E. D.

Secondly, ſince $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5$ is $= r^5$, and $5r^4\times r+10r^3\times r^2$

$\times r^2 - 10r^2 \times r^3 - 5r \times r^4 + r^5$ is alſo $= r^5$, it follows that $5r^4t + 10r^3t^2 - 10r^2t^3 - 5rt^4 + t^5$ will be $= 5r^4 \times r + 10r^3 \times r^2 - 10r^2 \times r^3 - 5r \times r^4 + r^5$. Therefore (adding $10r^2 \times r^3 + 5r \times r^4$ to both ſides,) we ſhall have $5r^4t + 10r^3t^2 + 10r^2 \times r^3 + 5r \times r^4 - 10r^2t^3 - 5rt^4 + t^5 = 5r^4 \times r + 10r^3 \times r^2 + r^5$, or $5r^4t + 10r^3t^2 + 10r^2 \times r^3 - 10r^2 \times t^3 + 5r \times r^4 - 5r \times t^4 + t^5 = 5r^4 \times r + 10r^3 \times r^2 + r^5$; and (ſubtracting $5r^4t + 10r^3t^2$ from both ſides,) we ſhall have $10r^2 \times r^3 - 10r^2 \times t^3 + 5r \times r^4 - 5r \times t^4 + t^5 = 5r^4 \times r - 5r^4 \times t + 10r^3 \times r^2 - 10r^3 \times t^2 + r^5$, and (ſubtracting t^5 from both ſides,) we ſhall have $10r^2 \times r^3 - 10r^2 \times t^3 + 5r \times r^4 - 5r \times t^4 = 5r^4 \times r - 5r^4 \times t + 10r^3 \times r^2 - 10r^3 \times t^2 + r^5 - t^5$; that is, $10r^2 \times \overline{r^3 - t^3} + 5r \times \overline{r^4 - t^4}$ will be $= 5r^4 \times \overline{r - t} + 10r^3 \times \overline{r^2 - t^2} + r^5 - t^5$. Therefore (dividing all the terms by $r - t$,) we ſhall have $10r^2 \times \frac{r^3 - t^3}{r - t} + 5r \times \frac{r^4 - t^4}{r - t} = 5r^4 \times \frac{r - t}{r - t} + 10r^3 \times \frac{r^2 - t^2}{r - t} + \frac{r^5 - t^5}{r - t}$, or $10r^2 \times \overline{r^2 + rt + t^2} + 5r \times \overline{r^3 + r^2t + rt^2 + t^3} = 5r^4 \times 1 + 10r^3 \times \overline{r + t} + r^4 + r^3t + r^2t^2 + rt^3 + t^4$, or $10r^4 + 10r^3t + 10r^2t^2 + 5r^4 + 5r^3t + 5r^2t^2 + 5rt^3 = 5r^4 + 10r^4 + 10r^3t + r^4 + r^3t + r^2t^2 + rt^3 + t^4$, or $15r^4 + 15r^3t + 15r^2t^2 + 5rt^3 = 16r^4 + 11r^3t + r^2t^2 + rt^3 + t^4$, and (ſubtracting $15r^4 + 11r^3t$ from both ſides,) $4r^3t + 15r^2t^2 + 5rt^3 = r^4 + r^2t^2 + rt^3 + t^4$, and (ſubtracting $r^2t^2 + rt^3$ from both ſides,) $4r^3t + 14r^2t^2 + 4rt^3 = r^4 + t^4$, and, laſtly, (ſubtracting t^4 from both ſides,) we ſhall have $4r^3t + 14r^2t^2 + 4rt^3 - t^4 = r^4$; which is a biquadratick equation that has two roots (that is, real and affirmative roots,) of which the leſſer is equal to the tangent of an arch of 9 Degrees, and the greater is equal to the third, or greateſt, root of the quinquinomial equation $5r^4t + 10r^3t^2 - 10r^2t^3 - 5rt^4 + t^5 = r^5$, from which the ſaid biquadratick equation has been derived.

And, if in this laſt biquadratick equation $4r^3t + 14r^2t^2 + 4rt^3 - t^4 = r^4$ we ſubſtitute 1, inſtead of r, becauſe in all the foregoing Computations the radius of the circle has been ſuppoſed to be $= 1$, this equation will be thereby converted into the equation $4t + 14t^2 + 4t^3 - t^4 = 1$.

Thirdly, this laſt equation $4t + 14t^2 + 4t^3 - t^4 = 1$ is now to be reſolved, by finding, by a few reaſonable conjectures and trials, a pretty good firſt near value of t, or the leſſer of it's two roots, and then proceeding to derive from this firſt near value of it a ſecond, and a third, and a fourth, and, if neceſſary,

neceſſary, a ſtill further near value of it by ſucceſſive proceſſes of Mr. Raphſon's Method of Approximation, till we have obtained a near value of the ſaid leſſer root that is exact to the intended number of decimal figures. And this reſolution of this equation has been performed with great care by Dr. Mackay, and the ſeveral arithmetical operations of it are ſet-down at length in pages 939, 940, 941, &c. - - - - 949 of this volume: and, after taking 0.158 for the firſt near value of it, he finds, by a proceſs of Mr. Raphſon's Method of Approximation, a ſecond near value of it to be = 0.158,384; and, by a ſecond proceſs of the ſame method of approximation, a third near value of it to be = 0.158,384,440,324; and by a third proceſs of the ſame method of approximation, a fourth near value of it to be = 0.158,384,440,324,536,293,838,883; which is, probably, exact in all it's 24 figures, as it agrees in all thoſe figures with the value of the leaſt root of the quinquinomial equation $5t+10t^2-10t^3-5t^4+t^5 = 1$, or the length of the tangent of an arch of 9 Degrees, that had been found before in art. 25, page 803, by reſolving the ſaid quinquinomial equation.

The thought of inveſtigating the length of the tangent of an arch of 9 Degrees, or the value of the leaſt of the three roots of the quinquinomial equation $5t+10t^2-10t^3-5t^4+t^5 = 1$, by, firſt, obſerving that it's ſecond, or middle, root would be equal to the radius, or 1, and then employing this middle root, when thus diſcovered, to reduce the ſaid quinquinomial equation to the more manageable quadrinomial equation $4t+14t^2+4t^3-t^4 = 1$, and reſolving the ſaid quadrinomial equation in order to find the leſſer of it's two roots, or the length of the tangent of an arch of 9 Degrees, was ſuggeſted to me by the ingenious Mr. William Frend, M. A. and Fellow of Jeſus College, Cambridge. And he recommended this method of computing the length of a tangent of an arch of 9 Degrees in preference to the former method of computing it by reſolving the quinquinomial equation $5t+10t^2-10t^3-5t^4+t^5 = 1$, becauſe it was much leſs laborious than that former method. And the difference of the two methods in this reſpect is, indeed, very conſiderable; ſince the reſolution of the quadrinomial equation $4t+14t^2+4t^3-t^4 = 1$ has taken-up only 11 pages of this volume, to wit, the pages from page 938 to

page 950, whereas the reſolution of the quinquinomial equation $5t+10t^2-10t^3-5t^4+t^5=1$ has taken-up 23 pages of it, to wit, the pages from page 780 to page 804.

As to the third, or greateſt, root of the quinquinomial equation $5t+10t^2-10t^3-5t^4+t^5=1$, or, (if we expreſs this equation in the firſt and more general manner without ſuppoſing the radius r to be $=1$,) of the quinquinomial equation $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5=r^5$, it is a third proportional to the leaſt root t and the middle root r, and therefore is $=\frac{rr}{t}$, and is equal to the co-tangent of the arch of which t is the tangent, or of an arch of 9 Degrees, or to the tangent of an arch of 81 Degrees. For, if b be put for the tangent or 81 Degrees, and be ſubſtituted inſtead of t in the quinquinomial quantity $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5$, which is equal to the abſolute term r^5, the new quinquinomial quantity ariſing from ſuch ſubſtitution, to wit, the quinquinomial quantity $5r^4b+10r^3b^2-10r^2b^3-5rb^4+b^5$, will alſo be equal to r^5; as may be ſhewn in the following manner.

Since b is the tangent of 81 Degrees, or of the complement of an arch of 9 Degrees to 90 Degrees, or the arch of the whole quadrant, the rectangle under the tangent b and the tangent t will be equal to the ſquare of the radius r, that is, to rr. Therefore t will be $=\frac{rr}{b}$, and t^2 will be $=\frac{r^4}{b^2}$, and t^3 will be $=\frac{r^6}{b^3}$, and t^4 will be $=\frac{r^8}{b^4}$, and t^5 will be $=\frac{r^{10}}{b^5}$; and conſequently the quinquinomial quantity $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5$ will be $\left(=5r^4\times\frac{rr}{b}+10r^3\times\frac{r^4}{b^2}-10r^2\times\frac{r^6}{b^3}-5r\times\frac{r^8}{b^4}+\frac{r^{10}}{b^5}\right)$ $=\frac{5r^6}{b}+\frac{10r^7}{b^2}-\frac{10r^8}{b^3}-\frac{5r^9}{b^4}+\frac{r^{10}}{b^5}$. But the quinquinomial quantity $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5$ is $=r^5$. Therefore the quinquinomial quantity $\frac{5r^6}{b}+\frac{10r^7}{b^2}-\frac{10r^8}{b^3}-\frac{5r^9}{b^4}+\frac{r^{10}}{b^5}$ will alſo be $=r^5$. Therefore (by multiplying all the terms of this equation into b^5,) we ſhall have $5r^6b^4+10r^7b^3-10r^8b^2-5r^9b+r^{10}=r^5b^5$; and (by dividing all the terms by r^5,) we ſhall have $5rb^4+10r^2b^3-10r^3b^2-5r^4b+r^5=b^5$; and (adding $10r^3b^2+5r^4b$ to both ſides,) we ſhall have $5rb^4+10r^2b^3+r^5=b^5+10r^3b^2+5r^4b$,

$5r^4b$, and (ſubtracting $5rb^4+10r^2b^3$ to both ſides,) we ſhall have $r^5 = b^5+10r^3b^2+5r^4b-5rb^4-10r^2b^3$, or (ranging the terms according to the powers of b,) $r^5 = 5r^4b+10r^3b^2-10r^2b^3-5rb^4+b^5$, or $5r^4b+10r^3b^2-10r^2b^3-5rb^4+b^5 = r^5$. Q. E. D.

Therefore b is a root of the quinquinomial equation $5r^4t+10r^3t^2-10r^2t^3-10rt^4+t^5 = r^5$ as well as t and r.

And in like manner it may be ſhewn that b, or $\frac{rr}{t}$, is likewiſe the greater of the two roots of the quadrinomial equation $4r^3t+14r^2t^2+4rt^3-t^4 = r^4$. For, ſince b is $= \frac{rr}{t}$, we ſhall have $t = \frac{rr}{b}$, and $t^2 = \frac{r^4}{b^2}$, and $t^3 = \frac{r^6}{b^3}$, and $t^4 = \frac{r^8}{b^4}$. Therefore the quadrinomial quantity $4r^3t+14r^2t^2+4rt^3-t^4$ will be $(= 4r^3 \times \frac{rr}{b} + 14r^2 \times \frac{r^4}{b^2} + 4r \times \frac{r^6}{b^3} - \frac{r^8}{b^4}) = \frac{4r^5}{b} + \frac{14r^6}{b^2} + \frac{4r^7}{b^3} - \frac{r^8}{b^4}$. But the quadrinomial quantity $4r^3t+14r^2t^2+4rt^3-t^4$ is $= r^4$. Therefore the quadrinomial quantity $\frac{4r^5}{b} + \frac{14r^6}{b^2} + \frac{4r^7}{b^3} - \frac{r^8}{b^4}$ will alſo be $= r^4$. Therefore, if we multiply all the terms into b^4, we ſhall have $4r^5b^3+14r^6b^2+4r^7b-r^8 = r^4b^4$, and (dividing both ſides by r^4,) we ſhall have $4rb^3+14r^2b^2+4r^3b-r^4 = b^4$, and (adding r^4 to both ſides,) we ſhall have $4rb^3+14r^2b^2+4r^3b = b^4+r^4$, and (ſubtracting b^4 from both ſides,) we ſhall have $4rb^3+14r^2b^2+4r^3b-b^4 = r^4$, or (ranging the terms according to the powers of b,) we ſhall have $4r^3b+14r^2b^2+4rb^3-b^4 = r^4$. Therefore b, or $\frac{rr}{t}$, is a root of the quadrinomial equation $4r^3t+14r^2t^2+4rt^3-b^4 = r^4$ as well as t, or the tangent of an arch of 9 Degrees. Q. E. D.

As all theſe leſſer Computations, that follow the firſt long and laborious Computation of the Tangent of an arch of 1 Minute by deriving it from the Tangent of an arch of 45 Degrees by repeated ſections of it, may be conſidered as appendixes of that firſt Computation, being only other and ſhorter methods of obtaining ſome of the tangents that had been obtained before in that firſt Computation, I have treated them as parts of the tract

which

which begins with that firſt Computation, and have therefore numbered the ſeveral articles into which they are divided by reference to the articles of the ſaid firſt Computation; ſo that the ſaid tract (which is the twenty-ſecond tract of this volume,) muſt be conſidered as extending from page 763, in which it begins, to page 950, or the end of the volume, and therefore as taking-up 187 pages.

FRANCIS MASERES.

Inner Temple, Nov. 27, 1806.

AN

EXAMINATION

OF

BARON MASERES'S SOLUTION

OF

M. BOUGUER'S EXAMPLE

TO

DR. HALLEY'S PROBLEM:

As given in Page 60th of the Fourth Volume of Baron MASERES's *Scriptores Logarithmici*,—

BY

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DOCTOR MACKAY'S EXAMINATION

OF

BARON MASERES'S SOLUTION OF M. BOUGUER'S EXAMPLE TO DR. HALLEY'S PROBLEM.

INTRODUCTION.

UPON comparing the result of the direct Solution of M. Bouguer's Example to Dr. Halley's Problem, as given by Baron Maseres in page 75 of the fourth Volume of his *Scriptores Logarithmici*, with the result of the Solution of the same Example as given by Dr. Mackay in page 285; they were found to disagree: and Baron Maseres, not knowing to which of the two to attribute the error, applied to Dr. Mackay to examine his own computation of the above-mentioned Example; which, upon trial, was found to be correct. —Baron Maseres, therefore, suspecting that there might be some errors in his own calculations; (which, from his not having taken into his computation a sufficient number of the terms of the original series, and also from the multiplicity of figures, is not to be wondered at:) and with a view to prove the accuracy of his direct method of solution, requested Dr. Mackay to examine his solution of that example; and to retain the same words which had been used by him, and only to alter the numbers: which direction has been adhered-to as nearly as possible.

In order to attain the utmost accuracy, it was judged necessary to take into account two more terms of Mr. James Gregory's general series; the investigations of which are given by Baron Maseres in the third Volume of his *Scriptores Logarithmici*, page 453. The angle of the course obtained from this complicated calculation is $33^\circ\ 9'$; and the latitude come-to is $43^\circ\ 15'.7$:— the former differing $43'$ from that found by the indirect method of solution; and the latter only about one minute; which, indeed, is as near an agreement as can reasonably be expected; especially when it is considered, that so many series are involved, and many terms of each of these series are required to obtain precision; also the numerous multiplications, divisions, extractions of roots, &c. all involving each other.

To this Examination Dr. Mackay has been induced to annex, upon account of it's extreme ſimplicity, another indirect method of ſolution, (different from that which he had given before, and which had been printed in the fourth Volume of the *Scriptores Logarithmici*,) deduced from the principles of Middle-latitude ſailing; which, it is preſumed, will be found acceptable to the practical navigator, in thoſe caſes where the data can be obtained with ſufficient accuracy.

BEFORE we begin the examination of this example, it will be neceſſary to inveſtigate the ſeveral ſerieſes exhibiting the values of the capital letters B, C, D, &c; taking into account two more terms of the general ſeries in page 53: and in order to ſimplify the operation, the letter r, and it's powers, being equal to 1, may, therefore, be omitted. The general ſeries in page 53 of the fourth Volume of the *Scriptores Logarithmici*, with the two additional terms, in page 453, Vol. III, will then be as follows, to wit,

$$m + dx + \frac{\overline{m+dx}^{3}}{6} + \frac{\overline{m+dx}^{5}}{24} + \frac{61 \times \overline{m+dx}^{7}}{5040} + \frac{277 \times \overline{m+dx}^{9}}{72,576} + \frac{50521 \times \overline{m+dx}^{11}}{39,916,800} + \frac{41581 \times \overline{m+dx}^{13}}{95,800,320}.$$

Now the third term of this ſeries, or the ſecond term in which x is involved, (to wit, the term $\frac{\overline{m+dx}^{3}}{6}$,) being expanded, is

$$= \frac{m^3 + 3m^2dx + 3md^2x^2 + d^3x^3}{6}.$$

The third term, $\frac{\overline{m+dx}^{5}}{24}$, being expanded, is

$$= \frac{m^5 + 5m^4dx + 10m^3d^2x^2 + 10m^2d^3x^3 + 5md^4x^4 + d^5x^5}{24}.$$

The fourth term, $\frac{61 \times \overline{m+dx}^{7}}{5040}$, is

$$= \frac{61}{5040} \times m^7 + 7m^6dx + 21m^5d^2x^2 + 35m^4d^3x^3 + 35m^3d^4x^4 + 21m^2d^5x^5 + 7md^6x^6 + d^7x^7.$$

The fifth term, $\frac{277 \times \overline{m+dx}^{9}}{72,576}$, is

$$= \frac{277}{72,576} \times m^9 + 9m^8dx + 36m^7d^2x^2 + 84m^6d^3x^3 + 126m^5d^4x^4 + 126m^4d^5x^5 + 84m^3d^6x^6 + 36m^2d^7x^7 + 9md^8x^8 + d^9x^9.$$

The fixth term, $\frac{50,521 \times \overline{m+dx}^{11}}{39,916,800}$, is

$$= \frac{50,521}{39,916,800} \times \overline{m^{11} + 11m^{10}dx + 55m^9d^2x^2 + 165m^8d^3x^3 + 330m^7d^4x^4} + \overline{462m^6d^5x^5 + 462m^5d^6x^6 + 330m^4d^7x^7 + 165m^3d^8x^8} + \overline{55m^2d^9x^9 + 11md^{10}x^{10} + d^{11}x^{11}}.$$

And the feventh term, $\frac{41,581 \times \overline{m+dx}^{13}}{95,800,320}$, is

$$= \frac{41,581}{95,800,320} \times \overline{m^{13} + 13m^{12}dx + 78m^{11}d^2x^2 + 286m^{10}d^3x^3 + 715m^9d^4x^4} + \overline{1287m^8d^5x^5 + 1716m^7d^6x^6 + 1716m^6d^7x^7 + 1287m^5d^8x^8} + \overline{715m^4d^9x^9 + 286m^3d^{10}x^{10} + 78m^2d^{11}x^{11} + 13md^{12}x^{12}} + \overline{d^{13}x^{13}}.$$

Hence, by collecting all the different terms in which the fame power of x is involved into one feries, the above infinite feries will be changed into another infinite feries, in which the co-efficients of the feveral powers of x will be equal to the following ferieſes, which we will denote by the capital letters C, D, F, G, H, I, K, L, M, N, O, P, as follows:

I. The terms in which x is involved.

$$C = d + \frac{m^2d}{2} + \frac{5m^4d}{24} + \frac{61m^6d}{720} + \frac{277m^8d}{8064} + \frac{50,521m^{10}d}{3,628,800} + \frac{540,553m^{12}d}{95,800,320} + \&c.$$

II. The terms in which x^2 is involved.

$$D = \frac{md^2}{2} + \frac{5m^3d^2}{12} + \frac{61m^5d^2}{240} + \frac{277m^7d^2}{2016} + \frac{50,521m^9d^2}{725,760} + \frac{3,243,318m^{11}d^2}{95,800,320} + \&c.$$

III. The terms in which x^3 is involved.

$$F = \frac{d^3}{6} + \frac{5m^2d^3}{12} + \frac{61m^4d^3}{144} + \frac{277m^6d^3}{864} + \frac{50,521m^8d^3}{241,290} + \frac{5,946,083m^{10}d^3}{47,900,160} + \&c.$$

IV. The terms in which x^4 is involved.

$$G = \frac{5md^4}{24} + \frac{61m^3d^4}{144} + \frac{277m^5d^4}{576} + \frac{50,521m^7d^4}{120,960} + \frac{41581 \times 143 \times m^9d^4}{19,160,064} + \&c.$$

V. The terms in which x^5 is involved.

$$H = \frac{d^5}{24} + \frac{61m^2d^5}{240} + \frac{277m^4d^5}{576} + \frac{50,521m^6d^5}{86,400} + \frac{41,581 \times 99m^8d^5}{967,680} + \&c.$$

VI. The

VI. The terms in which x^6 is involved.

$$I = \frac{61md^6}{720} + \frac{277m^3d^6}{864} + \frac{50,521m^5d^6}{86,400} + \frac{41,581 \times 13m^7d^6}{725,760} + \&c.$$

VII. The terms in which x^7 is involved.

$$K = \frac{61d^7}{5040} + \frac{277m^2d^7}{2016} + \frac{50,521m^4d^7}{120,960} + \frac{41,581 \times 13m^6d^7}{725,760} + \&c.$$

VIII. The terms in which x^8 is involved.

$$L = \frac{277md^8}{8064} + \frac{50,521m^3d^8}{241,920} + \frac{41,581 \times 99m^5d^8}{967,680} + \&c.$$

IX. The terms in which x^9 is involved.

$$M = \frac{277d^9}{72,576} + \frac{50,521m^2d^9}{725,760} + \frac{41,581 \times 143m^4d^9}{19,160,064} + \&c.$$

X. The terms in which x^{10} is involved.

$$N = \frac{50,521md^{10}}{3,628,800} + \frac{41,581 \times 143m^3d^{10}}{47,900,160} + \&c.$$

XI. The terms in which x^{11} is involved.

$$O = \frac{50,521d^{11}}{39,916,800} + \frac{41,581 \times 78m^2d^{11}}{95,800,320} + \&c.$$

XII. The terms in which x^{12} is involved.

$$P = \frac{41,581 \times 13md^{12}}{95,800,320} + \&c.$$

Art. 67, Vol. IV, page 60.

A degree of a great circle on the furface of the earth is 60 nautical, or geographical, miles; therefore the arch BM, or the latitude of the point B, which is $40^\circ\ 45'$, will be equal to $40^\circ\ 45' \times 60' = 2445$ nautical miles.

But

But the radius of the earth is 3437 nautical miles, and three-fourths of a mile nearly *; therefore the arch BM will be equal to $\frac{2445'}{3437{\cdot}75} = 0.711{,}222$ of the radius of the earth. Or, since the measure of an arch equal to the radius is $57^\circ\ 17'\ 44''.8$; if, therefore, the given latitude $40^\circ\ 45'$ be divided by $57^\circ\ 17'\ 44''.8$, the quotient 0.711,222 will be the measure of the given latitude, in such parts whereof the semidiameter contains 1; but m is equal to the arch BM; therefore m will be equal to 0.711,222.

In the next place, the loxodromick arch BA, or the distance run by the ship, is supposed to be 180 miles, which expressed in parts of the radius, that of the earth being assumed to be = 1, will be $= \frac{180'}{3437{\cdot}75}$, or $= \frac{3^\circ}{57^\circ\ 17'\ 44''.8}$ = 0.052,359,9: or, because $\frac{40^\circ\ 45'}{3^\circ} = 13^\circ.58$, hence $\frac{0.711{,}222}{13^\circ.58}$ will be = 0.052,359,9, as before. But d is = the loxodromick arch BA. Therefore d is = 0.052,359,9.

Thirdly, by art. 59, the capital letter C is equal to the infinite series $d + \frac{m^2 d}{2} + \frac{5m^4 d}{24} + \frac{61m^6 d}{720} + \frac{277m^8 d}{8064} + \frac{50{,}521m^{10} d}{3{,}628{,}800} + \frac{41{,}581 \times 13m^{12} d}{95{,}800{,}320} + \&c = d \times$ the infinite series $1 + \frac{m^2}{2} + \frac{5m^4}{24} + \frac{61m^6}{720} + \frac{277m^8}{8064} + \frac{50{,}521m^{10}}{3{,}628{,}800} + \frac{41{,}581 \times 13m^{12}}{95{,}800{,}320} + \&c$.

Now, since $m = 0.711{,}222$, we shall have

$m^2 = m \times m\ (= 0.711{,}222 \times 0.711{,}222) = 0.505{,}836$
$m^4 = m^2 \times m^2\ (= 0.505{,}836 \times 0.505{,}836) = 0.255{,}871$
$m^6 = m^4 \times m^2\ (= 0.255{,}871 \times 0.505{,}836) = 0.129{,}429$
$m^8 = m^4 \times m^4\ (= 0.255{,}871 \times 0.255{,}871) = 0.065{,}470$
$m^{10} = m^6 \times m^4\ (= 0.129{,}429 \times 0.255{,}871) = 0.033{,}117$
and $m^{12} = m^6 \times m^6\ (= 0.129{,}429 \times 0.129{,}429) = 0.016{,}752$.

And consequently the infinite series $1 + \frac{m^2}{2} + \frac{5m^4}{24} + \frac{61m^6}{720} + \frac{277m^8}{8064} + \frac{50{,}521m^{10}}{3{,}628{,}800} + \frac{41{,}581 \times 13m^{12}}{95{,}800{,}320} + \&c$, will be equal to $1{,}000{,}000 + \frac{0.505{,}836}{2}$

---

* For the circumference of the earth is 360°, or 21,600 geographical, or nautical, miles: hence the radius of the earth is equal to $\frac{10800 \times 113}{355} = 3437\frac{53}{71}$. or 3437 and three-fourths nearly.—The radius of the earth in English miles is $3958\frac{1}{4}$. See Dr. Mackay's Treatise on the Longitude, vol. i. page 7, second edition.

+

$+\frac{5\times 0.255,871}{24}+\frac{61\times 0.129,429}{720}+\frac{277\times 0.065,470}{8064}+\frac{50,521\times 0.033,117}{3,628,800}+\frac{41,581\times 13\times 0.016,752^2}{95,800,320}+$ &c $= 1.000,000 + 0.252,918 + 0.053,306 + 0.010,965 + 0.002,249 + 0.000,461 + 0.000,075 = 1.319,974$. Therefore $d\times$ the infinite series $1+\frac{m^2}{2}+\frac{5m^4}{24}+\frac{61m^6}{720}+\frac{277m^8}{8064}+\frac{50,521m^{10}}{3,628,800}+\frac{41,581\times 13\times m^{12}}{95,800,320}+$ &c, will be $= d\times 1.319,974 = 0.052,359,9\times 1.319,974 = 0.069,114$ nearly. Therefore the capital letter C will, in this case, be equal to 0.069,114.

Fourthly, the ſmall letter c, which is equal to the equatorial arch LM, or to $2^\circ\ 15'$, will be equal to $135'$, which expreſſed in parts of the radius, that of the earth being 1, will be equal to $\frac{135'}{3437.75}$, or $\frac{2^\circ\ 15'}{57^\circ\ 17'\ 44''.8} = 0.039,270$.

Therefore the equation ſet down above in art. 65, will in this caſe become

$$0.069,114 + Dx + Fx^2 + Gx^3 + Hx^4 + Ix^5 + Kx^6 + Lx^7 + Mx^8 + \text{\&c}$$
$$-\frac{Cx^2}{2}-\frac{Dx^3}{2}-\frac{Fx^4}{2}-\frac{Gx^5}{2}-\frac{Hx^6}{2}-\frac{Ix^7}{2}-\frac{Kx^8}{2}-\text{\&c}$$
$$-\frac{Cx^4}{8}-\frac{Dx^5}{8}-\frac{Fx^6}{8}-\frac{Gx^7}{8}-\frac{Hx^8}{8}-\text{\&c}$$
$$-\frac{Cx^6}{16}-\frac{Dx^7}{16}-\frac{Fx^8}{16}-\text{\&c}$$
$$-\frac{5Cx^8}{128}-\text{\&c}$$
$$= 0.039,270.$$

Art. 68. We muſt next find the values of the capital letters D, F, G, H, I, K, L, M, &c, which may be done as follows.

The capital letter D is equal to the infinite ſeries $\frac{md^2}{2}+\frac{5m^3d^2}{12}+\frac{61m^5d^2}{240}+\frac{277m^7d^2}{2016}+\frac{50,521m^9d^2}{725,760}+\frac{41,581\times 78m^{11}d^2}{95,800,320}+$ &c; and conſequently it will be equal to md^2 multiplied by the infinite ſeries $\frac{1}{2}+\frac{5m^2}{12}+\frac{61m^4}{240}+\frac{277m^6}{2016}+\frac{50,521m^8}{725,760}+\frac{41,581\times 78\times m^{10}}{95,800,320}+$ &c $= md^2$ multiplied by the infinite ſeries $\frac{1}{2}+\frac{5\times 0.505,836}{12}+\frac{61\times 0.255,871}{240}+\frac{277\times 0.129,429}{2016}+\frac{50,521\times 0.065,470}{725,760}+$

$\frac{41,581 \times 78 \times 0.033,117}{95,800,320}$ + &c = md^2 multiplied by the infinite series 0.500,000 + 0.210,765 + 0.065,034 + 0.017,784 + 0.004,557 + 0.001,121 + &c = md^2 multiplied by 0.799,261 = 0.711,222 × $\overline{0.052,359,9}^2$ × 0.799,261 = 0.001,549,8. Q. E. I.

The capital letter F is equal to the infinite series $\frac{d^3}{6} + \frac{5m^2d^3}{12} + \frac{61m^4d^3}{144} + \frac{277m^6d^3}{864} + \frac{50,521m^8d^3}{241,920} + \frac{5,946,083m^{10}d^3}{47,900,160}$ + &c = d^3 multiplied by the infinite series $\frac{1}{6} + \frac{5m^2}{12} + \frac{61m^4}{144} + \frac{277m^6}{864} + \frac{50,521m^8}{241,920} + \frac{41,581 \times 286 \times m^{10}}{95,800,320}$ + &c = d^3 multiplied by the infinite series $\frac{1}{6} + \frac{5 \times 0.505,836}{12} + \frac{61 \times 0.255,871}{144} + \frac{277 \times 0.129,429}{864} + \frac{50,521 \times 0.065,470}{241,920} + \frac{41,581 \times 286 \times 0.033,117}{95,800,320}$ + &c = d^3 multiplied by the infinite series 0.166,667 + 0.210,765 + 0.108,390 + 0.041,495 + 0.013,672 + 0.004,110 = a^3 multiplied by 0.545,100 = 0.000,143,5 × 0.545,100 = 0.000,078. Q. E. I.

The capital letter G is equal to the infinite series $\frac{5md^4}{24} + \frac{61m^3d^4}{144} + \frac{277m^5d^4}{576} + \frac{50,521m^7d^4}{120,960} + \frac{41,581 \times 715 \times m^9d^4}{95,800,320}$ + &c = md^4 multiplied by the infinite series $\frac{5}{24} + \frac{61m^2}{144} + \frac{277m^4}{576} + \frac{50,521m^6}{120,960} + \frac{41,581 \times 715 \times m^8}{95,800,320}$ + &c = md^4 multiplied by the infinite series $\frac{5}{24} + \frac{61 \times 0.505,836}{144} + \frac{277 \times 255,871}{576} + \frac{50,521 \times 0.129,429}{120,960} + \frac{41,581 \times 715 \times 0.065,470}{95,800,320}$ + &c = md^4 multiplied by the infinite series 0.208,333 + 0.214,278 + 0.123,049 + 0.054,058 + 0.020,318 = md^4 × 0.620,036 = 0.711,222 × 0.000,075 × 0.620,036 = 0.000,003. Q. E. I.

Since the computation is carried no lower than six decimal places, and becauſe the firſt ſix decimal places and upwards in the values of the other letters, to wit, H, I, K, L, M, &c. will be cyphers, the values of theſe letters may therefore be conſidered as equal to 0; ſee Vol. IV, page 64: hence the ſeveral terms that involve them may be omitted, and the ſaid equation will be as follows.

$$\left.\begin{array}{l} 0.069,114 + Dx + Fx^2 + Gx^3 + 0 + 0 + 0 + 0 + 0 + 0 \;\&c \\ \quad - \frac{Cx^2}{2} - \frac{Dx^3}{2} - \frac{Fx^4}{2} - \frac{Gx^5}{2} - 0 - 0 - 0 \;\&c \\ \quad - \frac{Cx^4}{8} - \frac{Dx^5}{8} - \frac{Fx^6}{8} - \frac{Gx^7}{8} - 0 \;\&c \\ \quad - \frac{Cx^6}{16} - \frac{Dx^7}{16} - \frac{Fx^8}{16} \;\&c \\ \quad - \frac{5Cx^8}{128} \;\&c \end{array}\right\} = 0.039,270.$$

Art. 69. In this equation we muſt now reduce the ſeveral compound co-efficients of x^2, x^3, x^4, x^5, x^6, x^7, and x^8, to wit, the compound quantities $+ F - \frac{C}{2}$, $+ G - \frac{D}{2}$, $- \frac{F}{2} - \frac{C}{8}$, $- \frac{G}{2} - \frac{D}{8}$, $- \frac{F}{8} - \frac{C}{16} - \frac{G}{8} - \frac{D}{16}$, $- \frac{F}{16} - \frac{5C}{128}$, into ſimple terms, which may be done as follows.

Since C is $=$ 0.069,114, we ſhall have $\frac{C}{2} \left(= \frac{0.069,114}{2}\right) = 0.034,557$, and $\frac{C}{8} \left(= \frac{0.069,114}{8}\right) = 0.008,639$, and $\frac{C}{16} \left(= \frac{0.069,114}{16}\right) = 0.004,320$, and $\frac{5C}{128} \left(= \frac{5 \times 0.069,114}{8} = \frac{0.345,570}{8}\right) = 0.002,699$.

And ſince D is $=$ 0.001,550, we ſhall have $\frac{D}{2} \left(= \frac{0.001,550}{2}\right) = 0.000,775$, and $\frac{D}{8} \left(= \frac{0.001,550}{8}\right) = 0.000,194$, and $\frac{D}{16} \left(= \frac{0.001,550}{16}\right) = 0.000,097$.

And ſince F is $=$ 0.000,078, we ſhall have $\frac{F}{2} \left(= \frac{0.000,078}{2}\right) = 0.000,039$, and $\frac{F}{8} \left(= \frac{0.000,078}{8}\right) = 0.000,010$, and $\frac{F}{16} \left(= \frac{0.000,078}{16}\right) = 0.000,005$.

And ſince G is $=$ 0.000,003, we ſhall have $\frac{G}{2} \left(= \frac{0.000,003}{2}\right) = 0.000,002$, and $\frac{G}{8} \left(= \frac{0.000,003}{8}\right) = 0.000,000$.

Therefore $+ F - \frac{C}{2}$ will be $(= + 0.000,078 - 0.034,557) = - 0.034,479$

And $+ G - \frac{D}{2}$ will be $(= + 0.000,003 - 0.000,775) = - 0.000,772$

And $- \frac{F}{2} - \frac{C}{8}$ will be $(= - 0.000,039 - 0.008,639) = - 0.008,678$

And

And $-\frac{G}{2}-\frac{D}{8}$ will be $(=-0.000,002-0.000,196)=-0.000,198$

And $-\frac{F}{8}-\frac{C}{16}$ will be $(=-0.000,010-0.004,320)=-0.004,330$

And $-\frac{G}{8}-\frac{D}{16}$ will be $(=-0.000,000-0.000,097)=-0.000,097$

And $-\frac{F}{16}-\frac{5C}{128}$ will be $(=-0.000,005-0.002,699)=-0.002,704$

Therefore, the foregoing complicated equation, ſet down in the preceeding article, will now become as follows: to wit, $0.069,114+0.001,550x-0.034,479x^2-0.000,772x^3-0.008,678x^4-0.000,198x^5-0.004,330x^6-0.000,097x^7-0,002,074x^8=0.039,270$.

Art. 70. Now let all the terms on the left-hand ſide of the equation that have the ſign — prefixed to them, or are ſubtracted from the other terms on the ſame ſide, be added to both ſides of the equation; and we ſhall then have,

$$0.069,114+0.001,550x=0.039,270+0.034,479x^2+0.000,772x^3+0.008,678x^4+0.000,198x^5+0.004,330x^6+0.000,097x^7+0.002,704x^8+\&c.$$

Therefore, (ſubtracting 0.039,270 from both ſides) we ſhall have

$$0.029,844+0.001,550x=0.034,479x^2+0.000,772x^3+0.008,678x^4+0.000,198x^5+0.004,330x^6+0.000,097x^7+0.002,704x^8+\&c.$$

And (ſubtracting $0.001,550x$ from both ſides) we ſhall have

$$-0.001,550x+0.034,479x^2+0.000,772x^3+0.008,678x^4+0.000,198x^5+0.004,330x^6+0.000,097x^7+0.002,704x^8=0.029,844.$$

Art. 71. In this equation it is remarkable that 0.034,479, the co-efficient of x^2, is greater than 0.001,550, the co-efficient of x; and that 0.008,678, the co-efficient of x^4, is greater than 0.000,772, the co-efficient of x^3; and that 0.004,330, the co-efficient of x^6, is greater than 0.000,198, the coefficient of x^5; and that 0.002,704, the co-efficient of x^8, is greater than 0.000,097, the co-efficient of x^7: But the co-efficients of x, x^3, x^5 and x^7, or the odd powers of x, (to wit, 0.001,550, 0.000,772, 0.000,198, and 0.000,097) are every one leſs than the next before it, or decreaſe when the powers of x increaſe; and the co-efficients of x^2, x^4, x^6 and x^8, or the even powers of x, (to

wit, 0.034,479, 0.008,678, 0.004,330 and 0.002,704) are alſo every one leſs than the next before it, or decreaſe when the powers of x increaſe, as well as the co-efficients of x, x^3, x^5 and x^7, or the odd powers of x.

Art. 72. We muſt now endeavour to reſolve this laſt equation obtained in art. 70, to wit, the equation

$$-0.001,550 \times x + 0.034,479 \times x^2 + 0.000,772 \times x^3 + 0.008,678 \times x^4 + 0.000,198 \times x^5 + 0.004,330 \times x^6 + 0.000,097 \times x^7 + 0.002,704 \times x^8 + \&c = 0.029,844$$

In this place we ſhall follow Mr. Raphſon's method of approximation as laid down by Baron Maſeres in page 66 of Vol. IV. of his *Scriptores Logarithmici*.

Art. 73. Now, ſince x, or the co-ſine of the courſe, is neceſſarily leſs than the radius, it follows that x^2 will be leſs than x, and x^3 leſs than x^2, and x^4 leſs than x^3, and that every following power of x will be leſs than that which preceeds it; we may, therefore, conclude that the two firſt terms, $0.001,550 \times x$, and $0.034,479 \times x^2$ will be greater than the two next terms, or than any other two terms on the ſame ſide of the equation: we will, therefore, neglect all the terms on the left-hand ſide of the equation, except the ſaid two firſt terms $-0.001,550 \times x + 0.034,479 \times x^2$; and will ſuppoſe thoſe two terms alone to be equal to the whole left hand ſide of the equation, and conſequently to the abſolute term 0.029,844, and will reſolve the quadratick equation $-0.001,550 \times x + 0.034,479 \times x^2 = 0.029,844$, or $0.034,479 \times x^2 - 0.001,550 \times x = 0.029,844$. This may be done in the following manner.

Divide all the terms by 0.034,479, the co-efficient of x^2; and we ſhall have $x^2 - \frac{0.001,550}{0.034,479} \times x = \frac{0.029,844}{0.034,479}$, that is $x^2 - 0.044,955 \times x = 0.865,570$. Divide the co-efficient 0.044,955 by 2, and the quotient will be 0.022,477, the ſquare of which is 0.000,505. Let this ſquare be added to both ſides of the equation; and we ſhall have $x^2 - 0.044,955 \times x + 0.000,505 = 0.865,570 + 0.000,505 = 0.866,075$: Therefore, extracting the ſquare roots of both ſides, we ſhall have $x - 0.0224 = 0.9306$, and conſequently $x = 0.9306 + 0.0224 = 0.9530$.

Art. 74. Now ſince 0.953, the value of x, is nearly equal to 1, the ſquare, cube, and other higher powers of x will form a decreaſing ſeries, of which a few of the firſt terms will be of a conſiderable magnitude; and, therefore, ſince only the two firſt terms of the ſeries in art. 72. were taken into the account and all the other terms of it are poſitive, or to be added to the value of the two firſt terms, it is hence evident that the value of x thus obtained will be greater than

the

the truth; and, therefore, for a firſt trial, and to avoid tedious multiplications, we ſhall ſubſtitute 0.9, inſtead of x in the compound quantity which forms the left-hand ſide of the equation ſet down in art. 70; to wit, the equation

$$- 0.001,550 \times x + 0.034,479 \times x^2 + 0.000,772 \times x^3$$
$$+ 0.008,678 \times x^4 + 0.000,198 \times x^5 + 0.004,330 \times x^6$$
$$+ 0.000,097 \times x^7 + 0.002,704 \times x^8 + \&c = 0.029,844,$$

in order to diſcover whether the value of the ſaid compound quantity reſulting from ſuch ſubſtitution will be greater or leſs than the abſolute term 0.029,844, of the ſaid equation, and whether its difference from the ſaid abſolute term will be great or ſmall.

Now, if we ſuppoſe to be 0.9, we ſhall have

$x^2 (= 0.900,000 \times 0.9) = 0.81$
And $x^3 (= 0.810,000 \times 0.9) = 0.729$
$x^4 (= 0.729,000 \times 0.9) = 0.6561$
$x^5 (= 0.656,100 \times 0.9) = 0.590,49$
$x^6 (= 0.590,490 \times 0.9) = 0.531,441$
$x^7 (= 0.531,441 \times 0.9) = 0.478,297$
And $x^8 (= 0.478,297 \times 0.9) = 0.430,467$

And conſequently

$0.001,550 \times x (= 0.001,550 \times 0.900,000) = 0.001,395$
$0.034,479 \times x^2 (= 0.034,479 \times 0.810,000) = 0.027,928$
$0.000,772 \times x^3 (= 0.000,772 \times 0.729,000) = 0.000,563$
$0.008,678 \times x^4 (= 0.008,678 \times 0.656,100) = 0.005,694$
$0.000,198 \times x^5 (= 0.000,198 \times 0.590,490) = 0.000,116$
$0.004,330 \times x^6 (= 0.004,330 \times 0.531,441) = 0.002,301$
$0.000,097 \times x^7 (= 0.000,097 \times 0.478,297) = 0.000,046$
$0.002,704 \times x^8 (= 0.002,704 \times 0.430,467) = 0.001,160.$

Therefore the compound quantity

$$- 0.001,550 \times x + 0.034,479 \times x^2 + 0.000,772 \times x^3$$
$$+ 0.008,678 \times x^4 + 0.000,198 \times x^5 + 0.004,330 \times x^6$$
$$+ 0.000,097 \times x^7 + 0.002,704 \times x^8 + \&c \text{ will be equal to}$$
$$(- 0.001,395 + 0.027,928 + 0.000,563 + 0.005,694$$
$$+ 0.000,116 + 0.002,301 + 0.000,046 + 0.001,160$$
$$= - 0.001,395 + 0.037,808) = 0.036,413;$$

which is greater than 0.029,844

0.029,844, or the abſolute term of the equation—0.001,550 × x + 0.034,479 × x^2 + 0.000,772 × x^3 + 0.008,678 × x^4 + 0.000,198 × x^5 + 0.004,330 × x^6 + 0.000,097 × x^7 + 0.002,704 × x^8 = 0.029,844. Therefore, 0.9 will be greater than the true value of x in the ſaid equation.

Art. 75. We will therefore in the next place ſuppoſe x to be equal to 0.9 − z, and ſubſtitute 0.9 − z inſtead of x in the terms of the foregoing equation, but with an omiſſion of all the terms that involve any higher powers of z than its ſimple power, or z itſelf, agreeably to the rules of Mr. Raphſon's method of approximation. This may be done in the manner following.

Since x is = 0.9 − z, we ſhall have

$x^2(=\overline{0.9-z}^2=\overline{0.9}^2 - 2\times 0.9 \times z + \&c) = 0.81 - 1.8\times + \&c$

and $x^3(=\overline{0.9-z}^3=\overline{0.9}^3 - 3\times\overline{0.9}^2\times z + \&c) = 0.729 - 2.43\times z + \&c$

and $x^4(=\overline{0.9-z}^4=\overline{0.9}^4 - 4\times\overline{0.9}^3\times z + \&c) = 0.6561 - 2.916\times z + \&c$

and $x^5(=\overline{0.9-z}^5=\overline{0.9}^5 - 5\times\overline{0.9}^4\times z + \&c) = 0.590,49 - 3.2805\times z + \&c.$

and $x^6(=\overline{0.9-z}^6=\overline{0.9}^6 - 6\times\overline{0.9}^5\times z + \&c) = 0.531,441 - 3.542,94\times z + \&c.$

and $x^7(=\overline{0.9-z}^7=\overline{0.9}^7 - 7\times\overline{0.9}^6\times z + \&c) = 0.478,297 - 3.720,087\times z$

and $x^8(=\overline{0.9-z}^8=\overline{0.9}^8 - 8\times\overline{0.9}^7\times z + \&c) = 0.430,467 - 3.826,376\times z$

Therefore, 0.001,550 × x will be $(= 0.001,550 \times \overline{0.9-z})$
$= 0.001,395 - 0.001,550 \times z.$

And 0.034,479 × x^2 will be $(= 0.034,479 \times \overline{0.81 - 1.8\times z}$
$= 0.034,479 \times 0.81 - 0.034,479 \times 1.8 \times z$
$= 0.027,928 - 0.062,062 \times z,$

And 0.000,772 × x^3 will be $(0.000,772 \times \overline{0.729 - 2.43 \times z}$
$= 0.000,772 \times 0.729 - 0.000,772 \times 2.43 \times z)$
$= 0.000,563 - 0.001,876 \times z,$

And 0.008,678 × x^4 will be $(= 0.008,678 \times \overline{0.6561 - 2.916\times z}$
$= 0.008,678 \times 0.6561 - 0.008,678 \times 2.916 \times z)$
$= 0.005,694 - 0.025,305 \times z.$

And 0.000,198 × x^5 will be $(= 0.000,198 \times \overline{0.590,49 - 3.2805 \times z}$
$= 0.000,198 \times 0.590,49 - 0.000,198 \times 3.2805 \times z)$
$= 0.000,116 - 0.000,649 \times z.$

And

And $0.004,330 \times x^6$ will be $(= 0.004,330 \times \overline{0.531,441 - 3.542,94 \times z}$
$= 0.004,330 \times 0.531,441 - 0.004,330 \times 3.542,94 \times z)$
$= 0.002,301 - 0.015,156 \times z.$

And $0.000,097 \times x^7$ will be $(= 0.000,097 \times \overline{0,478,297 - 3.720,087 \times z}$
$= 0.000,097 \times 0.478,297 - 0.000,097 \times 3.720,087 \times z)$
$= 0.000,046 - 0.000,361 \times z.$

And $0.002,704 \times x^8$ will be $(= 0.002,704 \times \overline{0.430,467 - 3.826,376 \times z}$
$= 0.002,704 \times 0.430,467 - 0.002,704 \times 3.826,376 \times z)$
$= 0.001,160 - 0.010,346 \times z.$

And consequently the compound quantity

$- 0.001,550 \times x + 0.034,479 \times x^2 + 0.000,772 \times x^3$
$+ 0.008,678 \times x^4 + 0.000,198 \times x^5 + 0.004,330 \times x^6$
$+ 0.000,097 \times x^7 + 0.002,704 \times x^8$, will be
$= - 0.001,395 + 0.001,550 \times z$
$+ 0.027,928 - 0.062,062 \times z$
$+ 0.000,563 - 0.001,876 \times z$
$+ 0.005,694 - 0.025,305 \times z$
$+ 0.000,116 - 0.000,649 \times z$
$+ 0.002,301 - 0.015,156 \times z$
$+ 0.000,046 - 0.000,361 \times z$
$+ 0.001,160 - 0.010,346 \times z$ &c
$= - 0.001,395 + 0.001,550 \times z$
$+ \underline{0.037,808} - \underline{0.115,755 \times z}$
$= + 0.036,413 - 0.114,205 \times z.$

But the compound quantity,

$- 0.001,550 \times x + 0.034,479 \times x^2 + 0.000,772 \times x^3$
$+ 0.008,678 \times x^4 + 0.000,198 \times x^5 + 0.004,330 \times x^6$
$+ 0.000,097 \times x^7 + 0.002,704, \times x^8$ is $= 0.029,844.$

Therefore, $0.036,413 - 0.114,205 \times z$ is alſo $= 0.029,844$;
And conſequently, (adding $0.114,205 \times z$ to both ſides) we ſhall have $0.036,413 = 0.029,844 + 0.114,205 \times z$, and (ſubtracting 0.029,844 from both ſides) we ſhall have $0.114,205 \times z = 0.006,569$, and $z = \frac{0.006,569}{0.114,205}$ $= 0.0575$, Therefore x, or $0.9 - z$ will be $(= 0.9 - 0.0575) = 0.8425$, or the ſecond near value of x in the equation

—

$$- 0.001,550 \times x + 0.034,479 \times x^2 + 0.000,772 \times x^3$$
$$+ 0.008,678 \times x^4 + 0.000,198 \times x^5 + 0.004,330 \times x^6$$
$$+ 0.000,097 \times x^7 + 0.002,704 \times x^8 = 0.029,844,$$ will be equal to 0.8425. Q. E. I.

Art. 76. It is probable, for the reaſon formerly aſſigned, that the value of x thus found is ſtill too great; and, therefore, we will ſubſtitute the two firſt figures only of this value, to wit, 0.84, inſtead of x in the compound quantity

$$- 0.001,550 \times x + 0.034,479 \times x^2 + 0.000,772 \times x^3$$
$$+ 0.008,678 \times x^4 + 0.000,198 \times x^5 + 0.004,330 \times x^6$$
$$+ 0.000,097 \times x^7 + 0.002,704 \times x^8,$$ (which forms the firſt, or left-hand, ſide of the foregoing equation,) in order to diſcover how near the value of the ſaid compound quantity reſulting from the ſaid ſubſtitution will approach to the true value of the ſaid compound quantity, or to the value of its equal, the abſolute term 0.029,844, of the ſaid equation; and conſequently to find how near the ſaid number 0.84 will approach to the true value of x in the ſaid equation. This may be done in the manner following.

If we ſuppoſe x to be equal to 0.84, we ſhall have

$$x^2 (= \overline{0.84}^2 = 0.7056$$
And $x^3 (= 0.705,600 \times 0.84) = 0.592,704$
$x^4 (= 0.592,704 \times 0.84) = 0.497,871$
$x^5 (= 0.497,871 \times 0.84) = 0.418,212$
$x^6 (= 0.418,212 \times 0.84) = 0.351,298$
$x^7 (= 0.351,298 \times 0.84) = 0.295,090$
$x^8 (= 0.295,090 \times 0.84) = 0.247,876$

And conſequently

$0.001,550 \times x (= 0.001,550 \times 0.84) = 0.001,302$
And $0.034,479 \times x^2 (= 0.034,479 \times 0.7056) = 0.024,328$
$0.000,772 \times x^3 (= 0.000,772 \times 0.592,704) = 0.000,457$
$0.008,678 \times x^4 (= 0.008,678 \times 0.497,871) = 0.004,320$
$0.000,198 \times x^5 (= 0.000,198 \times 0.418,212) = 0.000,082$
$0.004,330 \times x^6 (= 0.004,330 \times 0.351,298) = 0.001,521$
$0.000,097 \times x^7 (= 0.000,097 \times 0.295,090) = 0.000,029$
And $0.002,704 \times x^8 (= 0.002,704 \times 0.247,876) = 0.000,670$

Hence $- 0.001,550 \times x + 0.034,479 \times x^2 + 0.000,772 \times x^3$
$+ 0.008,678 \times x^4 + 0.000,198 \times x^5 + 0.004,330 \times x^6$

+

$+ 0.000,097 \times x^7 + 0.002,704 \times x^8$ will be $= - 0.001,302$
$+ 0.024,328 + 0.000,457 + 0.004,320 + 0.000,082$
$+ 0.001,521 + 0.000,029 + 0.000,670 = - 0.001,302$
$+ 0.031,407 = 0.030,105$; which is greater than 0.029,844, or the abſolute term of the equation

$- 0.001,550 \times x + 0.034,479 \times x^2 + 0.000,772 \times x^3$
$+ 0.008,678 \times x^4 + 0.000,198 \times x^5 + 0.004,330 \times x^6$
$+ 0.000,097 \times x^7 + 0.002,704 \times x^8 = 0.029,844$. Therefore, 0.84 is greater than the true value of x in that equation; but the difference between them, to wit, 0.000,261, is very ſmall.

Art. 77. We will, therefore, in the next place, ſuppoſe the value of x in the foregoing equation to be equal to 0.83, and ſubſtitute 0.83 inſtead of x in the compound quantity

$- 0.001,550 \times x + 0.034,479 \times x^2 + 0.000,772 \times x^3$
$+ 0.008,678 \times x^4 + 0.000,198 \times x^5 + 0.004,330 \times x^6$
$+ 0.000,097 \times x^7 + 0.002,704 \times x^8$, (which forms the firſt, or left-hand, ſide of the equation) in order to diſcover how nearly the value of the ſaid compound quantity reſulting from that ſubſtitution will approach to the true value of the ſaid compound quantity, or of its equal, the abſolute term 0.029,844 of the ſaid equation. This may be done in the manner following.

If we ſuppoſe x to be equal to 0.83, we ſhall have

$x^2 (= \overline{0.83}^2) = 0.6889$
And $x^3 (= 0.688,900 \times 0.83) = 0.571,787$
$x^4 (= 0.571,787 \times 0.83) = 0.474,583$
$x^5 (= 0.474,583 \times 0.83) = 0.393,904$
$x^6 (= 0.393,904 \times 0.83) = 0.326,940$
$x^7 (= 0.326,940 \times 0.83) = 0.271,361$
$x^8 (= 0.271,361 \times 0.83) = 0.225,229$

Therefore, $0.001,550 \times x$ will be $(= 0.001,550 \times 0.83 \quad) = 0.001,286$
And $0.034,479 \times x^2$ will be $(= 0.034,479 \times 0.6889 \quad) = 0.023,753$
$0.000,772 \times x^3$ will be $(= 0.000,772 \times 0.571,787) = 0.000,441$
$0.008,678 \times x^4$ will be $(= 0.008,678 \times 0.474,583) = 0.004,118$
$0.000,198 \times x^5$ will be $(= 0.000,198 \times 0.393,904) = 0.000,077$
$0.004,330 \times x^6$ will be $(= 0.004,330 \times 0.326,940) = 0.001,416$

0.000,097 × x^7 will be (= 0.000,097 × 0.271,361) = 0.000,026
0.002,704 × x^8 will be (= 0.002,704 × 0.225,229) = 0.000,609

And conſequently the compound quantity
— 0.001,550 × x + 0.034,479 × x^2 + 0.000,772 × x^3
+ 0.008,678 × x^4 + 0.000,198 × x^5 + 0.004,330 × x^6
+ 0.000,097 × x^7 + 0.002,704 × x^8, will be equal to
— 0.001,286 + 0.023,753 + 0.000,441 + 0.004,118
+ 0.000,077 + 0.001,416 + 0.000,026 + 0.000,609
= — 0.001,286 + 0.030,440 = 0.029,154, which is leſs than 0.029,844, or the abſolute term of the equation — 0.001,550 × x + 0.034,479 × x^2
+ 0.000,772 × x^3 + 0.008,678 × x^4 + 0.000,198 × x^5
+ 0.004,330 × x^6 + 0.000,097 × x^7 + 0.002,704 × x^8 + &c
= 0,029,844. Therefore, 0.83 will be nearly equal to, but a little leſs than, the true value of x in the ſaid equation. Q. E. I.

Art. 78. From the two laſt aſſumed values of x, with their correſponding abſolute terms, and alſo the abſolute term of the foregoing equation, another near value of x may be found by even proportions (or by what is often called *the Method of Trial and Error*, or *the Double Rule of Falſe Poſition*, and by Baron Maſeres *the Differential Method of Approximation*), as follows.

Subtract the abſolute term 0.029,154, obtained from the aſſumption of x being 0.83, from 0.030,105, the abſolute term found upon the ſuppoſition that x is equal to 0.84, and the remainder is 0.000,951; alſo, ſubtract 0.029,154 from 0.029,844, the abſolute term of the equation; and the remainder will be 0.000,690: And the difference between the two aſſumed values of x, to wit, 0.84, and 0.83, is = 0.01. Now $\frac{0.000,690 \times 0.01}{0.000,951} = 0.007,255$. Hence we ſhall have 0.83 + 0.007,255 = 0.837,255, for another near value of x; which we will ſubſtitute in the general equation, in order to ſee how near it will approach to the true value of x.

If we ſuppoſe x to be equal to 0.837,255, we ſhall have
x^2 (= 0.837,255 × 0.837,255) = 0.700,996
And x^3 (= 0.700,996 × 0.837,255) = 0.586,912
x^4 (= 0.586,912 × 0.837,255) = 0.491,395
x^5 (= 0.491,395 × 0.837,255) = 0.411,423
x^6 (=

x^6 (= 0.411,423 × 0.837,255) = 0.344,466
x^7 (= 0.344,466 × 0.837,255) = 0.288,406
x^8 (= 0.288,406 × 0.837,255) = 0.242,026

Therefore, 0.001,550 × x will be (= 0.001,550 × 0.837,255) = 0.001,297,7
And 0.034,479 × x^2 will be (= 0.034,479 × 0.700,996) = 0.024,169,6
0.000,772 × x^3 will be (= 0.000,772 × 0.586,912) = 0.000,453,1
0.008,678 × x^4 will be (= 0.008,678 × 0.491,395) = 0.004,264,3
0.000,198 × x^5 will be (= 0.000,198 × 0.411,423) = 0.000,081,4
0.004,330 × x^6 will be (= 0.004,330 × 0.344,466) = 0.001,491,5
0.000,097 × x^7 will be (= 0.000,097 × 0.288,406) = 0.000,028,0
0.002,704 × x^8 will be (= 0.002,704 × 0.242,026) = 0.000,652,9

And consequently the whole compound quantity
— 0.001,550 × x + 0.034,479 × x^2 + 0.000,772 × x^3
+ 0.008,678 × x^4 + 0.000,198 × x^5 + 0.004,330 × x^6
+ 0.000,097 × x^7 + 0.002,704 × x^8 will be = — 0.001,297,7
+ 0.024,169,6 + 0.000,453,1 + 0.004,264,3
+ 0.000,081,4 + 0.001,491,5 + 0.000,028,0
+ 0.000,652,9 = — 0.001,297,7 + 0.031,140,8
= 0.029,843,1, which being only 0.000,000,9 leſs than the abſolute term of the equation, to wit, 0.029,844; we may, therefore, conclude that the aſſumed value of x, to wit, 0.837,255 is very near the truth, but a little leſs: Let therefore 0.837,275 be aſſumed equal to x, and we ſhall have

x^2 (= 0.837,275 × 0.837,275) = 0.701,029
And x^3 (= 0.701,029 × 0.837,275) = 0.586,954
x^4 (= 0.586,954 × 0.837,275) = 0.491,442
x^5 (= 0.491,442 × 0.837,275) = 0.411,472
x^6 (= 0.411,472 × 0.837,275) = 0.344,515
x^7 (= 0.344,515 × 0.837,275) = 0.288,454
And x^8 (= 0.288,454 × 0.837,275) = 0.241,515

Therefore 0.001,550 × x (= 0.001,550 × 0.837,275) = 0.001,297,8
And 0.034,479 × x^2 (= 0.034,479 × 0.701,029) = 0.024,170,8
0.000,772 × x^3 (= 0.000,772 × 0.586,954) = 0.000,453,1
0.008,678 × x^4 (= 0.008,678 × 0.491,442) = 0.004,264,7
0.000,198 × x^5 (= 0.000,198 × 0.411,472) = 0.000,081,4
0.004,330 × x^6 (= 0.004,330 × 0.344,515) = 0.001,491,8
0.000,097 × x^7 (= 0.000,097 × 0.288,454) = 0.000,028,0
0.002,704 × x^8 (= 0.002,704 × 0.241,515) = 0.000,653,1

And consequently the compound quantity

$$-0.001,550 \times x + 0.034,479 \times x^2 + 0.000,772 \times x^3$$
$$+ 0.008,678 \times x^4 + 0.000,198 \times x^5 + 0.004,330 \times x^6$$
$$+ 0.000,097 \times x^7 + 0.002,704 \times x^8 \text{ will be} = -0.001,297,8$$
$$+ 0.024,170,8 + 0.000,453,1 + 0.004,264,7$$
$$+ 0.000,081,4 + 0.001,491,8 + 0.000,028,0$$
$$+ 0.000,653,1 = -0.001,297,8 + 0.031,142,9 = 0.029,845,1,$$

which exceeds the absolute term of the equation, to wit, 0.029,844, by only the small quantity 0.000,001,1. Therefore, 0.837,275 is very near the value of x, but a little greater.

Art. 79. We may now find a more exact value of x from the two last assumed values, and the corresponding approximated absolute terms, together with the given absolute term of the equation, by again making use of the Differential Method of Approximation, in the manner following.

The difference between the two assumed values of x, (to wit, 0.837,275 and 0.837,255,) is 0.000,020; and that between the two corresponding approximated absolute terms, (to wit, 0.029,845,1 and 0.029,843,1,) is 0.000,002,0; and that between the preceeding approximated absolute term, and the given absolute term, (to wit, 0.029,843,1 and 0.029,844,0,) is 0.000,000,9; Hence 0.000,002,0 : 0.000,000,9 : : 0.000,020 : 0.000,009; which added to 0.837,255 the preceeding assumed value of x, gives 0.837,264, for the true value of x. Q. E. I.

Now the above-found value of x, which is the co-sine of the course, will be found to be the co-sine of 33°, 9′. Hence, since the ship is supposed to have sailed upon a direct course between the N. and E. that course will, therefore, be N. 33°, 9′ E. or nearly N. E. by N.

The latitude of the parallel which the ship has arrived at, may be found by adding the difference of latitude to the latitude of the place sailed-from. Now the difference of latitude is equal to the distance multiplied by the co-sine of the course, the radius being 1; That is, to (60 leagues, or 180 nautical miles, or 180 minutes of a degree of a great circle of the Earth, multiplied into 0.837,264, or) $180' \times 0.837,264 = 150'.7 = 2°, 30'.7$. Hence the latitude come-to is $= 40°, 45' + 2°, 30'.7 = 43°, 15'.7$.

The truth of the operation may be verified by computing the difference of longitude with the meridian difference of latitude and the above-found course; which, if it agrees with the given difference of longitude; the course, and hence the latitude come-to, will be truely ascertained; If they do not agree, it will be necessary to re-examine the whole work. However, as that process would be attended with a considerable degree of labour, the true course may be very easily found

found by means of the given difference of longitude and the approximated meridian difference of latitude: Or by the approximated middle latitude, and the given diſtance and difference of longitude, as follows.

Since the latitude ſailed-from is 40°, 45′, of which the meridian parts are 2682, and the latitude come-to is 43°, 15′.7, whoſe meridian parts are 2885, the meridian difference of latitude is 203. Hence, becauſe the difference of longitude is equal to the meridian difference of latitude multiplied by the tangent of the courſe, we ſhall have the difference of longitude $= 203 \times .6531 = 132'.6$. But the given difference of longitude is 135′, which is therefore 2′.4 greater than that by computation: Hence, the computed courſe is leſs than the true courſe, and the latitude come-to is greater than the truth. Now the middle latitude is equal to $\frac{40° 45' + 43° 15'}{2} = 42°\ 0'$ nearly; and, becauſe, according to the principles of middle-latitude ſailing, the fine of the courſe is equal to the co-fine of the middle-latitude multiplied by the difference of longitude and the product divided by the diſtance; therefore $\frac{.74314 \times 135'}{180'} = \frac{.74314 \times 3}{4} = \frac{2.22942}{4} = .55737 =$ the fine of 33°, 52′, the courſe true to leſs than a minute: and the difference of latitude $= 180' \times .83034 = 1°, 49'.46 = 2°, 29'.46$. Hence, the latitude come-to is 43°, 14′.46; which is leſs than 43°, 15′.7, (or the Latitude come-to, as found before,) by only 1′.24, or leſs than 1 minute and a quarter.

Q. E. I.

Another and more eaſy Method of finding the Value of x in the foregoing Equation.

Before we conclude, it may be neceſſary to point-out another, and more eaſy method of finding the value of x in the given equation

$$\begin{aligned} &- 0.001{,}550 \times x + 0.034{,}479 \times x^2 + 0.000{,}772 \times x^3 \\ &+ 0.008{,}678 \times x^4 + 0.000{,}198 \times x^5 + 0.004{,}330 \times x^6 \\ &+ 0.000{,}097 \times x^7 + 0.002{,}704 \times x^8 = 0.029{,}844. \end{aligned}$$

From the nature of the queſtion the courſe cannot be greater than an angle whoſe fine is equal to the product of the multiplication of the co-fine of the latitude ſailed-from, by the difference of longitude divided by the diſtance; that is, equal to three fourths of the co-fine of the latitude left: nor leſs than three fourths of the co-fine of the ſum of the latitude ſailed-from and the diſtance: Hence, the extreme limits of the courſe in this example are 32°, 48′ and 34°, 37′.

Now

Now ſince the courſe is contained between the limits of 32°, 48′, and 34°, 37′, we may ſuppoſe it to be 33° 0′, and ſubſtitute its co-ſine for x in the above equation.

The co-ſine of 33° 0′ is 0.838,670,6

Hence x^2 (= 0.838,670,6 × 0.838,670,6) = 0.703,368
And x^3 (= 0.703,368 × 0.838,670,6) = 0.589,894
x^4 (= 0.589,894 × 0.838,670,6) = 0.494,727
x^5 (= 0.494,727 × 0.838,670,6) = 0.414,913
x^6 (= 0.414,913 × 0.838,670,6) = 0.347,975
x^7 (= 0.347,975 × 0.838,670,6) = 0.291,837
x^8 (= 0.291,837 × 0.838,670,6) = 0.244,755.

Therefore 0.001,550 × x (= 0.001,550 × 0.838,670,6) = 0.001,300
And 0.034,479 × x^2 (= 0.034,479 × 0.703,368) = 0.024,251
0.000,772 × x^3 (= 0.000,772 × 0.589,894) = 0.000,455
0.008,678 × x^4 (= 0.008,678 × 0.494,727) = 0.004,293
0.000,198 × x^5 (= 0.000,198 × 0.414,913) = 0.000,082
0.004,330 × x^6 (= 0.004,330 × 0.347,975) = 0.001,507
0.000,097 × x^7 (= 0.000,097 × 0.291,837) = 0.000,028
And 0.002,704 × x^8 (= 0.002,704 × 0.244,755) = 0.000,662

And conſequently the whole compound quantity

− 0.001,550 × x + 0.034,479 × x^2 + 0.000,772 × x^3
+ 0.008,678 × x^4 + 0.000,198 × x^5 + 0.004,330 × x^6
+ 0.000,097 × x^7 + 0.002,704 × x^8 will be = − 0.001,300
+ 0.024,251 + 0.000,455 + 0.004,293 + 0.000,082
+ 0.001,507 + 0.000,028 + 0.000,662 = − 0.001,300
+ 0.031,278 = 0.029,978.

Hence the value of x has been aſſumed too great, and the courſe too ſmall; Let, therefore, the courſe be ſuppoſed to be 33°, 20′; whence x is = 0.835,487,8.

And x^2 (= 0.835,487,8 × 0.835,487,8) = 0.698,040
x^3 (= 0.698,040 × 0.835,487,8) = 0.583,204
x^4 (= 0.583,204 × 0.835,487,8) = 0.487,260
x^5 (= 0.487,260 × 0.835,487,8) = 0.407,099
x^6 (= 0.407,099 × 0.835,487,8) = 0.340,127

x^7 (= 0.340,127 × 0.835,487,8) = 0.284,172
x^8 (= 0.284,172 × 0.835,487,8) = 0.237,422.

Therefore, 0.001,550 × x will be (= 0.001,550 × 0.835,488) = 0.001,295
and 0.034,479 × x^2 (= 0.034,479 × 0.698,040) = 0.024,068
and 0.000,772 × x^3 (= 0.000,772 × 0.583,204) = 0.000,450
and 0.008,678 × x^4 (= 0.008,678 × 0.487,260) = 0.004,228
and 0.000,198 × x^5 (= 0.000,198 × 0.407,099) = 0.000,080
and 0.004,330 × x^6 (= 0.004,330 × 0.340,127) = 0.001,473
and 0.000,097 × x^7 (= 0.000,097 × 0.284,172) = 0.000,028
and 0.002,704 × x^8 (= 0.002,704 × 0.237,422) = 0.000,642

And conſequently the whole compound quantity

− 0.001,550 × x + 0.034,479 × x^2 + 0.000,772 × x^3
+ 0.008,678 × x^4 + 0.000,198 × x^5 + 0.004,330 × x^6
+ 0.000,097 × x^7 + 0.002,704 × x^8 will be = − 0.001,295
+ 0.024,068 + 0.000,450 + 0.004,228
+ 0.000,080 + 0.001,473 + 0.000,028
+ 0.000,642 = − 0.001,295 + 0.030,969
= 0.029,674; hence x is aſſumed too ſmall, and therefore the courſe is too great: The true courſe may now be found by proportion, or the Differential Method of Approximation, in the manner following.

The given abſolute term of the equation is	0.029,844	diff. 0.000134
The courſe being 33°, 0′, the abſolute term is	0.029,978	
The courſe being 33°, 20′, the abſolute term is	0.029,674	diff. 0.000304
Difference, - - - 20′.		

Now 304 : 134 : : 20′ : (8′.81, or) nearly 9′; which being added to 33°, 0′, the ſum 33°, 9′ will be the courſe required, as before.

End of Dr. Mackay's *Examination of Baron* Maſeres's *Reſolution of Mr.* Bouguer's *Example to Dr.* Halley's *Nautical Problem*.

AN

AN
INDIRECT METHOD
OF SOLVING
DOCTOR HALLEY'S PROBLEM,
ACCORDING TO THE
PRINCIPLES OF MIDDLE-LATITUDE SAILING.

By ANDREW MACKAY, L.L.D. in the Univerſity of Aberdeen in Scotland.

AS the method of ſolving this problem by approximation, which we gave in the fourth volume of this work, page 285, is performed partly upon the principles of Mercator's ſailing; and, therefore, the courſe, and the latitude come-to, may be approximated to any required degree of accuracy; yet, as Middle-latitude ſailing is ſufficiently accurate in practice, eſpecially for any ſhort run;—and as it affords a very eaſy ſolution of this problem;—we ſhall therefore now proceed to reſolve it according to the principles of this method of ſailing.

PROBLEM.

Given the latitude ſailed-from; the diſtance run, upon a direct courſe; and the difference of longitude: to find the courſe, and the latitude come-to.

RULE.

When the ſhip is ſailing from the Equator, the middle latitude is contained between the latitude ſailed-from and the ſum of that latitude and half the diſtance; and when the ſhip is ſailing towards the Equator, the middle latitude is between the latitude ſailed-from and the difference between that latitude and half

the diſtance. Hence, a middle latitude may be aſſumed by conjecture that will not be far from the truth:—In general, it may be taken equal to the ſum, or difference, of the latitude ſailed-from and one-fourth of the diſtance, according as the ſhip is receeding from, or approaching towards, the Equator; which we will call the *firſt approximated middle latitude.*

Now, with this middle latitude, the given diſtance, and difference of longitude, find the firſt approximate courſe; with which, and the diſtance, compute the difference of latitude; and hence a ſecond approximated middle latitude will be obtained.

Again, with this ſecond middle latitude, the given diſtance, and the given difference of longitude, compute the courſe, and alſo the difference of latitude; and in this manner continue the operation until the laſt courſes, and conſequently the differences of latitude, agree; and hence the true courſe, and the latitude come-to, will be obtained.

EXAMPLE, (page 60. Vol. IV.)

A ſhip from latitude 40°, 45′ N. ſailed 180 miles upon a direct courſe between the N. and E., and, by obſervation, was found to have altered her longitude 135′. It is required to find the courſe ſteered, and the latitude come-to.

Latitude ſailed-from together with one fourth of the diſtance ($= 40°, 45' + \frac{180'}{4}$) $= 41°, 30'$, is $=$ the firſt approximated middle latitude.

Now: As the diſtance	180′	2.25527			2.25527
Is to the diff. of long.	135′	2.13033			
So is co-ſine of mid. lat.	41°, 30′	9.87446			
To ſine of 1ſt appr. courſe	34°, 10′	9.74952	co-ſine		9.91772
Firſt approx. diff. of latitude		149′			2.17299

Hence, $40°, 45' + \frac{149'}{2} = 41°, 59'\frac{1}{2}$, $=$ ſecond approximated middle latitude.

Then, log. of ratio of diſt. to diff. of long.		9.87506		
Second appr. mid. lat.	41°, 59′½; co-ſine	9.87113	diſt. 180′; log.	2.25527
Second appr. courſe	33°, 52′½; ſine	9.74619	co-ſine	9.91921
Second approx. diff. of latitude		149′.4		2.17448

Hence,

Hence, the courſe is N. 33°, 52′ E. or N. E. by N. nearly, and the latitude come-to is = 40°, 45′ + 2° 29′ = 43°, 14′ N; theſe quantities being aſcertained to a ſufficient degree of accuracy for any practical purpoſe whatever. However, by repeating the operation, the required quantities may be obtained with all the accuracy of which this method is ſuſceptible. Thus 40°, 45′ + $\frac{149'.4}{2}$ = 41°, 59.7 = the third approximated middle latitude.

Now, Log. of ratio of diſt. to diff. long. 9.875,0613

Third appr. m. lat.	41°, 59′.7;	co-ſine	9.871,1076,	diſt. 180′. log.	2.255,2725
True courſe	33°, 52′.35;	ſine	9.746,1689	co-ſine	9.919,2047
Difference of latitude			149′.44		2.174,4772

Hence, the latitude come-to is 43°, 14′, 26″.

According to the former ſolution of this problem given in page 285, Vol. IV. the courſe is 33°, 51′, 58″; which differs only about ſix-tenths of a minute from that found above; and the latitude come-to is 43°, 14′, 27″, the difference being only one ſecond. By repeating both methods a ſtill nearer agreement may be obtained; but this is abſolutely unneceſſary.

ADDITIONAL REMARKS

ON

Mr. GLENIE'S PROBLEM,

WHICH WAS SOLVED AND CONSTRUCTED IN VOL. IV. OF THE

SCRIPTORES LOGARITHMICI,

Pages 335, 336, &c to Page 412.

By FRANCIS MASERES, Esq. F. R. S.

CURSITOR BARON OF THE COURT OF EXCHEQUER.

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By FRANCIS MASERES, Esq. F.R.S.

CURSITOR BARON OF THE COURT OF EXCHEQUER.

ART. 1. IN Number V of the Tracts contained in the 4th volume of the *Scriptores Logarithmici*, the Problem propoſed by Mr. Glenie is reduced to an equation of the tenth order, or degree, which is as follows, to wit, $16{,}950\,a^9x + 12{,}435\,a^8x^2 - 1{,}304\,a^7x^3 - 9{,}162\,a^6x^4 - 8{,}748\,a^5x^5 - 4{,}762\,a^4x^6 - 1{,}848\,a^3x^7 - 501\,a^2x^8 - 90\,a\,x^9 - 9\,x^{10} = 12{,}625\,a^{10}$. And this equation is afterwards reſolved, and the value of it's leſſer root x, which is equal to the right line CH, (which is neceſſary to be found in order to the Solution of Mr. Glenie's Problem,) is diſcovered in two different methods; to wit, 1ſt by reducing the ſaid equation to a quadratick equation, to wit, the equation $xx + 2ax = \frac{5aa}{3}$, and then reſolving, and likewiſe conſtructing, the ſaid quadratick equation; and 2dly, by reſolving the ſaid high equation itſelf by Mr. Raphſon's method of approximation.

Art. 2. This ſecond method of reſolving the ſaid high equation, (which, upon the whole, I think preferable to the former, notwithſtanding the quantity of laborious calculation with which it is attended,) is ſo fully explained in that tract that I have nothing more to ſay concerning it. But the former method of reſolving it, by reducing it to a quadratick equation, and then reſolving and conſtructing the ſaid quadratick equation, will admit of a few additional remarks.

Art.

Art. 3. In order to reduce the equation $16{,}950\,a^9x + 12{,}435\,a^8x^2 - 1{,}304\,a^7x^3 - 9{,}162\,a^6x^4 - 8{,}748\,a^5x^5 - 4{,}762\,a^4x^6 - 1{,}848\,a^3x^7 - 501\,a^2x^8 - 90\,ax^9 - 9x^{10} = 12{,}625\,a^{10}$ to a quadratick equation, I, firſt, ſubtracted the abſolute term $12{,}625\,a^{10}$ from both ſides, and thereby obtained the equation $-12{,}625\,a^{10} + 16{,}950\,a^9x + 12{,}435\,a^8x^2 - 1{,}304\,a^7x^3 - 9{,}162\,a^6x^4 - 8{,}748\,a^5x^5 - 4{,}762\,a^4x^6 - 1{,}848\,a^3x^7 - 501\,a^2x^8 - 90\,ax^9 - 9x^{10} = 0$, and then I divided this compound quantity, conſiſting of eleven terms, by the compound quantity $-2525\,a^8 + 360\,a^7x + 1404\,a^6x^2 + 1640\,a^5x^3 + 978\,a^4x^4 + 408\,a^3x^5 + 124\,a^2x^6 + 24\,ax^7 + 3\,x^8$, (conſiſting of nine terms,) and found that the quotient was the trinomial quantity $5aa - 6ax - 3xx$, and that this diviſion was compleat, or without any remainder. And hence I concluded that, ſince the dividend in the foregoing diviſion was equal to 0, either the diviſor or the quotient, or both of them, muſt be equal to 0 likewiſe; becauſe if they had both been real, or finite, quantities, their product (which is equal to the ſaid dividend,) muſt have been equal to ſome finite quantity likewiſe. I therefore concluded that it was probable that the ſaid quotient $5aa - 6ax - 3xx$ was equal to 0, and conſequently that $3xx + 6ax$ was equal to $5aa$, and that $xx + 2ax$ was equal to $\frac{5aa}{3}$, and that the root of this quadratick equation $xx + 2ax = \frac{5aa}{3}$ would be a root of the original equation $16{,}950\,a^9x + 12{,}435\,a^8x^2 - 1{,}304\,a^7x^3 - 9{,}162\,a^6x^4 - 8{,}748\,a^5x^5 - 4{,}762\,a^4x^6 - 1{,}848\,a^3x^7 - 501\,a^2x^8 - 90\,ax^9 - 9x^{10} = 12{,}625\,a^{10}$, and, perhaps, would be that root of the ſaid equation (for the ſaid equation has evidently two roots,) which would anſwer the conditions of Mr. Glenie's Problem. I therefore reſolved the ſaid quadratick equation $xx + 2ax = \frac{5aa}{3}$, and found that it's root x was $= a \times \frac{\sqrt{24} - 3}{3}$, and conſequently that the line CH would be $= a \times \frac{\sqrt{24} - 3}{3}$, or $=$ FD $\times \frac{\sqrt{24} - 3}{3}$. And this value of CH was found to give us ſuch values of AB and AC, (the two ſides of the triangle ABC upon the given baſe BC, and of the given height $\frac{BC}{2}$, which Mr. Glenie's Problem requires us to find,) as will anſwer the conditions of the Problem.

Art. 4. But the reader will naturally aſk on this occaſion, "How did you diſcover that the firſt long compound quantity $-12{,}625\,a^{10} + 16{,}950\,a^9x + 12{,}435\,a^8x^2 - 1{,}304\,a^7x^3 - 9{,}162\,a^6x^4 - 8{,}748\,a^5x^5 - 4{,}762\,a^4x^6 - 1{,}848\,a^3x^7 - 501\,a^2x^8 - 90\,ax^9 - 9x^{10}$, conſiſting of eleven terms, was diviſible without a remainder by the ſecond long compound quantity $-2525\,a^8 + 360\,a^7x + 1404\,a^6x^2 + 1640\,a^5x^3 + 978\,a^4x^4 + 408\,a^3x^5 + 124\,a^2x^6 + 24\,ax^7 + 3x^8$, conſiſting of nine terms?" To this queſtion I have already given an anſwer in the 4th volume of the *Scriptores Logarithmici*, art. 36, page 359, by declaring "that I did not diſcover this long compound diviſor of the ſaid long "dividend till I had, firſt, diſcovered that the ſaid dividend was exactly diviſible by

" by the trinomial quantity $5aa - 6ax - 3xx$, and that the Quotient of this " Diviſion would be the ſaid ſecond long compound quantity, conſiſting of " nine terms;" from which it followed neceſſarily (from the nature of the operations of multiplication and diviſion,) that the ſaid Dividend muſt likewiſe be diviſible without a remainder by the ſaid long compound quantity, conſiſting of nine terms, which had been the Quotient of the firſt diviſion of it by the trinomial quantity $5aa - 6ax - 3xx$. And I added, that this diſcovery " that " the ſaid long Dividend, conſiſting of eleven terms, was exactly diviſible by " the ſaid trinomial quantity $5aa - 6ax - 3xx$," had been made only by the help of Mr. Glenie's conſtruction of the Problem, and that I did not yet know of any method of diſcovering with certainty *à priori*, and without the previous knowledge of Mr. Glenie's conſtruction, that the ſaid long compound quantity $- 12,625\,a^{10} + 16,950\,a^9x + 12,435\,a^8x^2 - 1,304\,a^7x^3 - 9,162\,a^6x^4 - 8,748\,a^5x^5 - 4,762\,a^4x^6 - 1,848\,a^3x^7 - 501\,a^2x^8 - 90\,ax^9 - 9x^{10}$ was diviſible without a remainder by the ſaid trinomial quantity $5aa - 6ax - 3xx$. But I added further, that we might, however, have conjectured with a great appearance of probability, that the ſaid long compound quantity was diviſible by ſome trinomial quantity of which the firſt term was $5aa$, and the laſt term was $3xx$, becauſe $- 12,625\,a^{10}$, (or the firſt term of the ſaid long compound quantity,) is diviſible by $5aa$, and $- 9x^{10}$, (or the laſt term of the ſaid compound quantity,) is diviſible by $3xx$. But how to diſcover, or even to form a conjecture, what would be the numeral co-efficient of ax in the middle term of the ſaid trinomial quantity (if there were really any ſuch trinomial quantity by which the ſaid long compound quantity might be divided without a remainder,) I confeſſed that I knew not.

Art. 5. But by a method of proceeding pointed-out to me by the learned and ingenious Mr. William Frend, (the author of the intelligible and uſefull treatiſe of Algebra in one volume octavo, intitled, *Principles of Algebra*, publiſhed in the year 1795) it is poſſible to find, by judicious and well-founded conjectures and trials, the ſaid numeral co-efficient of ax in the middle term of the ſaid trinomial quantity of which the firſt term is $5aa$ and the third, or laſt, term is $3xx$. This method I will now endeavour to explain.

A Method, ſuggeſted by Mr. Frend, of diſcovering, by judicious Conjectures and Trials, the Co-efficient of ax in the middle Term of the trinomial Quantity $5aa - 6ax - 3xx$, *by which the foregoing long compound Quantity conſiſting of eleven Terms, that is equal to* 0, *may be divided without a Remainder.*

Art. 6. Let the letter m be put for the unknown numeral co-efficient of ax in the ſaid trinomial quantity by which it is conjectured that the foregoing long

compound quantity is diviſible without a remainder, and of which $5aa$ is the firſt term, and $3xx$ is the laſt term: ſo that the ſaid trinomial quantity ſhall be $5aa - max - 3xx$.

Then it is evident, in the firſt place, that, if the ſaid trinomial quantity $5aa - max - 3xx$ will really divide the ſaid long compound quantity without a remainder, it may be concluded (for the reaſon given above in art. 3,) that it is equal to o, and conſequently that $5aa$ will be $= max + 3xx$. And hence it follows that, if we can find a tolerably near conjectural value of x, we may, by means of ſuch value, find a near value of the unknown co-efficient m, which will be $= \frac{5aa - 3xx}{ax}$, or to $\frac{5aa}{ax} - \frac{3xx}{ax}$, or to $\frac{5a}{x} - \frac{3x}{a}$. We will therefore endeavour to find a tolerably near value of x, which is the root of the quadratick equation $5aa = max + 3xx$ and likewiſe is one of the roots of the original high equation $16{,}950\, a^9x + 12{,}435\, a^8x^2 - 1{,}304\, a^7x^3 - 9{,}162\, a^6x^4 - 8{,}748\, a^5x^5 - 4{,}762\, a^4x^6 - 1{,}848\, a^3x^7 - 501\, a^2x^8 - 90\, ax^9 - 9x^{10} = 12{,}625\, a^{10}$. Now ſuch a near value of x may be derived from the contemplation of the ſaid original high equation in the manner following.

The ſaid original high equation may have two, but not more than two, roots, or, in the language of modern Algebräiſts, real and affirmative roots; and each of theſe roots muſt be leſs than the root of the quadratick equation $16{,}950\, a^9x + 12{,}435\, a^8x^2 = 12{,}625\, a^{10}$, which is derived from the foregoing high equation by omitting all thoſe terms of it which have the ſign — prefixed to them, or are ſubtracted from the two other terms $16{,}950\, a^9x + 12{,}435\, a^8x^2$; as is ſhewn in pages 358, 359, 360, 361, - - - 364 of the octavo volume, intitled, *Tracts on the Reſolution of affected Algebräick Equations by Dr. Halley's, Mr. Raphſon's, and Sir Iſaac Newton's Methods of Approximation*, publiſhed in the year 1800. We will therefore reſolve the ſaid quadratick equation $16{,}950\, a^9x + 12{,}435\, a^8x^2 = 12{,}625\, a^{10}$, in order to find a near value of x, or the leſſer root of the ſaid original high equation.

Art. 7. Now, in order to reſolve the quadratick equation $16{,}950\, a^9x + 12{,}435\, a^8x^2 = 12{,}625\, a^{10}$, we will, firſt, divide all the terms of it by a^8; and we ſhall thereby reduce it to the equation $16{,}950\, ax + 12{,}435\, x^2 = 12{,}625\, a^2$; and we will then divide this laſt equation by 12,435, the co-efficient of x^2, and we ſhall thereby obtain the equation $\frac{16{,}950\, ax}{12{,}435} + x^2 = \frac{12{,}625\, a^2}{12{,}435}$, or $1.36 \times ax + xx = 1.015 \times a^2$, or $xx + 1.36 \times ax = 1.015 \times aa$, of which the root is nearly $0.53 \times a$, as will appear from the following reſolution of it. Since $xx + 1.36 \times ax$ is $= 1.015 \times aa$, we ſhall have $xx + 1.36 \times ax + \overline{\frac{1.36}{2}}^2 \times aa = 1.015 \times aa + \overline{\frac{1.36}{2}}^2 \times aa$ $(= 1.015 \times aa + \overline{0.68}^2 \times aa = 1.015 \times aa + 0.4624 \times aa) = 1.4774 \times aa$, and conſequently (by extracting the ſquare-roots of both

both ſides,) $x + 0.68 \times a = 1.21 \times a$, and $x = 1.21 \times a - 0.68 \times a = 0.53 \times a$. Therefore the root of the quadratick equation $16,950\ a^9x + 12,435\ a^8xx = 12,625\ a^{10}$ will be $= 0.53 \times a$. Therefore the leſſer root of the long original equation $16,950\ a^9x + 12,435\ a^8x^2 - 1,304\ a^7x^3 - 9,162\ a^6x^4 - 8,748\ a^5x^5 - 4,762\ a^4x^6 - 1,848\ a^3x^7 - 501\ a^2x^8 - 90\ ax^9 - 9x^{10} = 12,625\ a^6$ will be greater than $0.53 \times a$. We will therefore ſuppoſe that it is nearly $= 0.6 \times a$; and, in order to find out whether $0.6 \times a$ will be greater, or leſs, than the true value of the ſaid leſſer root of the ſaid original equation, we will ſubſtitute $0.6 \times a$ inſtead of x in it's terms.

Art. 8. Now, if $0.6 \times a$ be ſubſtituted inſtead of x in the compound quantity $16,950\ a^9x + 12,435\ a^8x^2 - 1,304\ a^7x^3 - 9,162\ a^6x^4 - 8,748\ a^5x^5 - 4,762\ a^4x^6 - 1,848\ a^3x^7 - 501\ a^2x^8 - 90\ ax^9 - 9x^{10}$, the ſaid compound quantity will become $(= 16,950 \times a^9 \times 0.6 \times a + 12,435 \times a^8 \times 0.36 \times a^2 - 1,304 \times a^7 \times 0.216 \times a^3 - 9,162 \times a^6 \times 0.1296 \times a^4 - 8,748 \times a^5 \times 0.077,76 \times a^5 - 4,762 \times a^4 \times 0.046,656 \times a^6 - 1,843 \times a^3 \times 0.027,993,6 \times a^7 - 501 \times a^2 \times 0.016,796,16 \times a^8 - 90 \times a \times 0.010,077,696 \times a^9 - 9 \times 0.006,046,617,6 \times a^{10} = 16,950 \times 0.6 \times a^{10} + 12,435 \times 0.36 \times a^{10} - 1,304 \times 0.216 \times a^{10} - 9,162 \times 0.1296 \times a^{10} - 8,748 \times 0.077,76 \times a^{10} - 4,762 \times 0.046,656 \times a^{10} - 1,848 \times 0.027,993,6 \times a^{10} - 501 \times 0.016,796,16 \times a^{10} - 90 \times 0.010,077,696 \times a^{10} - 9 \times 0.006,046,617,6 \times a^{10} = 10,170.0 \times a^{10} + 4476.60 \times a^{10} - 281.664 \times a^{10} - 1187.3952 \times a^{10} - 680.244,48 \times a^{10} - 222.175,872 \times a^{10} - 51.732,172,8 \times a^{10} - 8.414,876,16 \times a^{10} - 0.906,992,640 \times a^{10} - 0.054,419,558,4 \times a = 14,646.60 \times a^{10} - 2432.588,013,158,4 \times a^{10}) = 12,214.011,986,841,6 \times a^{10}$; which is nearly equal to, but ſomething leſs than, $12,625\ a^{10}$, or the abſolute term of the ſaid long original equation. Therefore $0.6 \times a$ muſt be nearly equal to, but ſomething leſs than, the true value of x, or the leſſer root of the ſaid equation. Therefore $0.6 \times a$ will be nearly equal to, but ſomewhat leſs than, the true value of x in the quadratick equation $5aa = max + 3xx$; and conſequently $5aa$ will be nearly equal to $ma \times 0.6 \times a + 3 \times 0.36 \times aa$, or to $0.6 \times aa \times m + 1.08 \times aa$; and 5 will be nearly $= 0.6 \times m + 1.08$; and $0.6 \times m$ will be nearly $(= 5 - 1.08) = 3.92$; and m will be nearly $= \frac{3.92}{0.6} = 6.53$. Since therefore the unknown number m (which is the co-efficient of ax in the ſecond, or middle term of the trinomial quantity $5aa - max - 3xx$,) is found to be nearly equal to 6.53, it ſeems reaſonable to conjecture that it is equal to the whole number 6, and conſequently that the trinomial quantity $5aa - max - 3xx$ (by which we have reaſon to think that the long compound quantity $- 12,625\ a^{10} + 16,950\ a^9x + 12,435\ a^8x^2 - 1,304\ a^7x^3 - 9,162\ a^6x^4 - 8,748\ a^5x^5 - 4,762\ a^4x^6 - 1,848\ a^3x^7 - 501\ a^2x^8 - 90\ ax^9 - 9x^{10}$ is di-

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viſible without a remainder,) will be $5aa - 6ax - 3xx$. It will therefore be expedient to try whether the ſaid long compound quantity can be divided by the ſaid trinomial quantity $5aa - 6ax - 3xx$ without a remainder; and, upon trial, it will be found that it can be ſo divided. And, after we have thus found that this diviſion is practicable, we may conclude (for the reaſons mentioned above in art. 3,) that the ſaid trinomial quantity $5aa - 6ax - 3xx$ is equal to 0, or that $5aa$ is $= 6ax + 3xx$, or that $xx + 2ax$ is $= \frac{5aa}{3}$; which will lead us to a geometrical conſtruction of this quadratick equation, or a conſtruction performed by means of right lines and circles only, ſuch as that of Mr. Glenie, or that of art. 37 of the tract on this ſubject in the 4th Volume of the *Scriptores Logarithmici*, page 360.

Art. 9. By this method of proceeding, (which has been ſuggeſted by Mr. Frend,) it would have been poſſible, after reducing Mr. Glenie's Problem to the above-mentioned high equation of the tenth order, to find the trinomial Diviſor $5aa - 6ax - 3xx$ by which the ſaid high equation (after it's abſolute term $12{,}625\,a^{10}$ ſhould have been brought by ſubtraction to the firſt, or left-hand, ſide of the mark $=$ of equality, and the aggregate of all it's terms have been made equal to 0,) might have been converted into the quadratick equation $5aa - 6ax - 3xx = 0$, or $xx + 2ax = \frac{5aa}{3}$, which involves the leſſer of the two roots of the ſaid high original equation, or that root which is fitted for the ſolution of Mr. Glenie's Problem, and which leads to a Geometrical conſtruction of it. But this, Mr. Frend obſerves, will not often be practicable when ſuch high equations as this of the tenth power are derived from the conditions of Geometrical Problems, becauſe ſuch equations, for the moſt part, will not admit of ſuch reductions by Diviſion: and, when they do admit of ſuch reductions, he thinks that it will, in general, be eaſier and more convenient to reſolve them at once in their natural ſtate by the Methods of Approximation than to go through the various conjectures and trials that are neceſſary to be made in order to obtain the trinomial, or quadratick, quantities by which they may be divided without a remainder, and brought down to quadratick equations that are capable of a Geometrical conſtruction.

Art. 10. And we may obſerve further, that, when a Geometrical Problem naturally produces an equation of a high order, and, after a ſubtle and laborious inquiry (ſimilar to that by which we found above that the foregoing equation of the tenth power might be reduced by diviſion to the quadratick equation $5aa - 6ax - 3xx = 0$, or $xx + 2ax = \frac{5aa}{3}$,) we have found that the long compound quantity which is equal to 0, (ariſing from the ſubtraction of the known quantity, or abſolute term of the equation from both ſides of it,) is diviſible without a remainder by a certain quadratick, or trinomial, quantity, of which we know all the terms, it does not always follow that the ſaid quadratick, or trinomial, quantity

quantity will alſo be equal to o, and will therefore afford us a quadratick equation, by the reſolution of which we may hope to obtain the value of the unknown quantity x which will enable us to ſolve the Problem propoſed to us;—and, even, if the ſaid quadratick, or trinomial, quantity, (by which the ſaid multinomial quantity that is equal to o may be divided without a remainder,) will alſo be equal to o, and conſequently will produce a quadratick equation, it does not follow that the root of that equation will be *that* root of the original high equation which will enable us to ſolve the propoſed equation; for it may be another root of the ſaid original high equation, which relates to a different Problem from that which we were required to ſolve. And of this the foregoing high equation $16{,}950\,a^9x + 12{,}435\,a^8x^2 - 1{,}304\,a^7x^3 - 9{,}162\,a^6x^4 - 8{,}748\,a^5x^5 - 4{,}762\,a^4x^6 - 1{,}848\,a^3x^7 - 501\,a^2x^8 - 90\,ax^9 - 9x^{10} = 12{,}625\,a^{10}$ will afford us a notable example.

Art. 11. For let us ſuppoſe that the Geometrical Problem which we had been required to ſolve, had been a little different from Mr. Glenie's Problem, and had been as follows.

A PROBLEM.

In the Palace of one of the Kings of Perſia, it is ſaid, there was a triangular area of ſuch a ſhape and ſize that the Difference of the Cubes of two of it's ſides was equal to three times the Cube on the third ſide, or Baſe, of the triangle, (which was 200 feet in length,) and that it's area contained 10,000 ſquare feet. Suppoſing this to have been really the caſe, it is required to determine the lengths of the two ſides of the ſaid triangle.

Now, if ab, or the perpendicular altitude of this triangle, which is equal to 100 feet, or half the baſe BC, (in fig. 4, in the 4th volume of the *Scriptores Logarithmici*, page 396,) be denoted by the letter a, and Cb, or the external line intercepted between C, (the nearer extremity of the Baſe BC,) and the line ab, (which is drawn from a, the vertex of the triangle aBC, at right angles to the Baſe BC, produced,) be denoted by the letter x, it is ſhewn in art. 63 of the tract on Mr. Glenie's Problem in the ſaid 4th volume of the *Scriptores Logarithmici*, page 397, that the equation for determining the magnitude of the line x, or Cb, (upon which the poſition of the point a, or the vertex of the triangle aBC which we are required to deſcribe, and the magnitudes of the two ſides aB and aC, depend,) will be the very ſame equation of the tenth order which was obtained before for the ſolution of Mr. Glenie's Problem, to wit, the equation

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$16{,}950\ a^9x + 12{,}435\ a^8x^2 - 1{,}304\ a^7x^3 - 9{,}162\ a^6x^4 - 8{,}748\ a^5x^5 - 4{,}762\ a^4x^6 - 1{,}848\ a^3x^7 - 501\ a^2x^8 - 90\ ax^9 - 9x^{10} = 12{,}625\ a^{10}$; and the greater root of this equation will give us the magnitude of the said external line C*h*, (which is neceſſary to the ſolution of this Problem) juſt as the leſſer root of it gave us the magnitude of the external line CH, which was neceſſary to the ſolution of Mr. Glenie's Problem.

Art. 12. When we have thus reduced this ſecond Problem to the foregoing equation of the tenth order, we might proceed to inquire, whether, if we were to ſubtract the abſolute term $12{,}625\ a^{10}$ from both ſides of the equation, the long compound quantity reſulting from ſuch ſubtraction might not be divided without a remainder by ſome trinomial, quadratick, quantity, and we might find, by the method above-deſcribed, that it might be ſo divided by the trinomial quantity $5aa - 6ax - 3xx$, which diviſor might be made equal to 0, and thereby might furniſh us with the quadratick equation $5aa = 6ax + 3xx$, or $xx + 2ax = \frac{5aa}{3}$. But this quadratick equation would give us the value of the line CH, (which is equal to the leſſer of the two roots of the ſaid high equation, and is fitted for the ſolution of Mr. Glenie's Problem,) and would not give us the value of the line C*h*, which is equal to the greater root of the ſaid equation, and is fitted to the ſolution of the preſent Problem; and therefore it would not enable us to ſolve the preſent Problem either by the reſolution of it in numbers, or by a Geometrical conſtruction of it. And the only way in which the diſcovery of the ſaid trinomial quantity $5aa - 6ax - 3xx$ (after all the pains we had taken to find it,) could be of uſe to us in enabling us to find the value of the line C*h*, or of the greater of the two roots of the ſaid high equation, would be by furniſhing us with the Quotient of the diviſion of the ſaid long compound quantity, conſiſting of eleven terms, that was equal to 0, by the ſaid trinomial Diviſor $5aa - 6ax - 3xx$; which Quotient would be a quantity conſiſting of nine terms, to wit, the quantity $-2525\ a^8 + 360\ a^7x + 1404\ a^6x^2 + 1640\ a^5x^3 + 978\ a^4x^4 + 408\ a^3x^5 + 124\ a^2x^6 + 24\ ax^7 + 3x^8$, and would be equal to 0, as well as the trinomial diviſor $5aa - 6ax - 3xx$, and would therefore afford us the equation $-2525\ a^8 + 360\ a^7x + 1404\ a^6x^2 + 1640\ a^5x^3 + 978\ a^4x^4 + 408\ a^3x5 + 124\ a^2x^6 + 24\ ax^7 + 3x^8 = 0$, or the equation $3x^8 + 24\ ax^7 + 124\ a^2x^6 + 408\ a^3x^5 + 978\ a^4x^4 + 1640\ a^5x^3 + 1404\ a^6x^2 + 360\ a^7x = 2525\ a^8$. This equation can evidently have but one root; and this root will give us the value of the external line C*h*, which is neceſſary to the ſolution of the preſent Problem, as has been ſhewn in the 4th volume of the *Scriptores Logarithmici*, pages 396, 397, and 398; ſo that the only advantage we ſhall have obtained by diſcovering the trinomial Diviſor $5aa - 6ax - 3xx$, with reſpect to the ſolution of the preſent Problem, or the inveſtigation of the magnitude of the external line C*h*, is that we ſhall now have an equation of the eighth power, inſtead of the original equation of the tenth power, to reſolve in order to obtain the

ſaid

ſaid magnitude of C*b*, which is neceſſary to the ſolution of the preſent Problem. And this advantage will not be great, becauſe this equation of the eighth power would be almoſt as difficult to reſolve in the common and natural way, or by Mr. Raphſon's, or ſome other method, of approximation, as the original equation of the tenth power.

Art. 13. But, perhaps, it may be thought that this equation of the eighth power, to wit, $3x^8 + 24\,ax^7 + 124\,a^2x^6 + 408\,a^3x^5 + 978\,a^4x^4 + 1640\,a^5x^3 + 1404\,a^6x^2 + 360\,a^7x = 2525\,a^8$, may, in like manner as the original equation of the tenth power, be reduced to a quadratick equation by finding, in the manner above-deſcribed, a trinomial, quadratick, quantity, by which the compound quantity $-2525\,a^8 + 360\,a^7x + 1404\,a^6x^2 + 1640\,a^5x^3 + 978\,a^4x^4 + 408\,a^3x^5 + 124\,a^2x^6 + 24\,ax^7 + 3x^8$, (ariſing from the ſubtraction of $2525\,a^8$, the abſolute term of the ſaid equation, from both ſides of it,) may be divided without a remainder. I ſhall therefore now endeavour to find ſuch a quadratick, trinomial, quantity.

Art. 14. In the firſt place, then, we may reaſonably conjecture that the numeral co-efficient of *aa* in the firſt term of ſuch a trinomial quantity (if ſuch a trinomial quantity can exiſt,) will be the number 5, becauſe 5 is the ſmalleſt diviſor, except 1, of 2525, the co-efficient of a^8 in the firſt term, $-2525\,a^8$, of the ſaid Dividend; the other diviſors of the ſaid co-efficient being 1, 25, 101, 505, and 2525. And for a like reaſon we may conclude that the numeral co-efficient of *xx* in the third, or laſt, term of ſuch trinomial quantity muſt be either 1, or 3, becauſe they are the only numbers that are diviſors of the number 3, which is the co-efficient of x^8 in the laſt term, $3x^8$, of the ſaid Dividend; and of theſe two numbers, 1 and 3, it ſeems reaſonable to ſuppoſe that the ſmaller, to wit, 1, will be the ſaid co-efficient of *xx* in the ſaid trinomial quantity. We will therefore ſuppoſe 5 to be the co-efficient of *aa*, and 1 to be the co-efficient of *xx*, in the ſaid trinomial quantity, and we will put *m* for the unknown co-efficient of *ax* in the ſecond, or middle, term of the ſaid trinomial quantity: and, further, we will ſuppoſe that the two latter terms of the ſaid quantity are ſubtracted from it's firſt term, and are, together, exactly equal to it, ſo as to make the whole trinomial quantity be equal to 0. And then the ſaid trinomial quantity will be $5aa - max - xx$, and the quadratick equation by which we may hope to diſcover the value of *x* in the aforeſaid equation of the eighth power, or the magnitude of the line C*b*, by means of which the preſent Problem is to be ſolved, will be the equation $5aa - max - xx = 0$, or $xx + max = 5aa$. We muſt therefore, in the next place, endeavour to find the value of the unknown number *m*.

Art. 15. Now, in order to find the value of *m*, it will be neceſſary to find a tolerably near value of *x*, or one that ſhall agree with it's true value to one place of decimal figures; which may be done in the manner following.

We

We already know that the leſſer value of x in the foregoing original high equation of the tenth power, (which reſults equally from the conditions of Mr. Glenie's Problem and from thoſe of the preſent Problem,) or the leſſer root of the ſaid high equation is $=$ 0.632,993,15 &c $\times a$. Therefore the greater root of the ſaid high equation of the tenth power, or the only root of the foregoing equation of the eighth power, which we are now endeavouring to reduce to a quadratick equation, to wit, the equation $360\,a^7x + 1404\,a^6x^2 + 1640\,a^5x^3 + 978\,a^4x^4 + 408\,a^3x^5 + 124\,a^2x^6 + 24\,ax^7 + 3x^8 = 2525\,a^8$, muſt be greater than 0.632,993,15 &c $\times a$. And the ſaid root muſt be leſs than a; becauſe, if it were to be equal to a, the octinomial quantity $360\,a^7x + 1404\,a^6x^2 + 1640\,a^5x^3 + 978\,a^4x^4 + 408\,a^3x^5 + 124\,a^2x^6 + 24\,ax^7 + 3x^8$ (which forms the firſt, or left-hand, ſide of the ſaid equation, and is equal to $2525\,a^8$) would be $(= 360\,a^8 + 1404\,a^8 + 1640\,a^8 + 978\,a^8 + 408\,a^8 + 124\,a^8 + 24\,a^8 + 3\,a^8) = 4941\,a^8$, which is almoſt double of $2525\,a^8$, or the abſolute term of the ſaid equation; and conſequently a muſt be conſiderably greater than the true value of x in the ſaid equation. Therefore, ſince x is greater than 0.632 &c $\times a$, but leſs than a, or $1 \times a$, we will ſuppoſe it to be nearly equal to an arithmetical mean between 0.632 &c $\times a$ and $1 \times a$, or to $\frac{1.632}{2} \times a$, or 0.816 &c $\times a$; and will ſubſtitute $0.8 \times a$ inſtead of it in the quadratick equation $xx + max = 5aa$. And then we ſhall have $\overline{0.8}^2 \times aa + ma \times 0.8 \times a = 5aa$, or $0.64 \times aa + m \times 0.8 \times aa = 5aa$, and conſequently $0.64 + m \times 0.8 = 5$, and $m \times 0.8\ (= 5 - 0.64) = 4.36$, and $m = \frac{4.36}{0.8} = 5.45$. Therefore m (which muſt be a whole number, if the propoſed diviſion of the foregoing long compound quantity, conſiſting of nine terms, can take place,) muſt be either the number 5, or the number 6, which are the neareſt whole numbers to 5.45; and conſequently the quadratick equation ſought, to wit, the equation $xx + max = 5aa$, muſt be either $xx + 5ax = 5aa$, or $xx + 6ax = 5aa$. But neither of theſe quadratick equations will give us a value of x equal to that which we are ſeeking, or to that of the foregoing equation of the eighth power. For, if we ſuppoſe $xx + 5ax$ to be $= 5aa$, we ſhall have $xx + 5ax + \frac{25}{4} \times aa\ (= 5aa + \frac{25}{4} \times aa = \frac{20\,aa}{4} + \frac{25\,aa}{4}) = \frac{45\,aa}{4}$; and conſequently $x + \frac{5a}{2}$ will be $= \frac{\sqrt{45}}{2} \times a$, and x will be $(= \frac{\sqrt{45} - 5}{2} \times a = \frac{6.70 - 5}{2} \times a = \frac{1.70}{2} \times a) = 0.85 \times a$. And, if we ſuppoſe $xx + 6ax$ to be $= 5aa$, we ſhall have $xx + 6ax + 9aa\ (= 5aa + 9aa) = 14aa$; and conſequently $x + 3a$ will be $= \sqrt{14} \times a$, and x will be $= \sqrt{14} \times a - 3a = 3.72 \times a - 3a = 0.72 \times a$. But neither of theſe values of x is equal to the root of the foregoing equation of the eighth power, which has been found in the *Scriptores Logarithmici*, Vol. 4, pages 386, 387, &c - - - 395, to be $= 0.791,626,8$, &c $\times a$, which is leſs than $0.85 \times a$, or the root of the quadratick equation $xx + 5ax = 5aa$, and greater than $0.72 \times a$,

or the root of the quadratick equation $xx + 6ax = 5aa$. And moreover, neither of the two trinomial quantities $5aa - 5ax - xx$ and $5aa - 6ax - xx$ will divide the long compound quantity $- 2525a^8 + 360a^7x + 1404a^6x^2 + 1640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$ without a remainder. For, if we attempt to divide this quantity by the trinomial quantity $5aa - 5ax - xx$, we ſhall find the Quotient to be $- 505a^6 - \frac{2162}{5} a^5x$ &c; and, if we attempt to divide it by the trinomial quantity $5aa - 6ax - xx$, we ſhall find the Quotient to be $- 505a^6 - 534a^5x - 461a^4x^2 - 332a^3x^3 - 295a^2x^4 - \frac{1694ax^5}{5}$ &c; both which Quotients contain ſome terms that have fractional co-efficients, and therefore will not compleatly exhauſt the Dividend, or leave it without a remainder. This endeavour to reduce the ſaid equation of the eighth power to a quadratick equation by means either of the trinomial quantity $5aa - 5ax - xx$, or the trinomial quantity $5aa - 6ax - xx$, is therefore wholly unſucceſsful.

Art. 16. We will now ſuppoſe the co-efficient of xx in the ſaid trinomial quantity by which we hope to divide the foregoing long compound quantity, conſiſting of nine terms, without a remainder, to be 3, inſtead of 1, and will try the effect of that ſuppoſition.

Now, if 5 is, as before, the co-efficient of aa, and 3 is the co-efficient of xx, and m be put, as before, for the unknown co-efficient of ax, the ſaid trinomial quantity will be $5aa - max + 3xx$. And this trinomial quantity, in order to be uſeful to us in our preſent purſuit, muſt be ſuppoſed to be equal to 0; and conſequently $5aa$ will be $= max + 3xx$. Therefore max will be equal to $5aa - 3xx$, and m will be $= \frac{5aa - 3xx}{ax}$. Now we know that x is nearly $= 0.8 \times a$. Therefore xx will be nearly $= 0.64 \times aa$, and ax will be nearly $(= a \times 0.8 \times a) = 0.8 \times aa$, and m, or $\frac{5aa - 3xx}{ax}$ will be nearly $(= \frac{5aa - 3 \times 0.64 \times aa}{0.8 \times aa} = \frac{5aa - 1.92 \times aa}{0.8 \times aa} = \frac{5 - 1.92}{0.8} = \frac{3.08}{0.80}) = 3.85$. Therefore m muſt be the whole number that is neareſt to 3.85, which is 4. Therefore the trinomial quantity ſought will be $5aa - 4ax - 3xx$, and the value of x which we are ſeeking muſt be the root of the quadratick equation $5aa - 4ax - 3xx = 0$, or of the equation $5aa = 4ax + 3xx$, or of the equation $xx + \frac{4ax}{3} = \frac{5aa}{3}$. Now, if $xx + \frac{4ax}{3}$ is $= \frac{5aa}{3}$, we ſhall have $xx + \frac{4ax}{3} + \frac{4aa}{9} (= \frac{5aa}{3} + \frac{4aa}{9} = \frac{15aa}{9} + \frac{4aa}{9}) = \frac{19aa}{9}$; and conſequently $x + \frac{2a}{3}$ will be $= \frac{\sqrt{19}}{3} \times a = \frac{4.35}{3} \times a$, and x will

will be $= \frac{4.35}{3} \times a - \frac{2a}{3} = \frac{2.35}{3} \times a = 0.78 \times a$. But this value of x is leſs than the root of the equation of the eighth power which we are ſeeking, which has been ſhewn to be $= 0.791,626,8 \times a$. Therefore this trinomial quantity $5aa - 4ax - 3xx$ fails us as well as the two former trinomial quantities $5aa - 5ax - xx$ and $5aa - 6ax - xx$. And moreover, this new trinomial quantity $5aa - 4ax - 3xx$ will not divide the long compound quantity $- 2525a^8 + 360a^7x + 1404a^6x^2 + 1640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8$ without a remainder, any more than either of the two former trinomial quantities $5aa - 5ax - xx$ and $5aa - 6ax - xx$. For, if we attempt to divide this long compound quantity by the trinomial quantity $5aa - 4ax - 3xx$, we ſhall find the Quotient to be $- 505a^6 - 332a^5x - \frac{1217a^4x^2}{5}$ &c, of which the third term, $- \frac{1217a^4x^2}{5}$, contains the fractional number $\frac{1217}{5}$, or $243 + \frac{2}{5}$, and which therefore will not compleatly exhauſt the Dividend, or leave it without a remainder.

Art. 17. From the failure of all theſe endeavours to find a quadratick, trinomial, quantity that ſhall divide the ſaid long Dividend, (conſiſting of nine terms,) without a remainder, it ſeems highly probable that no ſuch Diviſor can exiſt, or, at leaſt, can be found by this method of Inveſtigation; though, to make this quite certain, it would be neceſſary to try all the following trinomial quantities, to wit, $1 \times aa - max - xx$, $1 \times aa - max - 3xx$, $25aa - max - xx$, $25aa - max - 3xx$, $101aa - max - xx$, $101aa - max - 3xx$, $505aa - max - xx$, $505aa - max - 3xx$, $2525aa - max - xx$, and $2525aa - max - 3xx$, in which all the poſſible diviſors of the number 2525, except the number 5, (which has been already tried,) are ſucceſſively taken for the co-efficient of aa in the firſt term of the ſaid trinomial quantity; which trials muſt be made by ſuppoſing each of the ſaid trinomial quantities ſucceſſively to be equal to 0, and ſubſtituting $0.8 \times a$ inſtead of x in each of the ſeveral equations thence reſulting, to wit, in the equations $1 \times aa = max + xx$, $1 \times aa = max + 3xx$, $25aa = max + xx$, $25aa = max + 3xx$, $101aa = max + xx$, $101aa = max + 3xx$, $505aa = max + xx$, $505aa = max + 3xx$, $2525aa = max + xx$, and $2525aa = max + 3xx$, and then reſolving the ſaid equations, ſo as thereby to obtain near values of the unknown co-efficient m, and then ſubſtituting for m in each of the ſaid equations the neareſt whole number to the value of m ſo obtained. But this would be a tedious taſk, and would give us more trouble than to find the root of the long equation $360a^7x + 1404a^6x^2 + 1640a^5x^3 + 978a^4x^4 + 408a^3x^5 + 124a^2x^6 + 24ax^7 + 3x^8 = 2525a^8$ to ſeven, or eight, places of decimal figures in the natural and common way, by Mr. Raphſon's, or ſome other, method of Approximation, as has been done in the 4th volume of the

Scriptores

Scriptores Logarithmici, in pages 386, 387, 388, 389, - - - - - 395; and we may be almoſt certain that it would be unſucceſsful, and only ſerve to ſhew that ſuch reduction of the ſaid equation of the eighth power to a quadratick equation by means of ſuch a trinomial Diviſor of the ſaid compound quantity, conſiſting of nine terms, is impoſſible. I am therefore inclined to acceed to Mr. Frend's opinion, that theſe attempts to reduce high equations to quadratick equations by means of ſuch quadratick, trinomial, Diviſors, are, for the moſt part, but of little uſe, and that the beſt way of reſolving thoſe equations, and conſequently of ſolving the Problems from which they have been derived, is the moſt obvious and natural way of doing it, to wit, to begin our reſolution of the ſaid equations by making judicious conjectures and trials concerning thoſe roots of them which are neceſſary to the Solution of the Problem that gave riſe to them, ſo as to obtain tolerably good firſt near values of the ſaid roots, and then to approach nearer and nearer to the true values of the ſaid roots by two, or three, proceſſes of Mr. Raphſon's, or ſome other, method of Approximation, till we have obtained the values of the ſaid roots to the degree of exactneſs that we think neceſſary.

FRANCIS MASERES.

INNER TEMPLE,
Oct. 30, 1801.

AN

INVESTIGATION

OF THE

DIFFERENTIAL SERIES

$$a - \frac{bx}{1+x} - \frac{D^{I}x^{2}}{\overline{1+x}|^{2}} - \frac{D^{II}x^{3}}{\overline{1+x}|^{3}} - \frac{D^{III}x^{4}}{\overline{1+x}|^{4}} - \frac{D^{IV}x^{5}}{\overline{1+x}|^{5}} - \frac{D^{V}x^{6}}{\overline{1+x}|^{6}} - \frac{D^{VI}x^{7}}{\overline{1+x}|^{7}} - \&c$$

ad infinitum, by which the Law of Continuation of it's Terms is ſhewn to extend to all it's Terms as well as to thoſe which have been actually inveſtigated.

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Mr. MANNING'S INVESTIGATION

OF THE

DIFFERENTIAL SERIES

$$a - \frac{bx}{1+x} - \frac{D^{I}x^{2}}{1+x)^{2}} - \frac{D^{II}x^{3}}{1+x)^{3}} - \frac{D^{III}x^{4}}{1+x)^{4}} - \frac{D^{IV}x^{5}}{1+x)^{5}} - \frac{D^{V}x^{6}}{1+x)^{6}} - \frac{D^{VI}x^{7}}{1+x)^{7}} - \text{\&c } \textit{ad infinitum},$$

by which the Law of Continuation of it's Terms is ſhewn to extend to all it's Terms as well as to thoſe which have been actually inveſtigated.

Article 1. IN the 3d volume of this Collection of Mathematical Tracts, intitled, *Scriptores Logarithmici*, there is a very uſeful Tract on the Summation of infinite Serieſes that converge too ſlowly to be ſummed in the common way, or by the mere computation of a moderate number of their firſt terms, and the addition of them together, or the addition of ſome of them together and the ſubtraction of others of them from the former; in which it is ſhewn that, if the ſlowly-converging Series be of the following form, to wit, $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + ix^8 - kx^9 +$ &c *ad infinitum*, the ſaid Series will be equal to the Differential Series $a - \frac{bx}{1+x} - \frac{D^{I}x^{2}}{1+x)^{2}} - \frac{D^{II}x^{3}}{1+x)^{3}} - \frac{D^{III}x^{4}}{1+x)^{4}} - \frac{D^{I}x^{5}}{1+x)^{5}} - \frac{D^{V}x^{6}}{1+x)^{6}} - \frac{D^{VI}x^{7}}{1+x)^{7}} -$ &c *ad infinitum*, which will always converge with a conſiderable degree of ſwiftneſs. The ſuppoſitions upon which the equality of theſe two ſerieſes to each other is founded are as follows.

In the firſt place the letter x (which enters into all the terms of the ſlowly-converging ſeries $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + ix^8 - kx^9$

+

+ &c, except the first term a,) must be either less than 1, or equal to 1, so that the series of it's powers, x, x^2, x^3, x^4, x^5, x^6, x^7, x^8, x^9, &c, shall either form a series of equal quantities, to wit, 1, 1, 1, 1, 1, 1, 1, 1, 1, &c, or a series of decreasing quantities, such as the series $\frac{9}{10}$, $\frac{9}{10}\Big|^2$, $\frac{9}{10}\Big|^3$, $\frac{9}{10}\Big|^4$, $\frac{9}{10}\Big|^5$, $\frac{9}{10}\Big|^6$, $\frac{9}{10}\Big|^7$, $\frac{9}{10}\Big|^8$ $\frac{9}{10}\Big|^9$, &c, or the Series $\frac{99}{100}$, $\frac{99}{100}\Big|^2$, $\frac{99}{100}\Big|^3$, $\frac{99}{100}\Big|^4$, $\frac{99}{100}\Big|^5$, $\frac{99}{100}\Big|^6$, $\frac{99}{100}\Big|^7$, $\frac{99}{100}\Big|^8$, $\frac{99}{100}\Big|^9$, &c.

In the second place, the several numbers a, b, c, d, e, f, g, h, i, k, &c must be a set of numbers that continually decrease, or of which every one is less than that which immediately preceeds it; to wit, b than a, c than b, d than c, e than d, and so on.

And in the 3d place, the several differences of these numbers, beginning from the second number b, to wit, the differences $b - c$, $c - d$, $d - e$, $e - f$, $f - g$, $g - h$, $h - i$, $i - k$, &c, also form a decreasing progression, as well as the numbers b, c, d, e, f, g, h, i, k, &c, of which they are the differences.

And, 4thly, the differences of these differences, to wit, the quantities $b - c - \overline{c - d}$, $c - d - \overline{d - e}$, $d - e - \overline{e - f}$, $e - f - \overline{f - g}$, $f - g - \overline{g - h}$, $g - h - \overline{h - i}$, and $h - i - \overline{i - k}$, &c, or the trinomial quantities $b - 2c + d$, $c - 2d + e$, $d - 2e + f$, $e - 2f + g$, $f - 2g + h$, $g - 2h + i$, and $h - 2i + k$, &c, or the *second* differences of the numbers b, c, d, e, f, g, h, i, k, &c, also form a decreasing progression; and the differences of these second differences, to wit, the quantities $b - 2c + d - \overline{c - 2d + e}$, $c - 2d + e - \overline{d - 2e + f}$, $d - 2e + f - \overline{e - 2f + g}$, $e - 2f + g - \overline{f - 2g + h}$, $f - 2g + h - \overline{g - 2h + i}$, and $g - 2h + i - \overline{h - 2i + k}$, &c, or the quadrinomial quantities $b - 3c + 3d - e$, $c - 3d + 3e - f$, $d - 3e + 3f - g$, $e - 3f + 3g - h$, $f - 3g + 3h - i$, and $g - 3h + 3i - k$, &c, or the *third* differences of the numbers b, c, d, e, f, g, h, i, k, &c also form a decreasing progression; and the differences of these third differences, to wit, the quantities $b - 3c + 3d - e - \overline{c - 3d + 3e - f}$, $c - 3d + 3e - f - \overline{d - 3e + 3f - g}$, $d - 3e + 3f - g - \overline{e - 3f + 3g - h}$, $e - 3f + 3g - h - \overline{f - 3g + 3h - i}$, and $f - 3g + 3h - i - \overline{g - 3h + 3i - k}$, &c, or the quinquinomial quantities $b - 4c + 6d - 4e + f$, $c - 4d + 6e - 4f + g$, $d - 4e + 6f - 4g + h$; $e - 4f + 6g - 4h + i$, and $f - 4g + 6h - 4i + k$, &c, or the *fourth* differences of the numbers b, c, d, e, f, g, h, i, k, &c, also form a decreasing progression; and, in like manner, the fifth differences, and the sixth differences, and the seventh differences, and all the other following differences, of the said numbers b, c, d, e, f, g, h, i, k, &c of other higher orders, continued to any number whatsoever, also form decreasing progressions.

And further, the first term, $b - c$, of the first series of differences of the said numbers $b, c, d, e, f, g, h, i, k$, &c, is denoted by D^{I}, or the capital letter D with the Roman numeral figure I placed above it; and the first term, $b - 2c + d$, of the second series of differences of the said numbers is denoted by D^{II}, or the same capital letter D with the Roman numeral figure II placed above it; and the first term, $b - 3c + 3d - e$, of the third series of differences of the said numbers is denoted by D^{III}, or the same capital letter D with the Roman numeral figure III placed above it; and the first term, $b - 4c + 6d + 4e - f$, of the fourth series of differences of the said numbers is denoted by D^{IV}, or the same capital letter D with the Roman numeral figure IV placed above it; and the first terms of the fifth, sixth, seventh, eighth, ninth, and tenth, and other following, series of differences of the said numbers are in like manner denoted by D^{V}, D^{VI}, D^{VII}, D^{VIII}, D^{IX}, D^{X}, &c, or by the same capital letter D with the Roman numeral figures that express the orders of differences to which they respectively belong, placed over it.

With this notation of the first terms of these several series of differences, and upon the suppositions above-mentioned concerning those series of differences, to wit, that each of them is a progression of decreasing quantities, it will be true that the slowly-converging series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 + ix^8 - kx^9 +$ &c *ad infinitum* will be equal to the Differential Series $a - \frac{bx}{1+x} - \frac{D^{I}x^2}{1+x|^2} - \frac{D^{II}x^3}{1+x|^3} - \frac{D^{III}x^4}{1+x|^4} - \frac{D^{IV}x^5}{1+x|^5} - \frac{D^{V}x^6}{1+x|^6} - \frac{D^{VI}x^7}{1+x|^7}$ — &c *ad infinitum*, in which Series all the terms after the first term a are marked with the sign —, or the sign of Subtraction, or are to be subtracted from the said first term a. The truth of this Proposition is demonstrated in the said Tract in the 3d volume of this Collection of Tracts, intitled *Scriptores Logarithmici*, which is intitled, *A Method of finding the Value of a slowly-converging infinite Series of decreasing Quantities of a certain Form, &c*, and begins in page 219 and ends in page 252 of the said volume. And several very notable and curious examples are given in the said Tract of the great usefulness of the latter Series (which, from it's involving in it's terms the several Differences D^{I}, D^{II}, D^{III}, D^{IV}, D^{V}, D^{VI}, &c of the co-efficients $b, c, d, e, f, g, h,$ &c of the several powers of x in the original Series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c, is called *The Differential Series*,) in finding the value of the original Series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c, when it's terms decrease with such excessive slowness as to make it incapable of being summed in the common way, or by the mere computation of a moderate number of it's initial terms, and a subtraction of the sum of those which are marked with the sign — from the sum of the first term a and the following terms cx^2, ex^4, gx^6, ix^8, &c, which are marked with the sign +, or are to be added to the said first term. But the method of investigating this Differential Series was but slightly touched upon in

that paper, to avoid making it too long: and I likewiſe thought, at the time of writing it, that I had made it ſufficiently intelligible. But in this opinion I was miſtaken; and I came afterwards to think that this omiſſion of a more ample explanation of the method of inveſtigating this Series was a defect of ſufficient importance to deſerve to be ſupplied in the beſt manner I was able. For a learned and intelligent friend of mine, who had a very clear head and a good capacity and turn for mathematical inquiries (though the buſineſs of his Profeſſion prevented him from employing much time and labour in the purſuit of them,) the late Henry Boult Cay, Eſquire, a Barriſter at Law of the Society of the Middle Temple, (who was afterwards made Sollicitor to the Exciſe in the year 1783,) upon reading that Tract, though he was greatly pleaſed with the Differential Series deſcribed in it, on account of it's great utility in finding the values of ſlowly-converging Serieſes that fall under the form of the general Series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7$ + &c *ad infinitum*, yet complained at the ſame time that the manner of obtaining it had not been therein ſufficiently explained for him to underſtand it thoroughly. This made me reſolve to give a further explanation of it in as full and clear a manner as I could, and was the occaſion of my drawing-up a ſecond Tract upon that Series, which is alſo printed in the 3d volume of this Collection immediately after the former Tract, in pages 255, 256, 257, &c, - - - - 312, under the Title of *The Inveſtigation of the foregoing Differential Series*, in which the ſubject is treated with very great care and exactneſs. In this ſecond Tract I have, in the firſt place, given a very full deſcription and demonſtration of the Principles by means of which I had inveſtigated the ſaid Differential Series, which are ſet forth in four Lemmas, or preliminary Propoſitions to the Inveſtigation itſelf of the ſaid Series; and I have there exhibited the ſaid Inveſtigation in a very diſtinct and particular manner, ſo far as to determine the firſt five terms of the ſaid Differential Series, which are found to be $a - \frac{bx}{1+x} - \sqrt{b-c} \times \frac{x^2}{\sqrt{1+x}^2} - \sqrt{b-2c+d} \times \frac{x^3}{\sqrt{1+x}^3} - \sqrt{b-3c+3d-e} \times \frac{x^4}{\sqrt{1+x}^4}$, or $a - \frac{bx}{1+x} - \frac{D^I x^2}{\sqrt{1+x}^2} - \frac{D^{II} x^3}{\sqrt{1+x}^3} - \frac{D^{III} x^4}{\sqrt{1+x}^4}$, of which all the terms after the firſt term a have the ſign — prefixed to them, or are ſubtracted from the ſaid firſt term. And I added, (in art. 46, page 282,) that I had, for my own ſatisfaction, inveſtigated in the ſame manner the ſixth, ſeventh, and eighth, terms of the ſame Differential Series, and found them to be equal to $\sqrt{b-4c+6d-4e+f} \times \frac{x^5}{\sqrt{1+x}^5}$, and $\sqrt{b-5c+10d-10e+5f-g} \times \frac{x^6}{\sqrt{1+x}^6}$, and $\sqrt{b-6c+15d-20e+15f-6g+h} \times \frac{x^7}{\sqrt{1+x}^7}$, or to $\frac{D^{IV} x^5}{\sqrt{1+x}^5}$, $\frac{D^V x^6}{\sqrt{1+x}^6}$, and $\frac{D^{VI} x^7}{\sqrt{1+x}^7}$, and that they alſo, as well as the four

preceeding

preceeding terms, were to be ſubtracted from the firſt term a, and conſequently to have the ſign — prefixed to them. And I then obſerved, that it ſeemed reaſonable to conjecture, from the analogy of thoſe ſeveral terms to each other, that the ſame Law of Continuation would take place in the following terms of the Series after the eighth term $\frac{D^{VI}x^7}{1+x)^7}$, and that thoſe terms would conſequently be $\frac{D^{VII}x^8}{1+x)^8}$, $\frac{D^{VIII}x^9}{1+x)^9}$, $\frac{D^{IX}x^{10}}{1+x)^{10}}$, $\frac{D^{X}x^{11}}{1+x)^{11}}$, $\frac{D^{XI}x^{12}}{1+x)^{12}}$, &c, and would all have the ſign — prefixed to them, or be ſubtracted from the firſt term a, like the other ſeven terms, immediately following the ſaid firſt term, which I had already inveſtigated. But I confeſſed at the ſame time that I was not able to ſee clearly and diſtinctly in my own mind, and much leſs to demonſtrate to others, that theſe things muſt of neceſſity be ſo. This therefore was a defect which ſtill remained in the Inveſtigation, and which I was ſorry to be forced to leave in it, and which the readers of it muſt, I thought, have wiſhed to ſee ſupplied, in order to make their knowledge of the ſaid very uſeful Differential Series quite compleat, and to enable them to uſe with perfect confidence as many of it's terms as they might at any time have occaſion for. Now this defect I was, about two years ago, enabled to ſupply by the aſſiſtance of my learned and ingenious friend Mr. John Hellins, Vicar of Potter's Pury in Northamptonſhire, who communicated to me another method of inveſtigating the values of the ſecond, third, fourth, fifth, and other following terms of the aſſumed ſeries P, $\frac{Qx}{1+x}$, $\frac{Rx^2}{1+x)^2}$, $\frac{Sx^3}{1+x)^3}$, $\frac{Tx^4}{1+x)^4}$, $\frac{Vx^5}{1+x)^5}$, $\frac{Wx^6}{1+x)^6}$, $\frac{Xx^7}{1+x)^7}$, &c *ad infinitum*, (which is equal to the original Series $a - bx + cx^2 - dx^3 + ex^4 - fx^5 + gx^6 - hx^7 +$ &c *ad infinitum*,) and of ſhewing that the ſame ſign — muſt be prefixed to all of them, which was different from that which had been ſet forth in the ſaid ſecond tract of Vol. 3 of this Collection, and which had the additional merit of being of ſuch a nature as to enable us to perceive that the Law of Continuation, or generation, of the terms of the ſaid aſſumed Series one from another, which is found to take place in thoſe terms of it that are actually inveſtigated, muſt likewiſe take place in all the following terms, which have not been inveſtigated, to whatever number the ſaid terms may be continued. And this method of inveſtigating the terms of the ſaid aſſumed Series, and the ſigns that were to be prefixed to them, which had been invented by Mr. Hellins, I afterwards ſet forth and explained at great length in the IXth tract of the fourth volume of the *Scriptores Logarithmici*, in pages 571, 572, 573, &c, - - - - 614, intitled " An Appendix to the Tract on " the Summation of infinite Serieſes contained in the 3d volume of this Col- " lection of tracts, called *Scriptores Logarithmici*, pages 255, 256, 257, &c, " - - - - 312, in which an Inveſtigation is given of the Differential Series *a*

" $-\frac{bx}{1+x}-\frac{D^{I}x^{2}}{\overline{1+x}^{2}}-\frac{D^{II}x^{3}}{\overline{1+x}^{3}}-\frac{D^{III}x^{4}}{\overline{1+x}^{4}}-\frac{D^{IV}x^{5}}{\overline{1+x}^{5}}-\frac{D^{V}x^{6}}{\overline{1+x}^{6}}-\frac{D^{VI}x^{7}}{\overline{1+x}^{7}}-$ &c *ad in-*
" *finitum*; containing an improvement on the ſaid Inveſtigation, that has lately
" been made by the Reverend Mr. John Hellins, B. D. Vicar of Potter's Pury
" in Northamptonſhire; in pages 571, 572, 573, &c, - - - 614."

This Inveſtigation of the ſaid Differential Series, communicated to me by Mr. Hellins, is very ſatisfactory, (when fully underſtood,) as to the proof that the Law of continuation of the terms which takes place in the terms that have been actually inveſtigated muſt likewiſe take place in all the following terms of the ſaid Series, to whatever number of terms the ſaid Series may be continued. But it is rather long, and requires a great deal of attention to underſtand it thoroughly. And therefore I have been very much pleas'd with the peruſal of another Inveſtigation of this Differential Series, which (though it bears a conſiderable reſemblance to that of Mr. Hellins, and is founded on the ſame principles,) ſeems to be ſomewhat ſhorter and eaſier than the other, and equally ſatisfactory with it as to the proof that the Law of continuation of the terms muſt extend to all the terms of the Series as well as to thoſe which have been actually inveſtigated. This Inveſtigation was communicated to me very lately, in December, 1801, by Mr. Thomas Manning, a young Gentleman of great ſkill in the Mathematicks, who was ſome years ago a ſtudent of Caius College, Cambridge, and who, in the year 1796, publiſhed a learned work in one volume octavo, intitled, "*An Introduction to Arithmetick and Algebra.*" This Inveſtigation I will therefore now proceed to deſcribe and explain in as full and clear a manner as I can.

Art. 2. It is evident that, if it be true that the Series $a - bx + cx^{2} - dx^{3} + ex^{4} - fx^{5} + gx^{6} - hx^{7} + ix^{8} - kx^{9} +$ &c *ad infinitum* is equal to the Differential Series $a - \frac{bx}{1+x} - \frac{D^{I}x^{2}}{\overline{1+x}^{2}} - \frac{D^{II}x^{3}}{\overline{1+x}^{3}} - \frac{D^{III}x^{4}}{\overline{1+x}^{4}} - \frac{D^{IV}x^{5}}{\overline{1+x}^{5}} - \frac{D^{V}x^{6}}{\overline{1+x}^{6}} - \frac{D^{VI}x^{7}}{\overline{1+x}^{7}} - \frac{D^{VII}x^{8}}{\overline{1+x}^{8}} - \frac{D^{VIII}x^{9}}{\overline{1+x}^{9}} -$ &c *ad infinitum*, it will alſo be true that the Series $bx - cx^{2} + dx^{3} - ex^{4} + fx^{5} - gx^{6} + hx^{7} - ix^{8} + kx^{9} -$ &c *ad infinitum* will be equal to the Differential Series $\frac{bx}{1+x} + \frac{D^{I}x^{2}}{\overline{1+x}^{2}} + \frac{D^{II}x^{3}}{\overline{1+x}^{3}} + \frac{D^{III}x^{4}}{\overline{1+x}^{4}} + \frac{D^{IV}x^{5}}{\overline{1+x}^{5}} + \frac{D^{V}x^{6}}{\overline{1+x}^{6}} + \frac{D^{VI}x^{7}}{\overline{1+x}^{7}} + \frac{D^{VII}x^{8}}{\overline{1+x}^{8}} + \frac{D^{VIII}x^{9}}{\overline{1+x}^{9}} +$ &c *ad infinitum*; and *vice verſâ*. And therefore, if we can prove the latter propoſition, the former propoſition, to wit, that the Series $a - bx + cx^{2} - dx^{3} + ex^{4} - fx^{5} + gx^{6} - hx^{7} +$ &c *ad infinitum* is equal to the Differential Series $a - \frac{bx}{1+x} - \frac{D^{I}x^{2}}{\overline{1+x}^{2}} - \frac{D^{II}x^{3}}{\overline{1+x}^{3}} - \frac{D^{III}x^{4}}{\overline{1+x}^{4}} - \frac{D^{IV}x^{5}}{\overline{1+x}^{5}} - \frac{D^{V}x^{6}}{\overline{1+x}^{6}} -$

$-\frac{D^{I}x^{7}}{1+x^{7}}-$ &c *ad infinitum* will alſo neceſſarily be true. Now the latter propoſition, to wit, that the Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 + kx^9 -$ &c *ad infinitum* is equal to the Differential Series $\frac{bx}{1+x} + \frac{D^{I}x^{2}}{1+x^{2}} + \frac{D^{II}x^{3}}{1+x^{3}} + \frac{D^{III}x^{4}}{1+x^{4}} + \frac{D^{IV}x^{5}}{1+x^{5}} + \frac{D^{V}x^{6}}{1+x^{6}} + \frac{D^{VI}x^{7}}{1+x^{7}} + \frac{D^{VII}x^{8}}{1+x^{8}} + \frac{D^{VIII}x^{9}}{1+x^{9}} +$ &c *ad infinitum*, is proved by Mr. Manning in the following manner.

Art. 3. Let the ſeveral Differences of the co-efficients, b, c, d, e, f, g, h, i, k, &c, of the powers of x in the Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 + kx^9 -$ &c *ad infinitum*, to wit, the Differences $b - c$, $c - d$, $d - e$, $e - f$, $f - g$, $g - h$, $h - i$, $i - k$, &c, be denoted by the letter D with the Roman numeral figure I placed above it and the Arabian numeral figures 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, denoting their ſeveral places in the ſaid Series of Differences of the firſt order, placed under the Roman numeral figure I that is placed above the ſaid letter; ſo that the ſaid Differences ſhall be denoted by the ſeveral expreſſions D^{I}_{1}, D^{I}_{2}, D^{I}_{3}, D^{I}_{4}, D^{I}_{5}, D^{I}_{6}, D^{I}_{7}, and D^{I}_{8}, &c, reſpectively.

And let the ſecond Differences of the ſaid co-efficients, or their Differences of the ſecond order, to wit, the trinomial quantities $b - 2c + d$, $c - 2d + e$, $d - 2e + f$, $e - 2f + g$, $f - 2g + h$, $g - 2h + i$, and $h - 2i + k$, &c, be denoted in like manner by the marked letters D^{II}_{1}, D^{II}_{2}, D^{II}_{3}, D^{II}_{4}, D^{II}_{5}, D^{II}_{6}, and D^{II}_{7}, &c, reſpectively. And let their third Differences, or Differences of the third order, to wit, the quadrinomial quantities $b - 3c + 3d - e$, $c - 3d + 3e - f$, $d - 3e + 3f - g$, $e - 3f + 3g - h$, $f - 3g + 3h - i$, and $g - 3h + 3i - k$, &c, be, in like manner, denoted by the marked letters D^{III}_{1}, D^{III}_{2}, D^{III}_{3}, D^{III}_{4}, D^{III}_{5}, and D^{III}_{6}, &c reſpectively. And let their fourth Differences, or Differences of the fourth order, to wit, the quinquinomial quantities $b - 4c + 6d - 4e + f$, $c - 4d + 6e - 4f + g$, $d - 4e + 6f - 4g + h$, $e - 4f + 6g - 4h + i$, and $f - 4g + 6h - 4i + k$, &c, be, in like manner, denoted by the marked letters D^{IV}_{1}, D^{IV}_{2}, D^{IV}_{3}, D^{IV}_{4}, and D^{IV}_{5}, &c, reſpectively. And let their Differences of the fifth, ſixth, ſeventh, eighth, and other following orders, be, in like manner, denoted by the Serieſes D^{V}_{1}, D^{V}_{2}, D^{V}_{3}, D^{V}_{4}, D^{V}_{5}, &c, and D^{VI}_{1}, D^{VI}_{2}, D^{VI}_{3}, D^{VI}_{4}, D^{VI}_{5}, &c, and D^{VII}_{1}, D^{VII}_{2}, D^{VII}_{3}, D^{VII}_{4}, D^{VII}_{5}, &c, and D^{VIII}_{1}, D^{VIII}_{2}, D^{VIII}_{3}, D^{VIII}_{4}, D^{VIII}_{5}, &c, *ad infinitum*.

This

This notation being premiſed, we may proceed to lay down and demonſtrate the following Lemma, or preliminary Propoſition, which is the foundation of Mr. Manning's inveſtigation of the Differential Series $\frac{bx}{1+x} + \frac{D^{I}x^{2}}{1+x)^{2}} + \frac{D^{II}x^{3}}{1+x)^{3}} + \frac{D^{III}x^{4}}{1+x)^{4}} + \frac{D^{IV}x^{5}}{1+x)^{5}} + \frac{D^{V}x^{6}}{1+x)^{6}} + \frac{D^{VI}x^{7}}{1+x)^{7}} +$ &c *ad infinitum.*

A LEMMA.

Art. 4. The ſeries $bx - cx^{2} + dx^{3} - ex^{4} + fx^{5} - gx^{6} + hx^{7} - ix^{8} + kx^{9}$ — &c is equal to the compound quantity $\frac{bx}{1+x}$ + the fraction $\frac{x}{1+x}$ × the Series $D^{I}_{1}x - D^{I}_{2}x^{2} + D^{I}_{3}x^{3} - D^{I}_{4}x^{4} + D^{I}_{5}x^{5} - D^{I}_{6}x^{6} + D^{I}_{7}x^{7} - D^{I}_{8} +$ &c *ad infinitum*, conſiſting of two parts, of which the firſt part is the fraction $\frac{bx}{1+x}$, or the firſt term bx of the firſt Series $bx - cx^{2} + dx^{3} - ex^{4} + fx^{5} - gx^{6} + hx^{7} - ix^{8} + kx^{9}$ — &c divided by $1 + x$, and the ſecond part is the product of the multiplication of the fraction $\frac{x}{1+x}$ into the Series $D^{I}_{1}x - D^{I}_{2}x^{2} + D^{I}_{3}x^{3} - D^{I}_{4}x^{4} + D^{I}_{5}x^{5} - D^{I}_{6}x^{6} + D^{I}_{7}x^{7} - D^{I}_{8}x^{8} +$ &c *ad infinitum*; in which Series the ſame powers of x occur as in the original Series $bx - cx^{2} + dx^{3} - ex^{4} + fx^{5} - gx^{6} + hx^{7} - ix^{8} +$ &c, and the terms have the ſame ſigns — and + prefixed to them reſpectively as in the ſaid original Series, but the co-efficients of the ſeveral powers of x, to wit, D^{I}_{1}, D^{I}_{2}, D^{I}_{3}, D^{I}_{4}, D^{I}_{5}, D^{I}_{6}, &c are equal to the Differences of the co-efficients $b, c, d, e, f, g, h, i,$ &c of the ſeveral powers of x in the ſaid original Series.

DEMONSTRATION.

Let the value of the whole original Series $bx - cx^{2} + dx^{3} - ex^{4} + fx^{5} - gx^{6} + hx^{7} - ix^{8} +$ &c be denoted by the capital letter S. And let both ſides of the equation $bx - cx^{2} + dx^{3} - ex^{4} + fx^{5} - gx^{6} + hx^{7} - ix^{8} +$ &c $=$ S be multiplied into the binomial quantity $1 + x$. And we ſhall then have the multinomial

tinomial quantity $\left\{\begin{matrix} bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 + \&c \\ + bx^2 - cx^3 + dx^4 - ex^5 + fx^6 - gx^7 + hx^8 - \&c \end{matrix}\right\} =$ $\overline{1+x} \times S.$

But the said multinomial quantity is $= bx + \overline{b-c} \times x^2 - \overline{c-d} \times x^3 + \overline{d-e} \times x^4 - \overline{e-f} \times x^5 + \overline{f-g} \times x^6 - \overline{g-h} \times x^7 + \overline{h-i} \times x^8 - \&c =$ (by the notation described in art. 3,) $bx + D_1^I x^2 - D_2^I \times x^3 + D_3^I \times x^4 - D_4^I x^5 + D_5^I x^6 - D_6^I x^7 + D_7^I x^8 - \&c.$

Therefore $bx + D_1^I x^2 - D_2^I x^3 + D_3^I x^4 - D_4^I x^5 + D_5^I x^6 - D_6^I x^7 + D_7^I x^8 - \&c$ will be $= \overline{1+x} \times S.$

Now let both sides of this equation be divided by $1 + x$. And we shall then have $\frac{bx}{1+x} + \frac{D_1^I x^2}{1+x} - \frac{D_2^I x^3}{1+x} + \frac{D_3^I x^4}{1+x} - \frac{D_4^I x^5}{1+x} + \frac{D_5^I x^6}{1+x} - \frac{D_6^I x^7}{1+x} + \frac{D_7^I x^8}{1+x} - \&c = S.$

But $\frac{bx}{1+x} + \frac{D_1^I x^2}{1+x} - \frac{D_2^I x^3}{1+x} + \frac{D_3^I x^4}{1+x} - \frac{D_4^I x^5}{1+x} + \frac{D_5^I x^6}{1+x} - \frac{D_6^I x^7}{1+x} + \frac{D_7^I x^8}{1+x} - \&c$ is $= \frac{bx}{1+x}$ + the fraction $\frac{x}{1+x} \times$ the Series $D_1^I x - D_2^I x^2 + D_3^I x^3 - D_4^I x^4 + D_5^I x^5 - D_6^I x^6 + D_7^I x^7 - \&c.$

Therefore $\frac{bx}{1+x}$ + the fraction $\frac{x}{1+x} \times$ the Series $D_1^I x - D_2^I x^2 + D_3^I x^3 - D_4^I x^4 + D_5^I x^5 - D_6^I x^6 + D_7^I x^7 - \&c$ will be $= S$, or $=$ the original Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 + \&c$ *ad infinitum*. Q. E. D.

From this Lemma Mr. Manning's Investigation of the Differential Series $\frac{bx}{1+x} + \frac{D^I x^2}{\overline{1+x}^2} + \frac{D^{II} x^3}{\overline{1+x}^3} + \frac{D^{III} x^4}{\overline{1+x}^4} + \frac{D^{IV} x^5}{\overline{1+x}^5} + \frac{D^V x^6}{\overline{1+x}^6} + \frac{D^{VI} x^7}{\overline{1+x}^7} + \&c$ *ad infinitum* (which is equal to the original Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 + kx^9 - \&c$ *ad infinitum*) may be derived in the following manner.

The

The Inveſtigation of the Differential Series $\frac{bx}{1+x} + \frac{D^{I}x^{2}}{1+x^{2}} + \frac{D^{II}x^{3}}{1+x^{3}} + \frac{D^{III}x^{4}}{1+x^{4}} + \frac{D^{IV}x^{5}}{1+x^{5}} + \frac{D^{V}x^{6}}{1+x^{6}} + \frac{D^{VI}x^{7}}{1+x^{7}}$ + &c ad infinitum, *which is equal to the ſlowly-converging Series* $bx - cx^{2} + dx^{3} - ex^{4} + fx^{5} - gx^{6} + hx^{7} - ix^{8}$ + &c ad infinitum.

Art. 5. The Series $D^{I}_{1}x - D^{I}_{2}x^{2} + D^{I}_{3}x^{3} - D^{I}_{4}x^{4} + D^{I}_{5}x^{5} - D^{I}_{6}x^{6} + D^{I}_{7}x^{7}$ — &c, which is obtained in the foregoing Lemma, and which, being multiplied into the fraction $\frac{x}{1+x}$, makes the ſecond part of the compound quantity which is there ſhewn to be equal to the original Series $bx - cx^{2} + dx^{3} - ex^{4} + fx^{5} - gx^{6} + hx^{7} - ix^{8}$ + &c, is a Series of exactly the ſame form with the ſaid original Series. For, in the 1ſt place, it involves in it's terms the ſame powers of x, to wit, x, x^{2}, x^{3}, x^{4}, x^{5}, x^{6}, x^{7}, &c in their natural order; and, ſecondly, it's ſecond, and third, and other following terms are marked with the ſigns — and + alternately, juſt as the ſecond, and third, and other following terms of the ſaid original Series are; and, thirdly, the co-efficients, D^{I}_{1}, D^{I}_{2}, D^{I}_{3}, D^{I}_{4}, D^{I}_{5}, D^{I}_{6}, D^{I}_{7}, &c, of the ſeveral powers of x in it's terms form a Series of decreaſing quantities, as well as the co-efficients, b, c, d, e, f, g, h, i, &c, of the ſame powers of x in the ſaid original Series. Therefore, ſince it has been ſhewn in the foregoing Lemma that the ſaid original Series $bx - cx^{2} + dx^{3} - ex^{4} + fx^{5} - gx^{6} + hx^{7} - ix^{8}$ + &c is equal to the compound quantity $\frac{bx}{1+x}$ + the fraction $\frac{x}{1+x}$ × the Series $D^{I}_{1}x - D^{I}_{2}x^{2} + D^{I}_{3}x^{3} - D^{I}_{4}x^{4} + D^{I}_{5}x^{5} - D^{I}_{6}x^{6} + D^{I}_{7}x^{7}$ — &c, in which Series the ſeveral co-efficients D^{I}_{1}, D^{I}_{2}, D^{I}_{3}, D^{I}_{4}, D^{I}_{5}, D^{I}_{6}, D^{I}_{7}, &c, of the powers of x are the Differences of the co-efficients, b, c, d, e, f, g, h, i, &c, of the powers of x in the original Series $bx - cx^{2} + dx^{3} - ex^{4} + fx^{5} - gx^{6} + hx^{7} - ix^{8}$ + &c, it follows that the ſaid Series $D^{I}_{1}x - D^{I}_{2}x^{2} + D^{I}_{3}x^{3} - D^{I}_{4}x^{4} + D^{I}_{5}x^{5} - D^{I}_{6}x^{6} + D^{I}_{7}x^{7}$ — &c muſt, in like manner, be equal to the compound quantity $\frac{D^{I}_{1}x}{1+x}$ + the fraction $\frac{x}{1+x}$ × the Series $\overline{D^{I}_{I} - D^{I}_{2}}$ × $\frac{x}{}$

$= \overbrace{D^{I}_{2} - D^{I}_{3}} \times x^{2} + \overbrace{D^{I}_{3} - D^{I}_{4}} \times x^{3} - \overbrace{D^{I}_{4} - D^{I}_{5}} \times x^{4} + \overbrace{D^{I}_{5} - D^{I}_{6}} \times x^{5} - \overbrace{D^{I}_{6} - D^{I}_{7}} \times x^{6} +$ &c, or (according to the notation described in art. 3,) to the compound quantity $\frac{D^{I}_{1} x}{1+x}$ + the fraction $\frac{x}{1+x} \times$ the Series $D^{II}_{1} x - D^{II}_{2} x^{2} + D^{II}_{3} x^{3} - D^{II}_{4} x^{4} + D^{II}_{5} x^{5} - D^{II}_{6} x^{6} +$ &c, in which Series the co-efficients, D^{II}_{1}, D^{II}_{2}, D^{II}_{3}, D^{II}_{4}, D^{II}_{5}, D^{II}_{6}, &c, of the several powers of x are the Differences of the co-efficients, D^{I}_{1}, D^{I}_{2}, D^{I}_{3}, D^{I}_{4}, D^{I}_{5}, D^{I}_{6}, D^{I}_{7}, &c, of the powers of x in the preceeding Series.

Therefore, if in the compound quantity $\frac{bx}{1+x}$ + the fraction $\frac{x}{1+x} \times$ the Series $D^{I}_{1} x - D^{I}_{2} x^{2} + D^{I}_{3} x^{3} - D^{I}_{4} x^{4} + D^{I}_{5} x^{5} - D^{I}_{6} x^{6} + D^{I}_{7} x^{7} -$ &c, (which has been shewn in the foregoing Lemma to be equal to the original Series $bx - cx^{2} + dx^{3} - ex^{4} + fx^{5} - gx^{6} + hx^{7} - ix^{8} +$ &c,) we substitute, instead of the Series $D^{I}_{1} x - D^{I}_{2} x^{2} + D^{I}_{3} x^{3} - D^{I}_{4} x^{4} + D^{I}_{5} x^{5} - D^{I}_{6} x^{6} +$ &c, it's value, the compound quantity $\frac{D^{I}_{1} x}{1+x}$ + the fraction $\frac{x}{1+x} \times$ the Series $D^{II}_{1} x - D^{II}_{2} x^{2} + D^{II}_{3} x^{3} - D^{II}_{4} x^{4} + D^{II}_{5} x^{5} - D^{II}_{6} x^{6} +$ &c, we shall have the said compound quantity $\frac{bx}{1+x}$ + the fraction $\frac{x}{1+x} \times$ the Series $D^{I}_{1} x - D^{I}_{2} x^{2} + D^{I}_{3} x^{3} - D^{I}_{4} x^{4} + D^{I}_{5} x^{5} - D^{I}_{6} x^{6} + D^{I}_{7} x^{7} -$ &c $= \frac{bx}{1+x}$ + the fraction $\frac{x}{1+x} \times \frac{D^{I}_{1} x}{1+x}$ + the fraction $\frac{x}{1+x} \times$ the fraction $\frac{x}{1+x} \times$ the Series $D^{II}_{1} x - D^{II}_{2} x^{2} + D^{II}_{3} x^{3} - D^{II}_{4} x^{4} + D^{II}_{5} x^{5} - D^{II}_{6} x^{6} +$ &c $= \frac{bx}{1+x} + \frac{D^{I}_{1} x^{2}}{(1+x)^{2}}$ + the fraction $\frac{x^{2}}{(1+x)^{2}} \times$ the Series $D^{II}_{1} x - D^{II}_{2} x^{2} + D^{II}_{3} x^{3} - D^{II}_{4} x^{4} + D^{II}_{5} x^{5} - D^{II}_{6} x^{6} +$ &c; and consequently this last compound quantity $\frac{bx}{1+x} + \frac{D^{I}_{1} x^{2}}{(1+x)^{2}}$ + the fraction $\frac{x^{2}}{(1+x)^{2}} \times$ the Se-

ries $D^{II}_1 x - D^{II}_2 x^2 + D^{II}_3 x^3 - D^{II}_4 x^4 + D^{II}_5 x^5 - D^{II}_6 x^6 +$ &c will be equal to the original Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 +$ &c *ad infinitum*; ſo that the two firſt terms of the compound quantity which is equal to the ſaid original Series will be $\frac{bx}{1+x} + \frac{D^{I}_1 x^2}{\overline{1+x}^2}$. Q. E. I.

Art. 6. And, in like manner, by transforming the laſt Series $D^{II}_1 x - D^{II}_2 x + D^{II}_3 x^3 - D^{II}_4 x^4 + D^{II}_5 x^5 - D^{II}_6 x^6 +$ &c, by means of the foregoing Lemma, into the compound quantity $\frac{D^{II}_1 x}{1+x}$ + the fraction $\frac{x}{1+x} \times$ the Series $D^{III}_1 x - D^{III}_2 x^2 + D^{III}_3 x^3 - D^{III}_4 x^4 + D^{III}_5 x^5 - D^{III}_6 x^6 +$ &c, we ſhall have the original Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 +$ &c $(= \frac{bx}{1+x} + \frac{D^{I}_1 x^2}{\overline{1+x}^2}$ + the fraction $\frac{x^2}{\overline{1+x}^2} \times$ the Series $D^{II}_1 x - D^{II}_2 x^2 + D^{II}_3 x^3 - D^{II}_4 x^4 + D^{II}_5 x^5 - D^{II}_6 x^6 +$ &c $= \frac{bx}{1+x} + \frac{D^{I}_1 x^2}{\overline{1+x}^2} +$ the fraction $\frac{x^2}{\overline{1+x}^2} \times \frac{D^{II}_1 x}{1+x}$ + the fraction $\frac{x^2}{\overline{1+x}^2} \times$ the fraction $\frac{x}{1+x} \times$ the Series $D^{III}_1 x - D^{III}_2 x^2 + D^{III}_3 x^3 - D^{III}_4 x^4 + D^{III}_5 x^5 - D^{III}_6 x^6 +$ &c$) = \frac{bx}{1+x} + \frac{D^{I}_1 x^2}{\overline{1+x}^2} + \frac{D^{II}_1 x^3}{\overline{1+x}^3}$ + the fraction $\frac{x^3}{\overline{1+x}^3} \times$ the Series $D^{III}_1 x - D^{III}_2 x^2 + D^{III}_3 x^3 - D^{III}_4 x^4 + D^{III}_5 x^5 - D^{III}_6 x^6 +$ &c; ſo that the three firſt terms of the compound quantity that is equal to the ſaid original Series will be $\frac{bx}{1+x} + \frac{D^{I}_1 x^2}{\overline{1+x}^2} + \frac{D^{II}_1 x^3}{\overline{1+x}^3}$. Q. E. I.

Art. 7. And, in like manner, by transforming the laſt Series $D^{III}_1 x - D^{III}_2 x^2 + D^{III}_3 x^3 - D^{III}_4 x^4 + D^{III}_5 x^5 - D^{III}_6 x^6 +$ &c, by means of the foregoing Lemma,

Lemma, into the compound quantity $\frac{D^{III}_{I}x}{1+x}$ + the fraction $\frac{x}{1+x}$ × the Series $D^{IV}_{1}x - D^{IV}_{2}x^2 + D^{IV}_{3}x^3 - D^{IV}_{4}x^4 + D^{IV}_{5}x^5 - D^{IV}_{6}x^6$ + &c, we ſhall have the original Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8$ + &c $(= \frac{bx}{1+x} + \frac{D^{I}_{I}x^2}{\overline{1+x}|^2} + \frac{D^{II}_{I}x^3}{\overline{1+x}|^3}$ + the fraction $\frac{x^3}{\overline{1+x}|^3}$ × the Series $D^{III}_{1}x - D^{III}_{2}x^2 + D^{III}_{3}x^3 - D^{III}_{4}x^4 + D^{III}_{5}x^5 - D^{III}_{6}x^6$ + &c $= \frac{bx}{1+x} + \frac{D^{I}_{I}x^2}{\overline{1+x}|^2} + \frac{D^{II}_{I}x^3}{\overline{1+x}|^3}$ + the fraction $\frac{x^3}{\overline{1+x}|^3} \times \frac{D^{III}_{I}x}{1+x}$ + the fraction $\frac{x^3}{\overline{1+x}|^3}$ × the fraction $\frac{x}{1+x}$ × the Series $D^{IV}_{1}x - D^{IV}_{2}x^2 + D^{IV}_{3}x^3 - D^{IV}_{4}x^4 + D^{IV}_{5}x^5 - D^{IV}_{6}x^6$ + &c) $= \frac{bx}{1+x} + \frac{D^{I}_{I}x^2}{\overline{1+x}|^2} + \frac{D^{II}_{I}x^3}{\overline{1+x}|^3} + \frac{D^{III}_{I}x^4}{\overline{1+x}|^4}$ + the fraction $\frac{x^4}{\overline{1+x}|^4}$ × the Series $D^{IV}_{1}x - D^{IV}_{2}x^2 + D^{IV}_{3}x^3 - D^{IV}_{4}x^4 + D^{IV}_{5}x^5 - D^{IV}_{6}x^6$ + &c; ſo that the four firſt terms of the compound quantity that is equal to the ſaid original Series will be $\frac{bx}{1+x} + \frac{D^{I}_{I}x^2}{\overline{1+x}|^2} + \frac{D^{II}_{I}x^3}{\overline{1+x}|^3} + \frac{D^{III}_{I}x^4}{\overline{1+x}|^4}$. Q. E. I.

Art. 8. And, in like manner, by transforming the laſt Series $D^{IV}_{1}x - D^{IV}_{2}x^2 + D^{IV}_{3}x^3 - D^{IV}_{4}x^4 + D^{IV}_{5}x^5 - D^{IV}_{6}x^6$ + &c, by means of the foregoing Lemma, into the compound quantity $\frac{D^{IV}_{I}x}{1+x}$ + the fraction $\frac{x}{1+x}$ × the Series $D^{V}_{1}x - D^{V}_{2}x^2 + D^{V}_{3}x^3 - D^{V}_{4}x^4 + D^{V}_{5}x^5 - D^{V}_{6}x^6$ + &c, we ſhall have the original Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8$ + &c $(= \frac{bx}{1+x} + \frac{D^{I}_{I}x^2}{\overline{1+x}|^2} + \frac{D^{II}_{I}x^3}{\overline{1+x}|^3} + \frac{D^{III}_{I}x^4}{\overline{1+x}|^4}$ + the fraction $\frac{x^4}{\overline{1+x}|^4}$ × the Series $D^{IV}_{1}x - D^{IV}_{2}x^2$ +

$+ D^{IV}_3 x^3 - D^{IV}_4 x^4 + D^{IV}_5 x^5 - D^{IV}_6 x^6 + \&c = \frac{bx}{1+x} + \frac{D^{I}_1 x^2}{\overline{1+x}|^2} + \frac{D^{II}_1 x^3}{\overline{1+x}|^3} + \frac{D^{III}_1 x^4}{\overline{1+x}|^4}$ + the fraction $\frac{x^4}{\overline{1+x}|^4} \times \frac{D^{IV}_1 x}{1+x}$ + the fraction $\frac{x^4}{\overline{1+x}|^4} \times$ the fraction $\frac{x}{1+x} \times$ the Series $D^{V}_1 x - D^{V}_2 x^2 + D^{V}_3 x^3 - D^{V}_4 x^4 + D^{V}_5 x^5 - D^{V}_6 x^6 + \&c =$

$\frac{bx}{1+x} + \frac{D^{I}_1 x^2}{\overline{1+x}|^2} + \frac{D^{II}_1 x^3}{\overline{1+x}|^3} + \frac{D^{III}_1 x^4}{\overline{1+x}|^4} + \frac{D^{IV}_1 x^5}{\overline{1+x}|^5}$ + the fraction $\frac{x^5}{\overline{1+x}|^5} \times$ the Series $D^{V}_1 x - D^{V}_2 x^2 + D^{V}_3 x^3 - D^{V}_4 x^4 + D^{V}_5 x^5 - D^{V}_6 x^6 + \&c$; so that the five firſt terms of the compound quantity that is equal to the ſaid original Series will be $\frac{bx}{1+x} + \frac{D^{I}_1 x^2}{\overline{1+x}|^2} + \frac{D^{II}_1 x^3}{\overline{1+x}|^3} + \frac{D^{III}_1 x^4}{\overline{1+x}|^4} + \frac{D^{IV}_1 x^5}{\overline{1+x}|^5}$. Q. E. I.

Art. 9. And by thus continually transforming the Series contained in the laſt compound quantity that has been ſhewn to be equal to the original Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 + kx^9 - \&c$, into the compound quantity conſiſting of two parts, of which the firſt is a fraction having the firſt term of the ſaid Series for it's numerator and $1 + x$ for it's denominator, and the ſecond part is the product of the multiplication of the fraction $\frac{x}{1+x}$ into the ſeries of terms involving the ſeveral ſucceſſive powers of x, and marked with the ſign — and the ſign + alternately, in which the co-efficients of the ſeveral powers of x in it's terms are the Differences of the co-efficients of the powers of x in the Series laſt obtained;—I ſay, by thus continually transforming the ſaid Series by means of the foregoing Lemma, it is eaſy to perceive that we ſhall obtain ſucceſſive compound quantities that will be equal to the ſaid original ſeries, and of which the ſixth, ſeventh, eighth, and ninth, and other following terms, to whatever number of terms we may chuſe to continue the ſaid compound quantities, will be $\frac{D^{V}_1 x^6}{\overline{1+x}|^6}, \frac{D^{VI}_1 x^7}{\overline{1+x}|^7}, \frac{D^{VII}_1 x^8}{\overline{1+x}|^8}, \frac{D^{VIII}_1 x^9}{\overline{1+x}|^9}$, &c, and that all theſe terms muſt be marked with the ſign +, or muſt be added to the five firſt terms $\frac{bx}{1+x} + \frac{D^{I}_1 x^2}{\overline{1+x}|^2} + \frac{D^{II}_1 x^3}{\overline{1+x}|^3} + \frac{D^{III}_1 x^4}{\overline{1+x}|^4} + \frac{D^{IV}_1 x^5}{\overline{1+x}|^5}$, and conſequently that the ſaid original Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 + kx^9 - \&c$ will be equal to

to the Differential Series $\frac{bx}{1+x} + \frac{D_1^{I}x^2}{\overline{1+x}|^2} + \frac{D_1^{II}x^3}{\overline{1+x}|^3} + \frac{D_1^{III}x^4}{\overline{1+x}|^4} + \frac{D_1^{IV}x^5}{\overline{1+x}|^5} + \frac{D_1^{V}x^6}{\overline{1+x}|^6}$
$+ \frac{D_1^{VI}x^7}{\overline{1+x}|^7} + \frac{D_1^{VII}x^8}{\overline{1+x}|^8} + \frac{D_1^{VIII}x^9}{\overline{1+x}|^9}$ + &c *ad infinitum*. Q. E. I.

Art. 10. That this law of continuation, or generation of the terms one from another, will take place in all the terms of this Differential Series, to whatever number of terms the ſaid Series may be continued, may be further made evident by denoting the order of Differences of the original co-efficients *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, *k*, &c which conſtitute the co-efficients of the powers of *x* in the Series laſt obtained, by the letter *n* inſtead of one of the Roman numeral figures I, II, III, IV, V, VI, VII, VIII, IX, X, &c, ſo that the ſaid Series ſhall be $D_1^n x - D_2^n x^2 + D_3^n x^3 - D_4^n x^4 + D_5^n x^5 - D_6^n x^6$ + &c. Then, by the foregoing Lemma, this Series will be $= \frac{D_1^n x}{1+x}$ + the fraction $\frac{x}{1+x} \times$ the Series $D_1^{n+1}x - D_2^{n+1}x^2 + D_3^{n+1}x^3 - D_4^{n+1}x^4 + D_5^{n+1}x^5 - D_6^{n+1}x^6$ + &c. Now the laſt compound quantity that will have been already ſhewn to be equal to the original Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 + kx^9$ — &c will be $\frac{bx}{1+x} + \frac{D_1^{I}x^2}{\overline{1+x}|^2} + \frac{D_1^{II}x^3}{\overline{1+x}|^3} + \frac{D_1^{III}x^4}{\overline{1+x}|^4} + \frac{D_1^{IV}x^5}{\overline{1+x}|^5}$ + &c continued to $\frac{D_1^{n-1}x^n}{\overline{1+x}|^n}$ + the fraction $\frac{x^n}{\overline{1+x}|^n} \times$ the Series $D_1^n x - D_2^n x^2 + D_3^n x^3 - D_4^n x^4 + D_5^n x^5 - D_6^n x^6 +$ &c. Therefore, if in this equation we ſubſtitute, inſtead of the Series $D_1^n x - D_2^n x^2 + D_3^n x^3 - D_4^n x^4 + D_5^n x^5 - D_6^n x^6$ + &c, it's equal, the compound quantity $\frac{D_1^n x}{1+x}$ + the fraction $\frac{x}{1+x} \times$ the Series $D_1^{n+1}x - D_2^{n+1}x^2 + D_3^{n+1}x^3 - D_4^{n+1}x^4 + D_5^{n+1}x^5 - D_6^{n+1}x^6$ + &c, we ſhall have the ſaid original Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 + kx^9$ — &c *ad infinitum* $(= \frac{bx}{1+x} + \frac{D_1^{I}x^2}{\overline{1+x}|^2} + \frac{D_1^{II}x^3}{\overline{1+x}|^3} + \frac{D_1^{III}x^4}{\overline{1+x}|^4} + \frac{D_1^{IV}x^5}{\overline{1+x}|^5}$ + &c continued

to

to $\frac{D^{n-1}_{1} x^n}{1+x)^n}$ + the fraction $\frac{x^n}{1+x)^n}$ × the Series $D^n_1 x - D^n_2 x^2 + D^n_3 x^3 - D^n_4 x^4 + D^n_5 x^5 - D^n_6 x^6 +$ &c $= \frac{bx}{1+x} + \frac{D^{I}_{1} x^2}{1+x)^2} + \frac{D^{II}_{1} x^3}{1+x)^3} + \frac{D^{III}_{1} x^4}{1+x)^4} + \frac{D^{IV}_{1} x^5}{1+x)^5} +$ &c continued to $\frac{D^{n-1}_{1} x^n}{1+x)^n}$ + the fraction $\frac{x^n}{1+x)^n} \times \frac{D^n_1 x}{1+x}$ + the fraction $\frac{x^n}{1+x)^n} \times$ the fraction $\frac{x}{1+x} \times$ the Series $D^{n+1}_{1} x - D^{n+1}_{2} x^2 + D^{n+1}_{3} x^3 - D^{n+1}_{4} x^4 + D^{n+1}_{5} x^5 - D^{n+1}_{6} x^6 +$ &c) $= \frac{bx}{1+x} + \frac{D^{I}_{1} x^2}{1+x)^2} + \frac{D^{II}_{1} x^3}{1+x)^3} + \frac{D^{III}_{1} x^4}{1+x)^4} + \frac{D^{IV}_{1} x^5}{1+x)^5} +$ &c continued to $\frac{D^{n-1}_{1} xn}{1+x)^n} + \frac{D^n_1 x^{n+1}}{1+x)^{n+1}}$ + the fraction $\frac{x^{n+1}}{1+x)^{n+1}} \times$ the Series $D^{n+1}_{1} x - D^{n+1}_{2} x^2 + D^{n+1}_{3} x^3 - D^{n+1}_{4} x^4 + D^{n+1}_{5} x^5 - D^{n+1}_{6} x^6 +$ &c; in which compound quantity (that is equal to the ſaid original Series,) the new term $\frac{D^n_1 x^{n+1}}{1+x)^{n+1}}$ is generated from the preceeding term $\frac{D^{n-1}_{1} x^n}{1+x)^n}$ in the ſame manner, or according to the ſame Law of Continuation, as every preceeding term, after the ſecond term $\frac{D^{I}_{1} x^2}{1+x)^2}$, is generated from the term that immediately preceeds it. And therefore the ſame Law of Continuation muſt take place in all the following new terms of the ſaid Differential Series that would be obtained by any further applications of the foregoing Lemma to the transformations of the Serieſes laſt-obtained, to whatever number the ſaid new applications of the ſaid Lemma may be continued. We may therefore now ſafely conclude that the ſaid original Series $bx - cx^2 + dx^3 - ex^4 + fx^5 - gx^6 + hx^7 - ix^8 + kx^9 -$ &c *ad infinitum* will be equal to the Differential Series

$$\frac{bx}{1+x} + \frac{D^{I}_{1} x^2}{1+x)^2} + \frac{D^{II}_{1} x^3}{1+x)^3} + \frac{D^{III}_{1} x^4}{1+x)^4} + \frac{D^{IV}_{1} x^5}{1+x)^5} + \frac{D^{V}_{1} x^6}{1+x)^6} + \frac{D^{VI}_{1} x^7}{1+x)^7} + \frac{D^{VII}_{1} x^8}{1+x)^8} + \frac{D^{VIII}_{1} x^9}{1+x)^9} +$$ &c *ad infinitum.* Q. E. D.

TRACTATUS

DE

LOGARITHMIS;

AUCTORE

ROBERTO SIMSON, M. D.,

A

IN ACADEMIA GLASGUENSI APUD SCOTOS NUPER MATHESEOS

A

PROFESSORE CELEBERRIMO.

EXCERPTUS

A

EX EDITIONE OPERUM EJUS RELIQUORUM IMPRESSA GLASGUÆ ANNO DOMINI 1776,

IMPENSIS *PHILIPPI* COMITIS *STANHOPE*, NUNC DEFUNCTI.

TRACTATUS

LOGARITHMIS;

AUCTORE

ROBERTO SIMSON, M. D.,

IN ACADEMIA GLASGUENSI APUD SCOTOS NUPER MATHESEOS

PROFESSORE ORDINARIO.

DE

LOGARITHMIS.

LEMMA I.

" QUÆVIS ratio AB ad BC componi poteſt ex rationibus inter ſe iiſdem, " quarum numerus major ſit numero quovis dato."

Inter AB et BC inveniatur media proportionalis BD; et rurſùs inter AB, BD, et inter BD, BC, inveniantur mediæ EB, FB, et ita deinceps; numerus rationum in quovis caſu duplus erit numeri earum in præcedente; atque ita major fieri poteſt quovis numero dato. Q. E. D.

A E D F C B

LEMMA II.

" Si fuerint duæ rationes, utraque majoris ad minorem; una autem earum " compoſita ſit ex rationibus quæ inter ſe ſunt eædem, et reliqua etiam ex iiſdem " componatur; erit ea quæ compoſita eſt ex majore numero rationum, major ra- " tione quæ componitur ex minore numero earundem. Et vice versâ, major ratio " componetur ex majore numero earundem rationum. Quæ quidèm manifeſta " ſunt.

" Contrarium autem eveniet ſi utraque ratio fuerit minoris ad majorem, ut " patet."

LEMMA III.

" Si una ratio major ſit alterâ, utraque verò ſit majoris ad minorem, et utraque " compoſita ſit ex eodem numero rationum inter ſe earundem ; erit una ex ra- " tionibus ex quibus major ratio compoſita eſt, major unâ ex iis ex quibus com- " poſita eſt reliqua ratio. Nam ſi eadem eſſet huic vel minor ipsâ, major ratio " eadem eſſet minori, vel ipsâ minor ; quod fieri non poteſt."

LEMMA IV.

" Sint duæ magnitudines, quarum AB minor et AC major ; poteſt ſeries " deinceps proportionalium, quarum prima eſt AB, et ſecunda AC, continuari " ita ut earum ultima major ſit quâvis magnitudine datâ AD."

Sumatur BE multiplex ipſius BC quæ major ſit BD ; et ipſis AB, AC, tertia proportionalis inveniatur AF ; et ipſis AC, AF inveniatur tertia proportionalis, et ita deinceps, donec tot ſumantur rationes, quarum prima eſt AB ad AC, quot ſunt partes in BE æquales ipſi BC ; ſítque AG ultima proportionalium. Et quoniam CF (exceſſus tertiæ ſuper ſecundam proportionalem) major eſt BC, (exceſſu ſecundæ ſuper primam,) et ita deinceps ; erit magnitudo BG, quæ continet tot harum proportionalium differentias quot BE continet partes æquales ipſi BC, major quàm BE, et multo quàm BD major. Q. E. D.

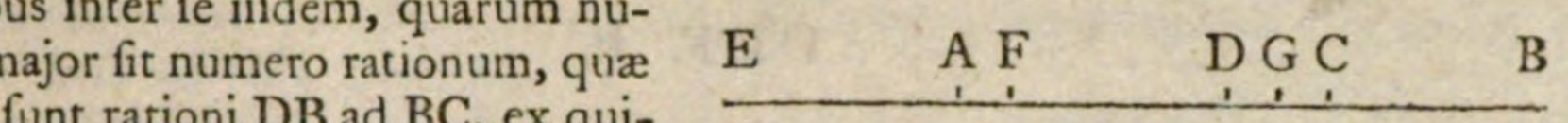
A BCF DEG

LEMMA V.

" Quævis ratio AB ad BC componi poteſt ex rationibus inter ſe iiſdem, qua- " rum una minor ſit quâvis ratione datâ DB ad BC."

Fiat ſeries deinceps proportionalium, quarum prima eſt BC, et ſecunda BD, et continuetur ita ut ultima earum BE major ſit quàm BA ; quod fieri poteſt per

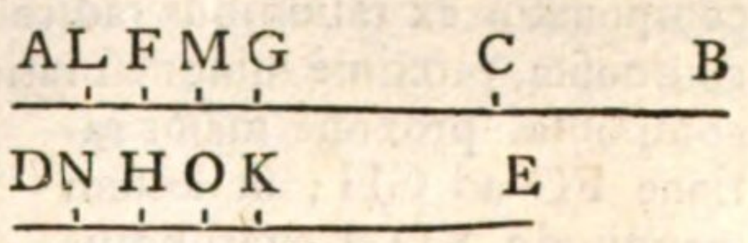
Lemma 4. et componatur (quod fieri poteſt per Lemma 1.) ratio AB ad BC ex rationibus inter ſe iiſdem, quarum numerus major ſit numero rationum, quæ eædem ſunt rationi DB ad BC, ex quibus compoſita eſt ratio EB ad BC. Sit ratio AB ad BF una ex iis ex quibus compoſita eſt ratio AB ad BC; ſit autem ratio AB ad BG ea, ex hiſce compoſita, quarum numerus idem eſt cum numero rationum ex quibus compoſita eſt ratio EB ad BC.



Quoniam igitùr ratio AB ad BC compoſita eſt ex majori numero earundem rationum quàm ratio AB ad BG; erit ratio AB ad BC major ratione AB ad BG, per Lemma 2; eſt igitùr BC minor quàm BG, [10. 5.] Et eſt EB major quàm AB. Igitùr ratio EB ad BC major eſt ratione AB ad BG. Et compoſitæ ſunt hæ rationes ex eodem numero rationum; quarè [per Lem. 3.] ratio DB ad BC, una ex iis ex quibus compoſita eſt ratio EB ad BC, major eſt ratione AB ad BF, quæ una eſt ex iis ex quibus compoſita eſt ratio AB ad BG. Ratio autem AB ad BC compoſita eſt ex hiſce, hoc eſt, ex rationibus quæ minores ſunt ratione datâ DB ad BC. Q. E. D.

DEFINITIO I.

Sit data ratio AB ad BC, et data magnitudo DE; componatur autem ratio AB ad BC ex quocunque numero rationum quæ eædem ſunt inter ſe; non autem ſint rationes æqualis ad æqualem; ex. gr. ex AB ad BF, FB ad BG, &c. Et ſecetur DE in tot partes æquales DH, HK, &c. quot ſunt rationes ex quibus compoſita eſt ratio AB ad BC. Vel, ſi ratio AB ad BC componatur ex rationibus quarum numerus major vel minor eſt quàm in præcedente exemplo, (ut ex rationibus AB ad BL, LB ad BF, &c.;) ſecetur ſemper DE in tot partes æquales DN, NH, HO, OK, &c. quot ſunt rationes [ex quibus compoſita eſt ratio AB ad BC.] Hoc facto, ratio AB ad BF, vel ratio AB ad BL, &c. una ſcil. ex iis ex quibus, in quovis caſu, componitur ratio AB ad BC, dicatur *ratio radicalis*; et pars DH, vel DN, &c. in eodem caſu, dicatur *pars radicalis*. Ratio autem data AB ad BC dicatur *ratio aſſumpta*, et data magnitudo DE dicatur *magnitudo aſſumpta*.

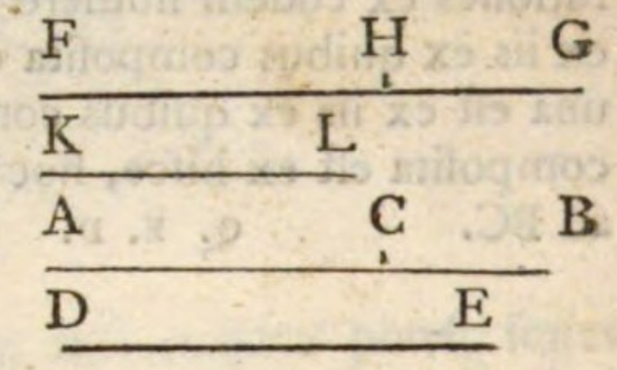


DEF. II.

Quid sit Logarithmus rationis duarum quantitatum inæqualium inter se.

Si ratio aliqua FG ad GH componatur ex rationibus radicalibus, hoc est, ex iisdem ex quibus composita est ratio AB ad BC; sítque magnitudo KL, quæ continet tot partes æquales iis in quas divisa est magnitudo DE, quot sunt rationes ex quibus composita est ratio FG ad GH; magnitudo KL dicatur *Logarithmus rationis FG ad GH*; quoniam, scilicet, KL continet tot partes radicales quot sunt rationes radicales ex quibus composita est ratio FG ad GH.

Quoniam verò ratio HG ad GF, quæ reciproca est rationis FG ad GH, componitur ex eodem numero rationum ex quibus ratio FG ad GH composita est, idem erit utriusque rationis Logarithmus. Ut verò cognoscatur quandò Logarithmus KL est rationis minoris ad majorem, convenit notam aliquam ei præfigere, ut posteà ostendetur.

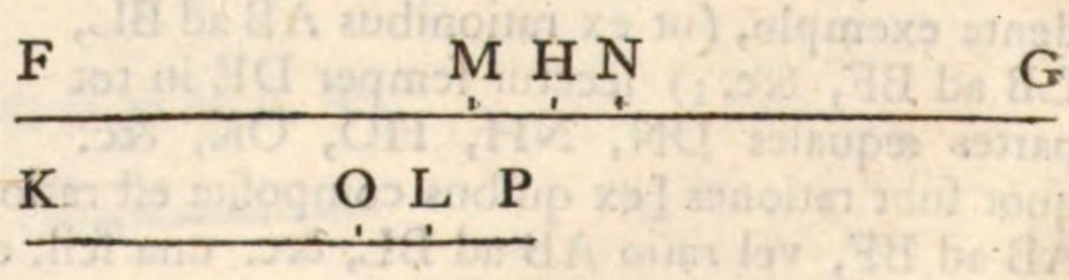


Cor. Et manifestum est DF Logarithmum esse rationis AB ad BC.

DEF. III.

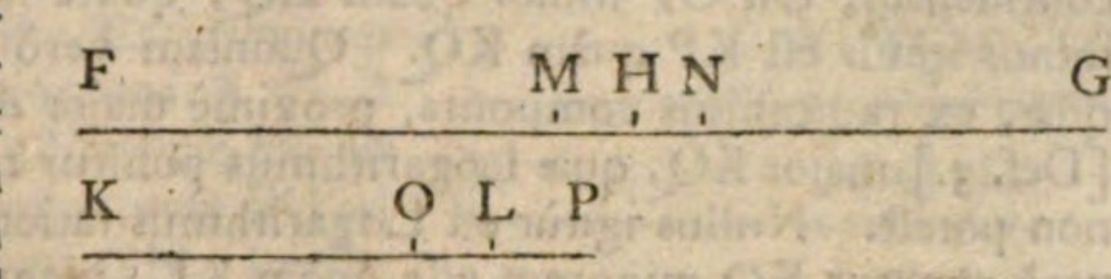
Quid sit Logarithmus rationis duarum quantitatum, quandò hæc ratio est rationi primò assumptæ inter lineam AB et lineam BC incommensurabilis.

Si autem ratio aliqua FG ad GH, quæ primò sit majoris ad minorem, non componatur ex rationibus radicalibus, sit ratio FG ad GM ea quæ, ex iisdem composita, proximè minor est ratione FG ad GH, et sit ratio FG ad GN, ex iisdem composita, proximè major ratione FG ad GH; sit autem magnitudo KO Logarithmus rationis FG ad GM, et KP Logarithmus rationis FG ad GN; magnitudo KL, quæ semper major est quàm KO et minor quàm KP, quicunque fuerit numerus rationum radicalium ex quibus assumpta ratio AB ad BC composita est, dicatur *Logarithmus rationis FG ad GH.*



Si

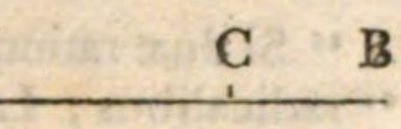
Si verò ſit ratio HG ad GF minoris ad majorem, ſumatur ratio MG ad GF, quæ, ex rationibus radicalibus compoſita, proximè major eſt ratione HG ad GF; et ratio NG ad GF, quæ, ex iiſdem compoſita, proximè minor eſt eâdem ratione; et ſint KO, KP, Logarithmi harum rationum, qui iidem ſunt cum Logarithmis rationum FG ad GM, et FG ad GN; et magnitudo KL, quæ ſemper major eſt KO et minor quàm KP, dicatur *Logarithmus rationis HG ad GF*, qui, ut patet, idem eſt cum Logarithmo rationis FG ad GH.



De Logarithmo rationis quam habet quantitas aliqua ad aliam quantitatem ſe ipſâ majorem, quandò iſta ratio eſt rationi aſſumptæ inter lineas AB et BC incommenſurabilis.

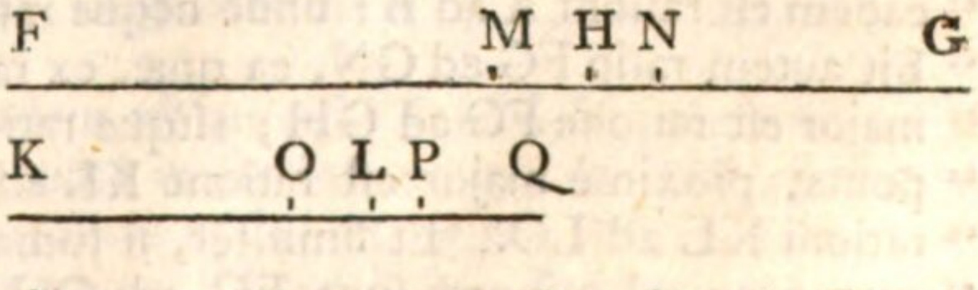
Cor. 1. Hinc Logarithmus rationis æqualis ad æqualem eſt nullius magnitudinis, ſive eſt nihil.

Sit enim ratio AB ad BA, æqualis ad æqualem; et, ſi ejus Logarithmus habeat ullam magnitudinem, ſit ea magnitudo DF, et componatur aſſumpta ratio AB ad BC ex rationibus, quarum numerus major ſit numero partium ipſi DF æqualium quæ in aſſumptâ magnitudine DE continentur. Igitùr, quoniam numerus partium radicalium major eſt numero partium in DE æqualium ipſi DF, una ex partibus radicalibus minor erit quàm DF. Sit ratio AB ad BG prima rationum radicalium, quæ propterea eſt prima quæ major eſt ratione AB ad BA, et ſit DH prima partium radicalium; erit igitùr DH minor DF. Ex Definitione verò 3. DH, Logarithmus ſc. rationis AB ad BG, major eſt quàm Logarithmus rationis AB ad BA, hoc eſt, quàm DF; quod fieri non poteſt. Logarithmus igitùr rationis AB ad BA, æqualis ſc. ad æqualem, nihil eſt.



Cor. 2. Ejuſdem rationis unicus eſt Logarithmus.

Si ratio propoſita componatur ex rationibus radicalibus, hoc manifeſtum eſt. Si autem ratio aliqua FG ad GH non ex iiſdem componatur, ſit KL ejus Logarithmus ſecundùm Def. 3. et, ſi alius poteſt eſſe ejuſdem Logarithmus, ſit is KQ, quæ primò major ſit quàm KL; et componatur aſſumpta ratio AB ad BC ex rationibus, quarum numerus major eſt numero partium ipſi LQ æqualium, quæ in aſſumptâ magnitudine DE continentur; ſit autem ratio FG ad GM ea quæ, ex rationibus illis compoſita, proximè minor eſt ratione FG ad GH, et ſit ratio FG ad GN, ex iiſdem compoſita, ea quæ proximè major eſt ratione FG ad GH; ſintque KO, KP harum rationum Logarithmi. Et, quoniam ratio FG ad GN componitur ex rationibus radicalibus quarum numerus unitate major eſt numero earundem ex quibus compoſita eſt ratio FG ad GM, numerus partium radicalium in KP unitate major erit numero earundem in KO; eſt igitùr OP una harum partium. Et, quoniam numerus rationum ex quibus



quibus ratio AB ad BC compoſita eſt, hoc eſt, numerus partium radicalium in DE, major, ex conſtructione, eſt numero partium ipſi LQ æqualium quæ in DE continentur, erit OP minor quàm LQ; quarè multò minor eſt LP ipſâ LQ; minor igitùr eſt KP quàm KQ. Quoniam verò KP Logarithmus eſt rationis quæ, ex radicalibus compoſita, proximè major eſt ratione FG ad GH, erit KP [Def. 3.] major KQ, quæ Logarithmus ponitur rationis FG ad GH; quod fieri non poteſt. Nullus igitùr eſt Logarithmus rationis FG ad GH prætèr KL. Si verò ponatur KQ minorem eſſe quàm KL, idem eodem modo oſtendetur.

PROP. I.

" Si duæ rationes ſint inter ſe eædem, et una earum componatur ex rationibus " radicalibus; Logarithmi harum rationum erunt inter ſe æquales."

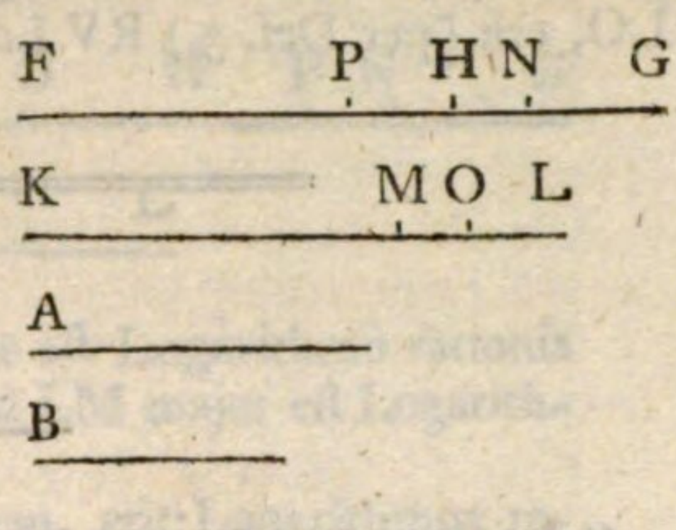
Quoniam enim rationes ſunt inter ſe eædem, compoſitæ erunt ex eodem numero earundem rationum, ut patet; quarè, per Def. 2. earum Logarithmi continebunt eundem numerum partium radicalium; et proptereà erunt inter ſe æquales. Q. E. D.

PROP. II.

" Sit ratio FG ad GH eadem rationi KL ad LM, sítque alia ratio A ad B; " non autem compoſita ſit ratio FG ad GH ex rationibus quarum unaquæque " eadem eſt rationi A ad B; unde neque ratio KL ad LM ex iiſdem componetur. " Sit autem ratio FG ad GN, ea quæ, ex rationibus A ad B compoſita, proximè " major eſt ratione FG ad GH; sítque ratio KL ad LO ea quæ, ex iiſdem com- " poſita, proximè major eſt ratione KL ad LM; erit ratio FG ad GN eadem " rationi KL ad LO. Et ſimilitèr, ſi ſumantur rationes ex prædictis compoſitæ " quæ proximè minores ſunt FG ad GH, et KL ad LM, erunt rationes illæ " inter ſe eædem."

Si

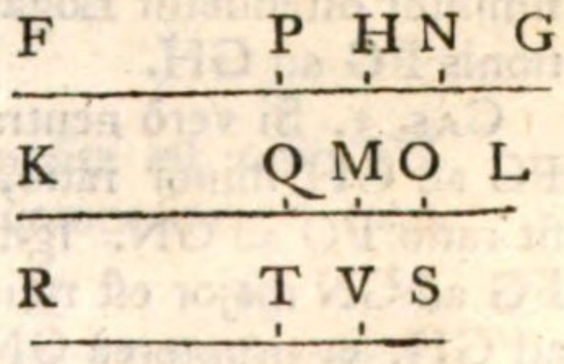
Si enim rationes FG ad GN, KL ad LO, non ſunt inter ſe eædem, una earum erit major alterâ. Sit ratio FG ad GN major; et ut KL ad LO, ita ſit FG ad GP. Igitùr [Cor. 13. 5.] ratio FG ad GN major eſt ratione FG ad GP. Et, quoniam ratio FG ad GN ea eſt, ex rationibus A ad B compoſita, quæ proximè major eſt ratione FG ad GH, erit ratio FG ad GH major ratione FG ad GP, quæ minor eſt ratione FG ad GN. Igitùr, quoniam ratio KL ad LM eadem eſt rationi FG ad GH, quæ major eſt ratione FG ad GP, erit [13. 5.] ratio KL ad LM major (ratione FG ad GP, hoc eſt) ratione KL ad LO. Sed et minor; quod fieri non poteſt. Igitùr ratio FG ad GN eadem eſt rationi KL ad LO. Similitèr altera pars Propoſitionis oſtendetur.



PROP. III.

" Si duæ rationes FG ad GH, et KL ad LM, utraque majoris ad minorem, " ſint inter ſe eædem, non autem compoſitæ ſint ex rationibus radicalibus ex " quibus, ſcilicet, compoſita eſt ratio aſſumpta AB ad BC; erit Logarithmus ra- " tionis FG ad GH idem cum Logarithmo rationis KL ad LM."

Sit enim ratio FG ad GN ea quæ, ex radicalibus compoſita, proximè major eſt ratione FG ad GH; ſítque ratio KL ad LO ea quæ, ex iiſdem compoſita, proximè major eſt ratione KL ad LM; et ſit ratio FG ad GP, proximè minor ratione FG ad GH; et ratio KL ad LQ ea quæ, ex iiſdem compoſita, proximè minor eſt ratione KL ad LM. Sit autem RS Logarithmus rationis FG ad GN, et RT Logarithmus rationis FG ad GP; ſit verò RV Logarithmus rationis FG ad GH, qui ſcilicet, per Def. 3. ſemper major eſt quàm RT, et minor quàm RS. Erit etiam RV Logarithmus rationis KL ad LM.



Quoniam enim ratio FG ad GH eadem eſt rationi KL ad LM, et eſt ratio FG ad GN, ex radicalibus compoſita, proximè major ratione FG ad GH, ratio verò KL ad LO, ex iiſdem compoſita, proximè major eſt ratione KL ad LM; erit, per Prop. 2. ratio FG ad GN, eadem rationi KL ad LO. Igitùr, per Prop. 1. Logarithmi harum rationum ſunt inter ſe æquales. Eſt autem RS Logarithmus rationis FG ad GN; quarè et RS Logarithmus eſt rationis KL ad LO. Similitèr oſtendetur

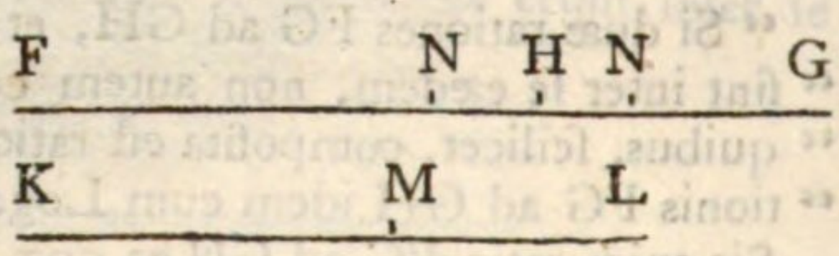
oſtendetur RT, Logarithmum, ſcilicet, rationis FG ad GP, eſſe etiam Logarithmum rationis KL ad LQ. Quoniam igitùr RV ſemper major eſt quàm RT Logarithmus rationis KL ad LQ, et minor quàm RS Logarithmus rationis KL ad LO, erit [per Def. 3.) RV Logarithmus rationis KL ad LM. Q. E. D.

PROP. IV.

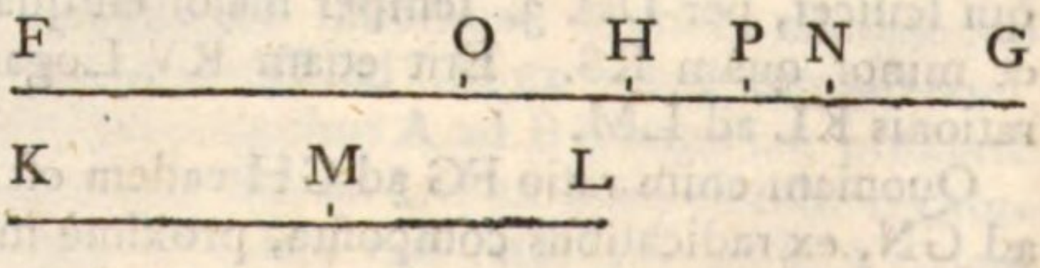
"Logarithmus majoris rationis major eſt logarithmo minoris; utraque autem "ratio ſit majoris ad minorem."

Cas. 1. Si utraque ratio compoſita ſit ex rationibus radicalibus, numerus rationum ex quibus compoſita eſt major ratio, major eſt [per Lem. 2.] numero earundem, ex quibus minor ratio compoſita eſt; et propterea Logarithmus majoris rationis major eſt Logarithmo minoris, ut patet ex Def. 2.

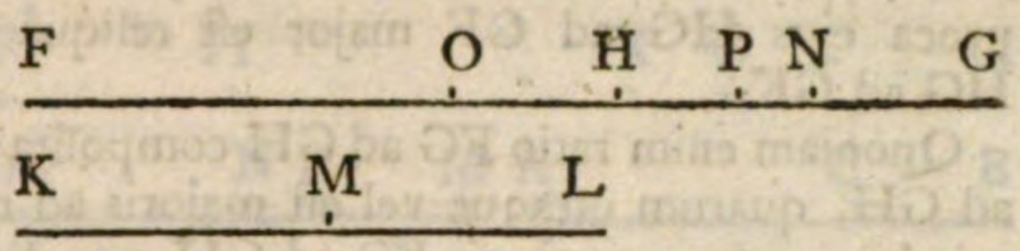
Cas. 2. Si una ratio, viz. ratio KL ad LM, compoſita ſit ex rationibus radicalibus, reliqua autem ratio FG ad GH ex iiſdem non ſit compoſita; [ſit verò ratio KL ad LM major ratione FG ad GH.] Sit ratio FG ad GN eadem rationi KL ad LM; et primò, ſit ratio KL ad LM, hoc eſt ratio FG ad GN, quæ ex rationibus radicalibus eſt compoſita, vel ea quæ proximè major eſt ratione FG ad GH, vel major eâ quæ proximè major eſt; et in primo Caſu, per Def. 3. in altero, per Caſ. 1. et Def. 3. Logarithmus rationis FG ad GN, hoc eſt rationis KL ad LM [per Prop. 1.] major erit Logarithmo rationis FG ad GH. Si verò ratio KL ad LM minor ſit ratione FG ad GH, ſimilitèr oſtendetur Logarithmus rationis KL ad LM eſſe minor Logarithmo rationis FG ad GH.



Cas. 3. Si verò neutra rationum ſit compoſita ex rationibus radicalibus, ſit FG ad GH minor ratio, et KL ad LM major; et rationi KL ad LM eadem ſit ratio FG ad GN. Igitùr ratio FG ad GN major eſt ratione FG ad GH, et propterea GN minor eſt quàm GH. [10. 5.] Componatur verò ratio aſſumpta AB ad BC ex rationibus inter ſe iiſdem, quarum una quævis minor eſt ratione HG ad GN, quod fieri poteſt [per Lem. 5.]; et ſit ratio FG ad GO ea quæ, ex hiſce compoſita, proximè minor eſt ratione FG ad GH. Igitùr ratio OG ad GP una eſt ex hiſce rationibus, quæ



quæ propterea minor eſt ratione HG ad GN; quarè multo magis ratio HG ad GP minor eſt ratione HG ad GN, et erit GP major GN. [10. 5.] Ratio igitur FG ad GN major eſt ratione FG ad GP. Et, quoniam ratio FG ad GP compoſita eſt ex rationibus radicalibus, Logarithmus rationis FG ad GN major erit eo rationis FG ad GP [per Caſ. 2.] et per eundem, Logarithmus rationis FG ad GP major eſt Logarithmo rationis FG ad GH. Multò igitur Logarithmus rationis KL ad LM major eſt Logarithmo rationis FG ad GH.



Cor. 1. Et ſi utraque ratio fuerit minoris ad majorem, erit Logarithmus rationis, cujus reciproca major eſt alterius reciprocâ, major Logarithmo alterius rationis.

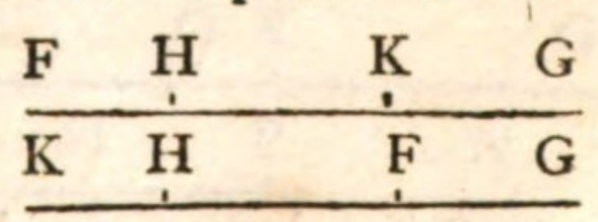
Cor. 2. Et ſi una ratio fuerit majoris ad minorem, et reliqua minoris ad majorem, erit Logarithmus rationis quæ, vel cujus reciproca, ſi fuerit minoris ad majorem, major eſt reliquâ vel ejuſdem reciprocâ, ſi fuerit reliqua minoris ad majorem, major Logarithmo reliquæ; idem enim eſt Logarithmus rationis et ejus reciprocæ, per Def. 3.

Cor. 3. Hinc Logarithmus major eſt majoris, et minor eſt Logarithmus minoris rationis, ſi, ſcilicet, utraque ratio ſit majoris ad minorem; vel, in aliis caſibus, ſumendo rationes quæ ſunt reciprocæ rationum minoris ad majorem.

PROP. V.

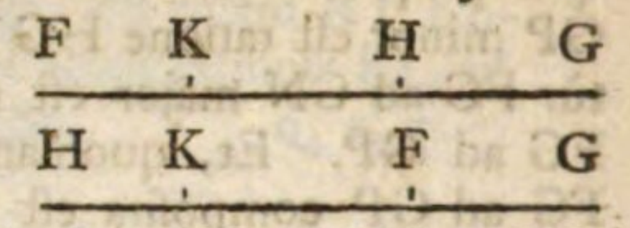
"Si utraque duarum rationum compoſita ſit ex rationibus radicalibus, utra-
"que autem ſit vel majoris ad minorem, vel minoris ad majorem; Logarithmus
"rationis ex duabus hiſce compoſitæ æqualis eſt ſummæ Logarithmorum ra-
"tionum ex quibus eſt compoſita."

Sint rationes FG ad GH, et HG ad GK ex quibus compoſita eſt ratio FG ad GK. Numerus igitur rationum radicalium, ex quibus compoſita eſt ratio FG ad GK, æqualis eſt numero earundem ex quibus componitur ratio FG ad GH unà cum numero earundem ex quibus componitur ratio HG ad GK; igitur, per Def. 2. Logarithmus rationis FG ad GK æqualis eſt ſummæ Logarithmorum rationum FG ad GH, et HG ad GK. Q. E. D.



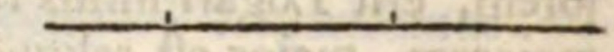
Cor. Si verò una ex rationibus fuerit majoris ad minorem, reliqua verò minoris ad majorem, erit Logarithmus rationis ex hiſce compoſitæ æqualis exceſſui [ſeu differentiæ] Logarithmorum rationum ex quibus eſt compoſita.

Cas. 1. Quando ratio FG ad GH, majoris, ſcilicet, ad minorem, major eſt reciprocâ rationis reliquæ HG ad GK; vel quando ratio FG ad GH eſt minoris ad majorem, et reciproca ejus HG ad GF major eſt reliquâ ratione HG ad GK.



Quoniam enim ratio FG ad GH compoſita eſt ex rationibus FG ad GK, et KG ad GH, quarum utraque vel eſt majoris ad minorem, vel minoris ad majorem; erit Logarithmus rationis FG ad GH æqualis ſummæ Logarithmorum rationum FG ad GK, et KG ad GH, per hanc, ſcilicet, Propoſitionem. Igitùr exceſſus Logarithmi rationis FG ad GH ſupra Logarithmum rationis KG ad GH, qui idem eſt cum Logarithmo rationis HG ad GK [Def. 2.] æqualis eſt Logarithmo rationis FG ad GK.

Cas. 2. Quando ratio FG ad GK compoſita eſt ex ratione FG ad GH, minoris ad majorem, et ex ratione HG ad GK, majoris ad minorem, quæ major eſt reciprocâ primæ, major, ſcilicet, ipſâ HG ad GF; idem ſequetur. Nam, quoniam ratio HG ad GK compoſita eſt ex ratione HG ad GF, et FG ad GK, quarum utraque eſt majoris ad minorem, Logarithmus rationis HG ad GK æqualis erit ſummæ Logarithmorum rationum HG ad GF, et FG ad GK. Igitùr exceſſus Logarithmi rationis HG ad GK ſuprà Logarithmum rationis HG ad GF, qui idem eſt cum Logarithmo rationis FG ad GH, æqualis eſt Logarithmo rationis FG ad GK.



Cas. 3. Quando ratio FG ad GH majoris ad minorem, minor eſt reciprocâ reliquæ rationis HG ad GK.

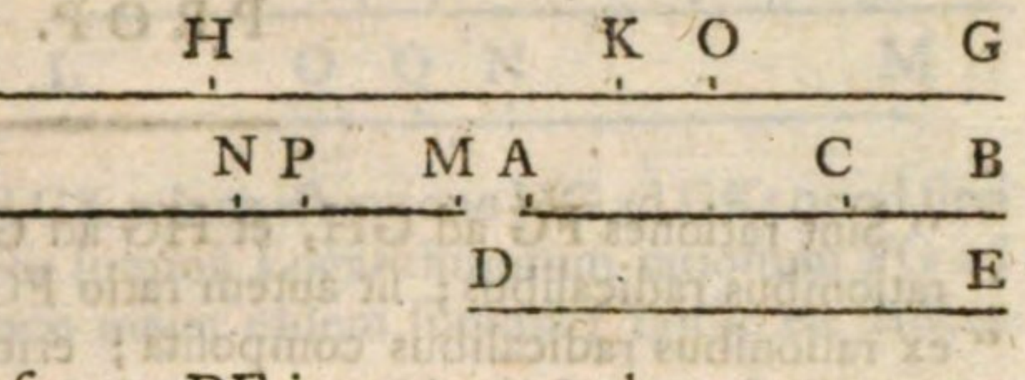
Quoniam utraque ratio HG ad GF, et FG ad GK eſt minoris ad majorem, erit [per Prop. 5.] Logarithmus rationis HG ad GK æqualis ſummæ Logarithmorum rationum HG ad GF, et FG ad GK. Igitùr exceſſus Logarithmi rationis HG ad GK ſuprà Logarithmum rationis HG ad GF, qui idem eſt cum Logarithmo rationis FG ad GH, æqualis eſt Logarithmo rationis FG ad GK.

PROP. VI.

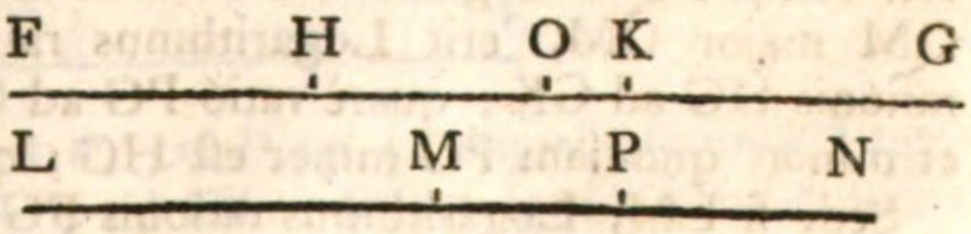
" Si ſint duæ rationes FG ad GH, et GH ad GK, utraque majoris ad mino-
" rem, quarum altera FG ad GH compoſita eſt ex rationibus radicalibus, reliqua
" verò HG ad GK ex iis non eſt compoſita; erit Logarithmus rationis FG ad GK
" quæ ex duabus illis eſt compoſita, æqualis ſummæ Logarithmorum rationum
" FG ad GH, et HG ad GK."

Si

Si enim magnitudines hæ non ſint æquales, altera earum major erit; et primò, ſit LM, ſumma, ſcilicet, Logarithmorum rationum FG ad GH, et HG ad GK, major LN Logarithmo rationis FG ad GK, ſítque NM ipſarum exceſſus. Et componatur ratio aſſumpta AB ad BC ex rationibus radicalibus, quarum numerus major ſit numero partium ipſi NM æqualium, quæ in datâ magnitudine DE continentur; et ſecetur DE in partes æquales, quarum numerus æqualis eſt numero rationum ex quibus compoſita eſt ratio AB ad BC; igitùr una ex híſce partibus minor eſt ipſâ NM. Sit ratio FG ad GO ea, quæ, ex rationibus radicalibus compoſita, proximè major eſt ratione FG ad GK; et ſit LP Logarithmus rationis FG ad GO. Et quoniam [Def. 3.] LN, Logarithmus rationis FG ad GK, cadit inter Logarithmum LP rationis FG ad GO, quæ proximè major eſt ratione FG ad GK, et Logarithmum rationis quæ, ex radicalibus compoſita, proximè minor eſt ratione FG ad GK, erit NP minor exceſſu horum Logarithmorum qui eſt una partium radicalium, et propterea minor eſt NM; multò igitùr NP minor eſt quàm NM. Igitùr LP, Logarithmus, ſcilicet, rationis FG ad GO, minor eſt quàm LM, ſumma, ſcilicet, Logarithmorum rationum FG ad GH, et HG ad GK. Quoniam verò ratio FG ad GO, quæ eſt compoſita ex rationibus radicalibus, eadem eſt rationi compoſitæ ex rationibus FG ad GH et HG ad GO, quarum ratio FG ad GH compoſita eſt ex rationibus radicalibus, reliqua ratio HG ad GO compoſita erit ex iiſdem. Igitùr, per Prop. 5. hujus, Logarithmus LP rationis FG ad GO eſt æqualis ſummæ Logarithmorum rationum FG ad GH, et HG ad GO. Eſt autem LM ſumma Logarithmorum rationum FG ad GH, et HG ad GK. Et quoniam LP minor eſt LM, erit prima ſumma minor reliquâ, et, ablato communi Logarithmo rationis FG ad GH, erit reliquus Logarithmus rationis HG ad GO minor reliquo Logarithmo rationis HG ad GK. Igitùr [Cor. 3. Prop. 4.] ratio HG ad GO minor eſt ratione HG ad GK. Sed et major; quod fieri non poteſt. Non igitùr eſt LM major LN.



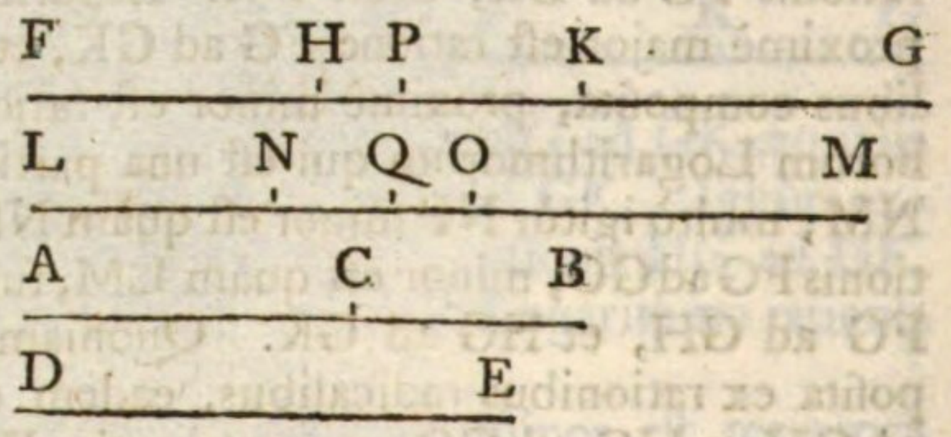
Sed ſi dicatur LM minor quàm LN, ſumptâ ratione FG ad GO quæ, ex radicalibus compoſita, proximè minor eſt ratione FG ad GK; ſimilitèr oſtendetur non poſſe LM minorem eſſe quàm LN. Æqualis igitùr eſt LN, Logarithmus rationis FG ad GK, ipſi LM, ſummæ, ſcilicet, Logarithmorum rationum FG ad GH, et HK ad GK. Q. E. D.



PROP. VII.

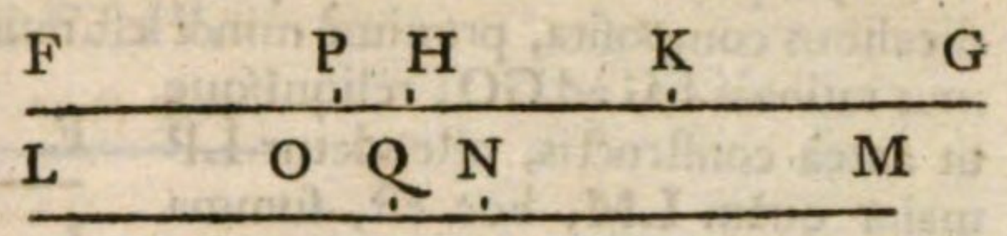
" Sint rationes FG ad GH, et HG ad GK, quarum neutra compoſita eſt ex " rationibus radicalibus; ſit autem ratio FG ad GK, quæ ex illis compoſita eſt, " ex rationibus radicalibus compoſita; erit Logarithmus rationis FG ad GK " æqualis ſummæ Logarithmorum rationum FG ad GH, et HG ad GK."

Si enim non ſit æqualis ſummæ huic, erit vel major, vel minor. Primò, ſit Logarithmus LM rationis FG ad GK major ſummâ Logarithmorum rationum FG ad GH, et HG ad GK; et ſit LN Logarithmus rationis FG ad GH. Igitùr NM major eſt Logarithmo rationis HG ad GK. Sit hujus rationis Logarithmus OM, minor, ſcilicet, quàm NM; et componatur ratio aſſumpta AB ad BC ex rationibus radicalibus, quarum numerus major ſit numero partium ipſi NO æqualium, quæ in datâ magnitudine DE continentur; et ſecetur DE in partes æquales, quarum numerus æqualis eſt numero rationum ex quibus ratio AB ad BC eſt compoſita; igitùr una harum partium minor eſt ipsâ NO. Sit ratio FG ad GP ea quæ, ex rationibus radicalibus compoſita, proximè major eſt ratione FG ad GH, ſítque LQ Logarithmus rationis FG ad GP; erit igitùr NQ minor quàm una partium radicalium, quòd, ut ſimile in præcedente Propoſitione, oſtendetur: multò igitùr NQ minor eſt quàm NO; quarè QM major eſt quàm OM. Et, ut ſimile in præcedente, oſtendetur Logarithmus rationis FG ad GK æqualis ſummæ Logarithmorum rationum FG ad GP, et PG ad GK. Igitùr, quoniam LM eſt Logarithmus rationis FG ad GK, et LQ Logarithmus eſt rationis FG ad GP, erit reliqua QM Logarithmus rationis HG ad GK. Quoniam igitùr oſtenſa eſt QM major OM, erit Logarithmus rationis PG ad GK major Logarithmo rationis HG ad GK; quarè ratio PG ad GK major eſt ratione HG ad GK. Sed et minor, quoniam PG minor eſt HG; quod fieri non poteſt.



Sed, ſi LM, Logarithmus rationis FG ad GK, dicatur minor quàm ſumma Logarithmorum rationum FG ad GH, et HG ad GK; ſit LM Logarithmus rationis FG ad GH. Igitùr NM minor eſt Logarithmo rationis HG ad GK. Sit OM, Logarithmus hujus rationis, major, ſcilicet, quàm NM; et, ſumptâ ratione

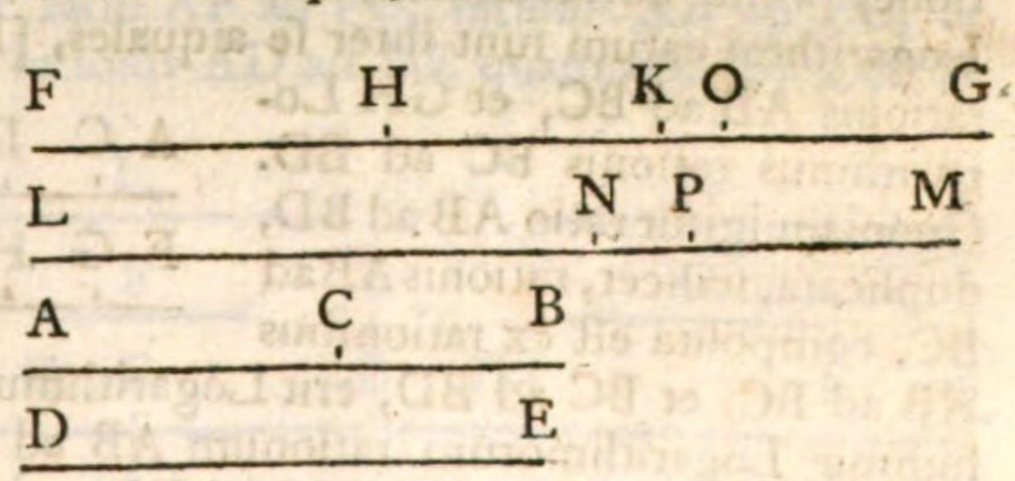
FG ad GP, quæ, ex radicalibus compoſita, proximè minor eſt ratione FG ad GH, reliquíſque ut in præcedente caſu conſtructis, oſtendetur QN minor eſſe quàm ON; quarè et QM, Logarithmus rationis PG ad GK, minor erit quàm OM, Logarithmus rationis HG ad GK. Ratio igitùr PG ad GK minor eſt ratione HG ad GK; quod fieri non poteſt. Igitùr LM non eſt minor ſummâ Logarithmorum rationum FG ad GH, et HG ad GK: et oſtenſa eſt non major eâdem ſummâ; igitùr eſt eidem æqualis.



PROP. VIII.

" Sint rationes FG ad GH, et HG ad GK, quarum neutra eſt compoſita ex
" rationibus radicalibus; erit ſumma Logarithmorum rationum FG ad GH, et
" HG ad GK æqualis Logarithmo rationis FG ad GK, quæ eſt ex illis com-
" poſita."

Si enim non ſit; primò, ſit LM, ſumma Logarithmorum rationum FG ad GH, et HG ad GK, major quàm LN, Logarithmus rationis FG ad GK. Et componatur ratio aſſumpta AB ad BC ex rationibus, quarum numerus major ſit numero partium ipſi NM æqualium, quæ in magnitudine aſſumptâ DE continentur. Secetur DE in partes æquales, quarum numerus idem ſit cum numero rationum ex quibus compoſita eſt ratio AB ad BC; igitùr una harum partium minor erit quàm MN. Sit ratio FG ad GO ea quæ, ex rationibus radicalibus compoſita, proximè major eſt ratione FG ad GK; ſítque LP Logarithmus rationis FG ad GO, qui proptereà æqualis erit ſummæ Logarithmorum rationum FG ad GH, et HG ad GO. Erit igitùr NP minor quàm una partium radicalium, ut in Prop. 6. oſtenſum, et proptereà multò quàm NM minor; quarè LP minor eſt LM. Eſt autem LP, per Prop. 7. æqualis ſummæ Logarithmorum rationum FG ad GH, et HG ad GK. Prima igitùr ſumma minor eſt reliquâ; et, ablato communi Logarithmo rationis FG ad GH, erit reliquus, Logarithmus rationis, ſcilicet, HG ad GO, minor reliquo Logarithmo rationis HG ad GK. Igitùr ratio HG ad GO minor eſt ratione HG ad GK. Sed et major eſt, quoniam GO minor eſt GK; quod fieri non poteſt. Igitùr LM non eſt major quàm LN.



Sed

Sed, ſi ponatur LM minor quàm LN, ſumatur ratio FG ad GO, quæ, ex radicalibus compoſita, proximè minor eſt ratione FG ad GK; ſitque LP Logarithmus rationis FG ad GO; reliquiſque ut anteà conſtructis, oſtendetur LP major quàm LM, hoc eſt, ſumma Logarithmorum rationum FG ad GH, et HG ad GO, major ſummâ Logarithmorum rationum FG ad GH, et HG ad GK; igitùr, ablato communi, Logarithmus rationis HG ad GO major eſt Logarithmo rationis HG ad GK; quarè et ratio HG ad GO major eſt ratione HG ad GK; quod fieri non poteſt. Igitùr LM non minor eſt quàm LN; et oſtenſa fuit non major; æqualis igitùr eſt LM ipſi LN.

F H O K G
L M P N

Cor. Idem Corollarium hinc ſequitur, quod ex Prop. 5. oſtenſum eſt.

PROP. IX.

"Si una ratio fuerit multiplex quæcunque alterius rationis, quàm multiplex "eſt prima ratio ſecundæ, tàm multiplex erit Logarithmus primæ rationis Lo- "garithmi ſecundæ."

Sint enim AB, BC, BD, BE, &c. deinceps proportionales. Erunt igitùr rationes primæ ad ſecundam, ſecundæ ad tertiam, &c. inter ſe eædem; quarè et Logarithmi earum ſunt inter ſe æquales, [Prop. 2. vel 3.] Sit FG Logarithmus rationis AB ad BC, et GH Logarithmus rationis BC ad BD. Quoniam igitùr ratio AB ad BD, duplicata, ſcilicet, rationis AB ad BC, compoſita eſt ex rationibus AB ad BC, et BC ad BD, erit Logarithmus rationis AB ad BD æqualis ipſi FH ſummæ Logarithmorum rationum AB ad BC, et BC ad BD, [Prop. 5, aut 6, 7, 8.] hoc eſt, duplo Logarithmi FG rationis AB ad BC. Similitèr, quoniam ratio AB ad BE, triplicata, ſcilicet, rationis AB ad BC, compoſita eſt ex ratione AB ad BD, et ratione BD ad BE, erit Logarithmus rationis AB ad BE æqualis ſummæ Logarithmorum rationum AB ad BD, et BD ad BE. Logarithmus autem FH rationis AB ad BD, duplus oſtenſus eſt Logarithmi FG rationis AB ad BC, et HK Logarithmus rationis BD ad BE æqualis eſt ipſi FG; igitùr FK Logarithmus rationis AB ad BE triplus eſt ipſius FG Logarithmi rationis AB ad BC; et ita deinceps.

A C D E B
F G H K

DEF.

DEF. IV.

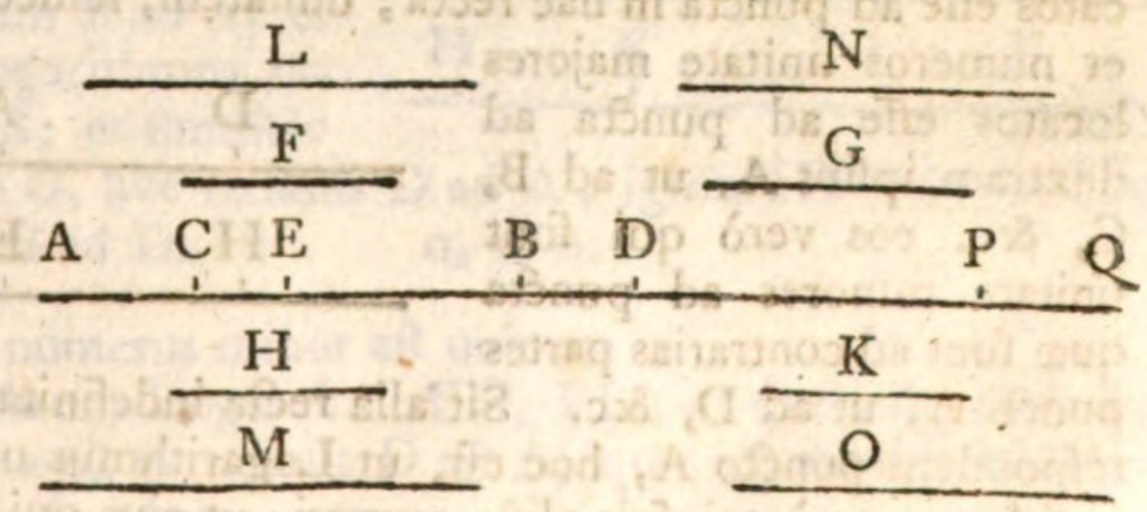
Logarithmi omnium rationum ex assumptâ ratione AB ad BC, et assumptâ magnitudine DE provenientes, secundùm Def. 2. et Def. 3. dicuntur esse *ejusdem speciei*. Sed si sumatur eadem ratio AB ad BC, et sumatur alia magnitudo DE, Logarithmi ex hisce, secundùm Def. 2. et 3. provenientes dicuntur esse *diversæ speciei* à præcedente.

PROP. X.

" Logarithmi duarum rationum unius speciei eadem inter se habent rationem " quàm Logarithmi earundem rationum cujuslibet aliûs speciei.

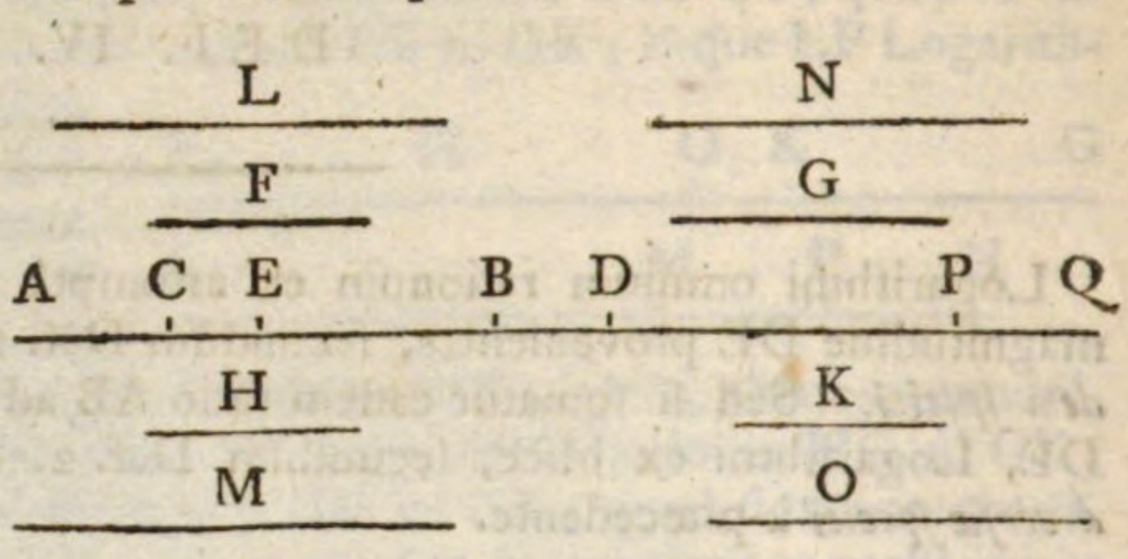
" Sint duæ rationes AB ad BC, et AD ad DE, et sint F, G earum Logarith- " mi unius speciei, et H, K earundem Logarithmi alterius speciei; erit F ad G, " ut H ad K."

Sumantur ipsarum F, H æquemultiplices utcunque L, M; et ipsarum G, K aliæ utcunque æquemultiplices N, O. Quam multiplex autem est L ipsius F, vel M ipsius H, tam multiplicata sit ratio AP ad PC, rationis AB ad BC, et tam multiplicata sit ratio AQ ad QE rationis AD ad DE quam multiplex est N ipsius G, vel O ipsius K.



Quoniam igitùr F est Logarithmus rationis AB ad BC, et est L tam multiplex ipsius F, quàm ratio AP ad PC est multiplicata rationis AB ad BC, erit L Logarithmus rationis AP ad PC [Prop. 9.]; eâdem ratione erit N Logarithmus rationis AQ ad QE, uterque, scilicet, in alterâ specie. Si igitùr Logarithmus L major sit Logarithmo N, erit ratio AP ad PC major ratione AQ ad QE [Cor. 3. Prop. 4.]; et quoniam ratio AP ad PC major est ratione AQ ad QE, erit [Prop. 4.] Logarithmus M major Logarithmo O in alterâ specie; igitùr, si L major sit N, erit et

M major O; eâdem ratione, ſi L æqualis fuerit ipſi N, vel eadem minor, erit M æqualis ipſi O, vel minor quàm ea. Quoniam igitùr quatuor ſunt magnitudines F, G, H, K, et ſumptæ ſunt ipſarum F, H utcunque æquemultiplices L, M; et ipſarum G, K, aliæ utcunque æquemultiplices N, O; et oſtenſum eſt ſi L major ſit N, et M majorem eſſe quàm O; et ſi æqualis æqualem; et ſi minor minorem eſſe; erit [5. Def. 5.] F ad G, ut H ad K.

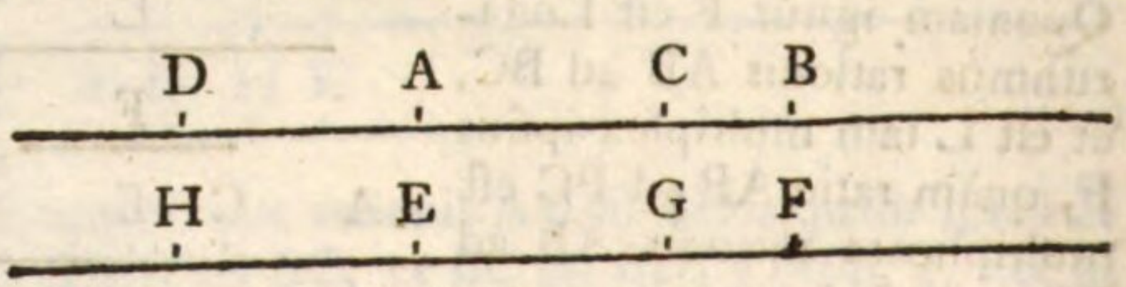


DEF. V.

Logarithmus rationis cujuſvis numeri ad unitatem dicitur etiam Logarithmus illius numeri.

Cor. Quoniam Logarithmus rationis æqualis ad æqualem eſt nihil [Cor. 1. Def. 3.] igitùr Logarithmus unitatis eſt nihil.

Ad ſequentia meliùs explicanda, ſit recta linea AB ad utraſque partes indefinita, et concipiantur omnes numeri, tum unitate majores, tum eâ minores, locatos eſſe ad puncta in hâc rectâ; unitatem, ſcilicet, locatam eſſe ad punctum A, et numeros unitate majores locatos eſſe ad puncta ad dextram ipſius A, ut ad B, C, &c. eos verò qui ſunt unitate minores ad puncta quæ ſunt ad contrarias partes puncti A, ut ad D, &c. Sit alia recta indefinita EF, in quâ ſit punctum E, reſpondens puncto A, hoc eſt, ut Logarithmus unitatis ſit ad punctum E, Logarithmus verò cujuſvis aliûs numeri, ut ejus qui eſt ad punctum B, ſumatur à puncto E, ad eas partes ejus ad quas eſt B reſpectu puncti A, ut EF, quæ ſemper æqualis ſit ipſi AB; et ita EG æqualis ipſi AC, ſit Logarithmus numeri ad punctum C, et EH æqualis ipſi AD ſit Logarithmus numeri ad D, et ita deinceps.



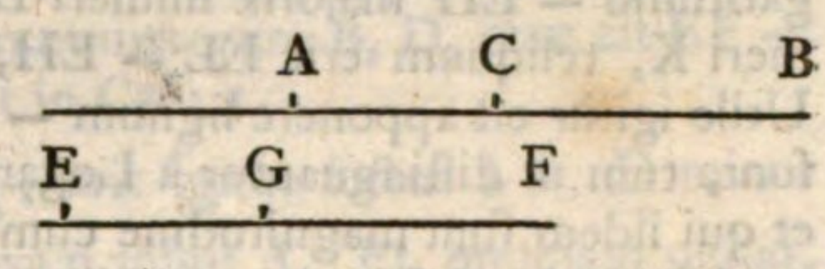
PROP.

PROP. XI.

" Si ſumantur à puncto E Logarithmi duorum quorumvis numerorum, ut " eorum qui locati ſunt ad puncta B, C, (qui, brevitatis gratiâ, in ſequentibus " vocentur numeri B, C,) qui Logarithmi ſint EF, EG; diſtantia GF inter " ipſorum terminos erit Logarithmus rationis numeri B ad numerum C."

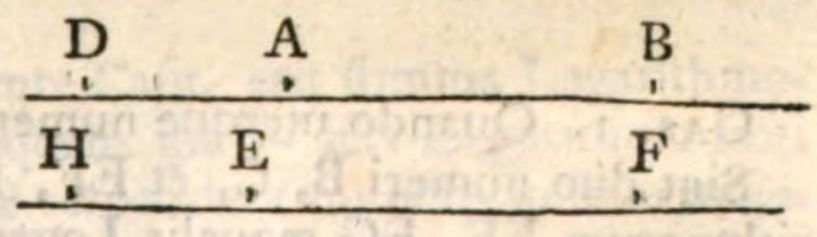
Cas. 1. Quando uterque numerus eſt major unitate.

Sint numeri B, C, et ipſorum Logarithmi EF, EG; et quoniam ratio numeri B ad numerum C compoſita eſt ex ratione B ad numerum A, ſive unitatem, et ratione A ad C, quarum prima eſt ratio majoris ad minorem, reliqua verò eſt ratio minoris ad majorem; erit Logarithmus rationis B ad C, per Cor. Prop. 5. vel Cor. Prop. 8. æqualis exceſſui Logarithmi rationis B ad A, hoc eſt, Logarithmi numeri B, ſuprà Logarithmum rationis A ad C, hoc eſt, ſuprà Logarithmum numeri C. Eſt autem EF Logarithmus numeri B, et EG Logarithmus ipſius C; GF verò eſt horum exceſſus, qui propterea eſt Logarithmus rationis B ad C. Q. E. D.



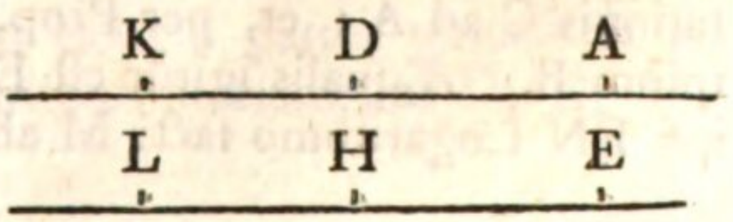
Cas. 2. Quando alter numerus major eſt unitate, reliquus verò minor.

Sit numerus B major unitate, numerus verò D ad contrarias partes ipſius A, minor ſit unitate; et quoniam ratio numeri B ad ipſum D compoſita eſt ex ratione B ad A, ſive unitatem, et ratione A ad D, quarum utraque eſt majoris ad minorem, erit, per Prop. 5, aut 6, 7, 8. Logarithmus rationis B ad D æqualis ſummæ Logarithmorum rationum B ad A, et A ad D; eſt autem EF Logarithmus rationis B ad A, ſive numeri B; et ſimilitèr EH eſt Logarithmus numeri D, ſive rationis D ad A; igitùr HF ipſorum ſumma eſt Logarithmus rationis B ad D. Q. E. D.



Cas. 3. Quando uterque numerus minor eſt unitate.

Sint numeri D, K, et ipſarum Logarithmi EH, EL; et quoniam ratio numeri D ad ipſum K compoſita eſt ex ratione D ad A, quæ eſt minoris ad majorem, et ratione A ad K, majoris ad minorem, quæ major eſt reciprocâ rationis D ad A, hoc eſt, major ratione A ad D; erit, per Caſ. 2. Cor. Prop. 5. Logarithmus rationis D ad K æqualis exceſſui Logarithmi rationis A ad K,



ſive

ſive Logarithmi ipſius K, ſuprà Logarithmum rationis D ad A, ſive Logarithmum ipſius D. Eſt autem EL Logarithmus ipſius K, et EH Logarithmus ipſius D, quorum exceſſus eſt HL; qui proptereà eſt Logarithmus rationis D ad K.

Q. E. D.

Cor. Si Logarithmis rationum minoris ad majorem, vel numerorum qui ſunt unitate minores, apponatur ſignum ſubtractionis —, Logarithmus rationis numeri cujuſvis ad numerum minorem, invenietur auferendo, ſecundùm regulas in Algebrâ traditas, Logarithmum minoris numeri à Logarithmo majoris. In ſecundo enim caſu præcedente, ſi à Logarithmo EF numeri B auferatur — EH, Logarithmus numeri D unitate minoris, appoſito ſigno ſubtractionis, reliquum erit EF + EH, hoc eſt HF, prorſùs ut in eo caſu oſtenſum fuit. Et in caſu tertio, ſi Logarithmi numerorum D, K unitate minorum fiant — EH — EL, et à Logarithmo — EH majoris numeri D auferatur — EL Logarithmus minoris numeri K, reliquum erit EL — EH, hoc eſt LH, ut in Caſ. 3. oſtenſum fuit. Utile igitùr eſt apponere ſignum — Logarithmis numerorum qui unitate minores ſunt, tum ut diſtinguantur à Logarithmis numerorum qui ſunt eorum reciproci, et qui iidem ſunt magnitudine cum illis, tum ut regulæ pro operationibus arithmeticis de Logarithmis generaliores fiant, ut in ſequentibus pleniùs apparebit.

PROP. XII. PROB.

Invenire Logarithmum facti, ſive producti, à duobus numeris.

Cas. 1. Quando uterque numerus major eſt unitate.

Sint duo numeri B, C, et EF, EG ipſorum Logarithmi; erit ſumma Logarithmorum EF, EG æqualis Logarithmo facti à numeris B, C. Sit enim M numerus factus ab ipſis B, C, et Logarithmus ejus ſit EN; et, quoniam ut unitas, ſeu A, ad alterum ex numeris C, ita reliquus B ad numerum M ab ipſis factum; hoc eſt, quoniam ratio C ad A eadem eſt rationi M ad B, erunt [per Prop. 1. aut 3.] Logarithmi harum rationum inter ſe æquales. Eſt autem EG Logarithmus numeri C, ſive rationis C ad A; et, per Prop. 11. eſt FN Logarithmus rationis numeri M ad ipſum B. Æqualis igitùr eſt EG ipſi FN, quarè EG, unà cum EF, æqualis eſt ipſi EN Logarithmo facti M ab ipſis B, C numeris.

A	C	B	M
E	G	F	N

Cas. 2. Quando alter numerus eſt major, reliquus autem minor unitate.

Sit

Sit numerus B major, D verò minor, unitate; et primò, ſit B major reciproco numero ipſius D, hoc eſt major quàm $\frac{1}{D}$; erit igitùr factus ab ipſis B, D major unitate. Sit factus numerus M. Cadit igitùr M ad dextram unitatis quæ eſt in A. Et, quoniam unitas, ſeu A, eſt ad B, ut D ad factum M, erit ratio B ad A eadem rationi M ad D; igitùr Logarithmi harum rationum erunt inter ſe æquales. Sit EF Logarithmus rationis D ad A, EN verò Logarithmus rationis M ad A; unde, per Prop. 11. erit HN Logarithmus rationis M ad D. Igitùr EF æqualis eſt ipſi HN; et, ablato communi HE, erit exceſſus EF ſuprà HE æqualis ipſi EN, hoc eſt, erit exceſſus Logarithmi numeri B ſuprà Logarithmum numeri D, æqualis Logarithmo facti ab ipſis M.

D	A	M	B
H	E	N	F

Si verò ipſi EH Logarithmo numeri D unitate minoris apponatur ſignum — ſive ſubtractionis, erit ſumma Logarithmorum numerorum B, D, quæ eſt EF — EH æqualis ipſi EN Logarithmo facti M, ut in Caſu 1.

Secundò, ſit numerus B minor quàm — $\frac{1}{D}$; erit igitùr factus à B, D minor unitate. Sit factus M; cadit igitùr M ad lævam ipſius A. Et, quoniam unitas, ſeu A, eſt ad B, ut D ad factum M, erit ratio B ad A eadem rationi M ad D; igitùr Logarithmi harum rationum ſunt inter ſe æquales. Sit EF Logarithmus rationis B ad A, et EH Logarithmus rationis D ad A, EN verò Logarithmus rationis M ad A; igitùr, per Prop. 11. erit HN Logarithmus rationis M ad D; et propterea EF eſt æqualis ipſi HN, quibus ex HE ablatis, erit reliquum, exceſſus ſc. ipſius EH ſuprà EF, æquale reliquo EN; hoc eſt, exceſſus Logarithmi ipſius D ſuprà Logarithmum numeri B æqualis eſt Logarithmo facti M.

D	M	A	B
H	N	E	F

Et, ſi EH notetur ſigno —, ut in præcedente Caſu, erit ſumma Logarithmorum numerorum B, D, ſcilicet, EF — EH, æqualis ipſi — EN Logarithmo facti M. Et quoniam Logarithmi in tabulis æquè reſpondent numeris unitate majoribus quibus in tabulis apponuntur, ac eorum reciprocis; igitùr ad inveniendum numerum M, qui unitate minor eſt, ope tabularum, ſumendus eſt reciprocus numerus ejus cui convenit Logarithmus numeri M in tabulis; quod in omni caſu obſervandum.

Cas. 3. Quando uterque numerus minor eſt unitate.

Sit uterque numerus D, K, minor unitate; et ſit M factus ab iiſdem, qui propterea minor etiam erit unitate; quarè M cadit ad eaſdem partes unitatis A ad quas ſunt D, K. Sint verò EH, EL, EN Logarithmi numerorum D, K, M; erit EN æqualis ſummæ ipſarum EH, EL.

Quoniam enim eſt unitas, ſeu A, ad D, ut K ad M, erit ratio A ad D eadem rationi K ad M, et rationum harum æquales erunt Logarithmi; rationis verò A ad D, ſive numeri D, Logarithmus eſt EH; et, per Prop. 11. rationis K ad M Logarithmus eſt NL. Eſt igitùr NL æqualis ipſi HE, et propterea NE æqualis eſt ſummæ ipſarum EH, EL; hoc eſt, Logarithmus facti M æqualis eſt ſummæ Logarithmorum numerorum D, K, à quibus factus eſt.

M	K	D	A
N	L	H	E

Et, ſi ipſis EH, EL apponatur ſignum —, erit — EH — EL æqualis ipſi — EN Logarithmo facti. In omnibus igitùr caſibus Logarithmus facti æqualis eſt ſummæ Logarithmorum numerorum à quibus factus eſt, dummodò Logarithmis numerorum qui unitate ſunt minores, apponatur ſignum — ſubtractionis. Q. E. D.

PROP. XIII. PROB.

Invenire Logarithmum quotientis, qui oritur dividendo unum numerum per alium.

CAS. 1. Quando uterque numerus major eſt unitate.

Sit numerus dividendus B, et ſit diviſor C, qui, primò, minor ſit quàm B; igitùr quotiens major erit unitate; ſit M quotiens. Erit Logarithmus ipſius M æqualis exceſſui Logarithmi numeri B ſuprà Logarithmum ipſius C.

A	M	C	B
E	N	G	F

Sint EF, EG, EN Logarithmi numerorum B, C, M; et quoniam unitas, ſeu A, eſt ad quotientem M, ut diviſor C ad dividendum B, erit ratio M ad A eadem rationi B ad C. Igitùr Logarithmus rationis M ad A, hoc eſt ipſius M, æqualis erit, per Prop. 1. aut 3. Logarithmo rationis B ad C, hoc eſt, [per Prop. 11.] ipſi GF. Eſt autem GF exceſſus ipſius EF, hoc eſt, Logarithmi dividendi B, ſuprà EG Logarithmum diviſoris C.

Sit autem diviſor C major dividendo B; igitùr quotiens M minor erit unitate, et cadet M ad contrarias partes A ad quas eſt B; erit autem EN, Logarithmus ipſius M, æqualis FG, exceſſui EG, Logarithmi diviſoris C, ſuprà E, Logarithmum dividendi B. Quoniam enim eſt unitas, ſeu A, ad quotientem M, ut diviſor C ad dividendum B, erit ratio M ad A eadem rationi B ad C; igitùr Logarithmus rationis M ad A, hoc eſt, numeri M, æqualis erit [per Prop. 1.

M	A	B	C
N	E	F	G

aut

aut 3.] Logarithmo rationis B ad C, hoc eſt [Prop. 11.] ipſi FG. Quoniam verò M eſt minor unitate, Logarithmi verò in tabulis aptantur numeris qui ſunt majores unitate; igitùr, in hoc et ſimilibus caſibus, numerus M erit reciprocus numeri cujus Logarithmus in tabulis eſt FG. Idem enim eſt Logarithmus cujuſvis numeri et ejus reciproci.

Et ſi à Logarithmo EF dividendi B auferatur EG Logarithmus diviſoris C, reliquus erit — FG, qui, ſigno — notatus, indicat numerum, cujus eſt Logarithmus, minorem eſſe unitate; atque ita regula præcedentis Caſûs huic etiam inſervit.

Cas. 2. Quando numerus dividendus B major eſt unitate, diviſor verò D minor eſt unitate, et propterea quotiens M unitate major eſt. Logarithmus verò ipſius M æqualis erit ſummæ Logarithmorum dividendi B et diviſoris D.

Sint enim ipſorum B, D, M Logarithmi EF, EH, EN; et quoniam eſt A ad M, ut D ad B, erit Logarithmus rationis M ad A, hoc eſt ipſius M Logarithmus, æqualis Logarithmo rationis D ad B. Eſt igitùr EN, Logarithmus ſc. ipſius M, æqualis HF Logarithmo ſc. rationis D ad B, [Prop. 11.] hoc eſt ſummæ Logarithmorum EF, EH dividendi B et diviſoris D.

D	A	B	M
H	E	F	N

Et ſi EH Logarithmus diviſoris D habeat præfixum ſibi ſignum —, et auferatur ab EF Logarithmo dividendi B, reliquus erit HE + EF, hoc eſt HF; atque ita regula Caſ. 1. huic etiam inſervit.

Cas. 3. Quando uterque numerus minor eſt unitate.

Sit numerus dividendus D, et diviſor ſit K, qui primò minor ſit quàm D; igitùr quotiens M major eſt unitate. Erit Logarithmus ipſius M æqualis exceſſui Logarithmi diviſoris K ſuprà Logarithmum dividendi D.

K	D	A	M
L	H	E	N

Sint EH, EL, EN Logarithmi numerorum D, K, M; et quoniam eſt ut A ad M, ita K ad D, erit Logarithmus rationis M ad A, hoc eſt numeri M Logarithmus, æqualis Logarithmo rationis D ad K, hoc eſt ipſi LH [Prop. 11.]; eſt igitùr EN æqualis ipſi LH, hoc eſt, exceſſui Logarithmi EL diviſoris K ſuprà EH Logarithmum dividendi D.

Et, ſi ipſis EH, EL præfigatur ſignum —, et auferatur — EL ab — EH, reliquum erit EL — EH, hoc eſt LH; igitùr regula pro Caſ. 1. huic etiam inſervit.

Sit autem diviſor K major dividendo D; igitùr quotiens M minor eſt unitate. Erit autem EN Logarithmus ipſius M æqualis exceſſui HL Logarithmi dividendi EH ſuprà EL Logarithmum diviſoris K. Quoniam enim eſt A ad M, ut K ad D, erit Logarithmus rationis M ad A, hoc eſt numeri M, æqualis Logarithmo rationis D ad K,

D	M	K	A
H	N	L	E

hoc

hoc eſt [Prop. 11.] ipſi HL, qui eſt exceſſus Logarithmi EH, dividendi, ſuprà EL Logarithmum diviſoris.

Si autem ipſis EH, EL præfigatur ſignum —, et — EL auferatur à — EH, reliquum erit — HL, Logarithmus, ſcilicet, ipſius M. Quoniam verò M minor eſt unitate, ſumendus eſt numerus reciprocus ejus qui habet Logarithmum HL in tabulis, ut anteà dictum.

In omnibus igitùr caſibus Logarithmus quotientis eſt exceſſus Logarithmi dividendi ſuprà Logarithmum diviſoris, dummodò Logarithmis numerorum, qui unitate ſunt minores, apponatur ſignum — ſubtractionis.

PROP. III. PROB.

Invenire Logarithmum numeri qui eſt quartus proportionalis tribus numeris datis.

Quoniam quartus proportionalis eſt quotiens qui oritur dividendo numerum factum à ſecundo et tertio per primum proportionalem, invenietur Logarithmus quarti auferendo Logarithmum primi à Logarithmo facti à ſecundo et tertio, præfixo ſigno — Logarithmis numerorum qui unitate ſunt minores, ut conſtat ex Prop. 13. Logarithmus verò facti eſt ſumma Logarithmorum ſecundi et tertii, præfixo ſigno — Logarithmis numerorum qui unitate ſunt minores [per Prop. 12.]; igitùr, præfixo ſigno —, ut dictum fuit, ſi à ſummâ Logarithmorum ſecúndi et tertii auferatur Logarithmus primi, reliquus erit Logarithmus quarti proportionalis qui inveniendus fuit.

FINIS.

DE

LIMITIBUS QUANTITATUM ET RATIONUM.

FRAGMENTUM.

AUCTORE

ROBERTO SIMSON, M. D.,

A

IN ACADEMIA GLASGUENSI APUD SCOTOS NUPER MATHESEOS PROFESSORE CELEBERRIMO.

EXCERPTUS

A

EX EDITIONE OPERUM EJUS RELIQUORUM IMPRESSA GLASGUÆ ANNO DOMINI 1776,

IMPENSIS *PHILIPPI* COMITIS *STANHOPE*, NUNC DEFUNCTI.

DE LIMITIBUS QUANTITATUM ET RATIONUM.

DEFINITIONES.

I. QUANTITAS *conſtans*, ſeu invariabilis, eſt quæ ſemper ejuſdem magnitudinis eſt: ut diameter ejuſdem Circuli, ipſiúſque dimidium, &c.; ſumma rectarum à quovis puncto in eâdem Ellipſi ad focos ipſius ductarum, et ejuſmodi: et patet omnem quantitatem datam conſtantem eſſe; non autem, contrà, omnem conſtantem datam eſſe.* Quantitas conſtans.

II. Quantitas *mutabilis* eſt quæ augeri vel diminui poteſt. Harum quantitatum variæ ſpecies in Prop. 31, 32, 33, 34, Pappi enumerantur *. Quantitas mutabilis.

III. Si quantitas mutabilis ſemper minor fuerit quantitate datâ, ſed ita augeri poterit, ut major fiat quâcunque quantitate datâ quæ minor eſt primâ quantitate datâ; vel ſi quantitas mutabilis ſemper major fuerit quantitate datâ, ſed ita minui poterit, ut minor fiat quâcunque quantitate datâ quæ major eſt primâ quantitate datâ; in utroque caſu quantitas prima data dicatur *Limes* quantitatis mutabilis †. Limes quantitatis mutabilis.

IV. Si ratio mutabilis ſemper minor fuerit quàm ratio data, ſed ita augeri poterit, ut major fiat ratione quâcunque datâ quæ minor eſt ratione primâ datâ; vel ſi ratio mutabilis ſemper major fuerit quàm ratio data, ſed ita minui poterit ut Limes rationis mutabilis.

* Eodem modo definiuntur *ratio conſtans*, et *ratio mutabilis*.

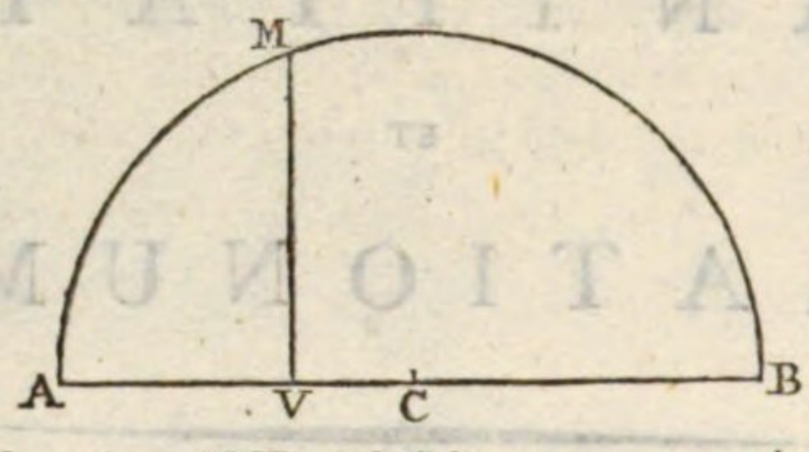
† Exempla horum Limitum ad quos continuò accedunt quantitates mutabiles, ſeu variabiles, ſed quos nunquam omninò attingunt ſeu æquant, ex lineis quibuſdam circà circulum ductis derivari poſſunt hoc modo.

Exemplum Primum.

Si ſupèr datâ quâdam rectâ lineâ AB deſcribatur ſemicirculus AMBC, cujus centrum ſit punctum C; et à quovis puncto M in arcu AMB, inter ejus puncta extrema A et B ſito, ducatur recta linea MV ad rectos angulos diametro AB occurrens in puncto V, eámque dividens in duas

ut minor fiat ratione quâcunque datâ quæ major eſt ratione primâ datâ: in utroque caſu dicatur ratio prima data *Limes rationis mutabilis*; in primo, ſcilicet, caſu dicatur *Limes creſcentis rationis*, in altero, *Limes decreſcentis rationis*.

partes AV et VB; notum eſt, quòd recta AV vocatur apud ſcriptores Trigonometriæ *Sinus verſus arcûs circularis A M.*



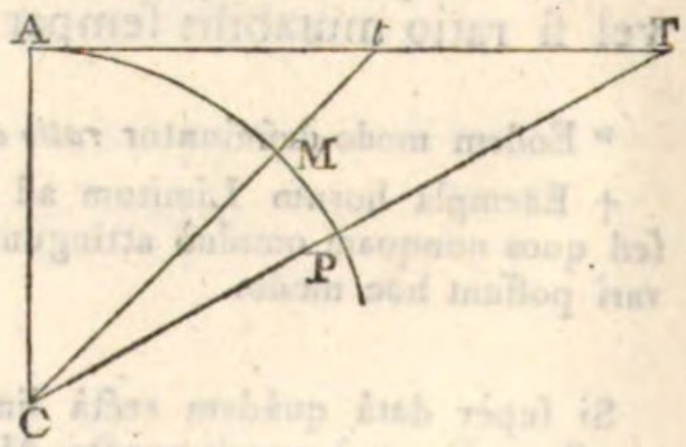
Ponamus jam ſemicircumferentiam AMB deſcribi per motum continuum puncti M à primo ejus puncto A ad extremum ejus punctum B, et rectam MV, ſeu ſinum rectum arcûs AM, ſimùl ferri ab A verſùs B motu ita ordinato ut ſit ſempèr ſibi ipſi parallela, et occurrat ſempèr diametro AB ad rectos angulos. Et manifeſtum erit quòd, dum arcus AM continuò creſcit à nihilo uſque ad AMB, ſeu arcum ſemicirculi, ſinus verſus AV ejuſdem arcûs ſimùl creſcet à nihilo donec fiat quàm proximè æqualis diametro AB; ſed nunquam eidem fiet omninò æqualis, quia, per præcedentem definitionem ſinûs verſi alicujus arcûs, neceſſe eſt ut punctum M (à quo ducitur recta MV quæ ſinum verſum AV determinat,) ſitum ſit inter puncta A et B in arcu AMB, et non omninò co-incidat cum puncto ejus extremo B; nam, ſi hoc fieret, non poſſemus ducere rectam lineam MV à puncto M ad diametrum AB. Poteſt tamen ſinus verſus AV ita augeri, dum arcus AM creſcit, ut fiat major quâcunque lineâ datâ quæ ſit minor Diametro AB. Ergò Diameter AB (quæ eſt quantitas conſtans,) erit *Limes*, ſeu *Limes magnitudinis*, ſinûs verſi AV, qui eſt quantitas mutabilis, ſeu variabilis, quæ continuò creſcit. Hoc igitùr eſt exemplum quantitatis conſtantis quæ eſt Limes ad quem continuò accedit quantitas mutabilis, ſeu variabilis, quæ continuò creſcit.

Exemplum Secundum.

Exemplum quantitatis conſtantis quæ ſit Limes ad quem continuò accedit quantitas mutabilis, ſeu variabilis, quæ continuò decreſcit, à radio dati circuli et ſecante arcûs ejuſdem qui ſit minor arcu quadrantis, derivari poteſt. Centro enim C, et datâ quâvis lineâ rectâ CA, deſcribatur arcus quivis circularis AP, qui ſit arcu quadrantis ejuſdem circuli minor. Et per primum hujus arcûs punctum A ducatur recta linea AT contingens circulum in A, et per extremum ejuſdem arcûs punctum P ducatur à centro C recta CP, quæ, producta, occurrat tangenti AT in puncto T. Hæc linea CPT, ſeu CT, vocatur apud Scriptores Trigonometriæ *Secans arcûs circularis* AP.

Porrò, ſit M punctum quodvis in arcu AP, ſitum inter puncta A et P; et per hoc punctum M ducatur à centro C recta CM, quæ, producta, occurrat tangenti AT in puncto t. Hæc recta CMt, ſeu Ct, erit ſecans arcûs circularis AM.

Ponamus jam arcum AM primò coincidere cum arcu AP, ſed poſteà decreſcere à primâ magnitudine AP ad nihilum motu continuo puncti M à puncto P ad punctum A. Et manifeſtum erit quòd recta Ct, ſeu Secans arcûs circularis AM, decreſcet in eodem tempore à primâ ejus magnitudine CT ad magnitudinem quæ erit quàm proximè æqualis radio CA, ſed eodem ſemper aliquanto major, tam diù quàm arcus AM aliquam retinet magnitudinem, quantumvis parvam. Poteſt tamen ita minui Secans Ct, decreſcente arcu MA, ut fiat minor quâcunque rectâ lineâ datâ quæ ſit major radio CA. Ergò radius CA (quæ eſt quantitas conſtans) erit *Limes*, ſeu *Limes magnitudinis*, Secantis Ct, quæ eſt quantitas mutabilis, vel variabilis, quæ continuò decreſcit.



Nota Editoris F. M.

PROP. I.

" Sit AB recta data, AC autem ſemper minor ſit ipſâ AB, ſed quæ augeri " poterit ita ut exceſſus AB ſuprà ipſam minor poterit fieri quâvis rectâ datâ, " hoc eſt, ſit AB limes creſcentis rectæ AC. Oſtendendum eſt, rationem æqua- " litatis limitem eſſe rationis AC ad AB; hoc eſt, rationem AC ad AB, poſſe " majorem fieri quâvis ratione datâ quæ minor eſt ratione æqualitatis."

Demonſtratio.

Sit enim ratio data ea quam habet AD ad AB. Eſt igitùr, ex hypotheſi, AD minor AB. Et, quoniam data eſt AB et ratio quam ad ipſam habet AD, dabitur AD; quare et DB; poterit igitùr AC augeri ita ut exceſſus AB ſuprà ipſam minor fiat rectâ DB: augeatur et fiat AE. Eſt igitùr AE major AD, quare et ratio AE ad AB major eſt ratione datâ AD ad AB.* Q. E. D.

A C D E B

PROP. II.

" Sit AB recta data, AC autem ſemper minor ſit ipſâ AB, ſed quæ augeri " poterit ita ut ratio ejus ad AB major poterit fieri quâvis ratione datâ minoris " ad majorem; hoc eſt, ſit ratio æqualitatis limes rationis AC ad AB. Oſten- " dendum eſt poſſe exceſſum AB ſuprà AC minorem fieri quâvis rectâ datâ."

Demonſtratio.

Sit DB recta data. Quoniam igitùr ratio AD ad AB eſt minoris ad majorem, poterit, ex hypotheſi, AC augeri ita ut ratio ejus ad AB major ſit ratione AD ad AB. Augeatur, et fiat AC æqualis ipſi AE. Eſt igitùr (per El. 5. 10.) AE major AD, et propterea EB minor DB. Q. E. D.

A C D E B

Oſtendatur etiam utraque harum propoſitionum in caſu quando AC ſemper major eſt AB, ſed quæ ita diminui poterit, &c.

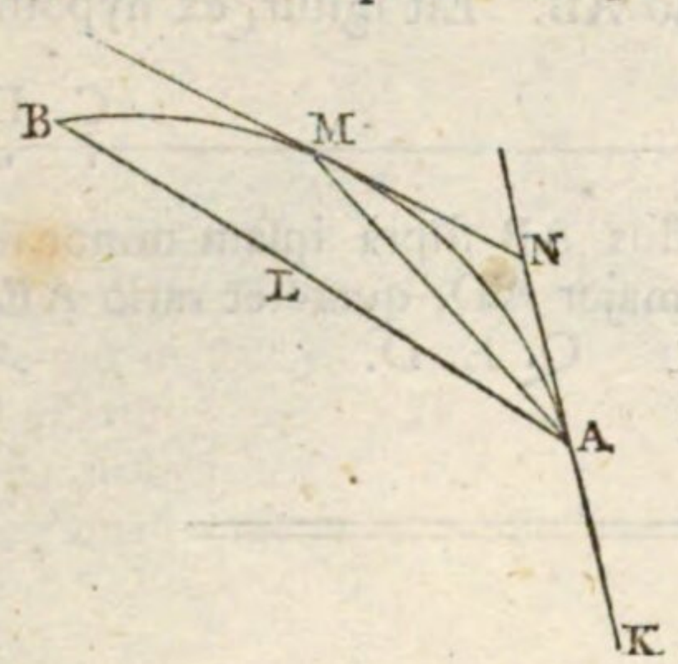
* Propoſitiones hæ, quatuor priores, materiem tractatûs hujus ſpectantes, hic, in principio, commodiſſimè locantur.

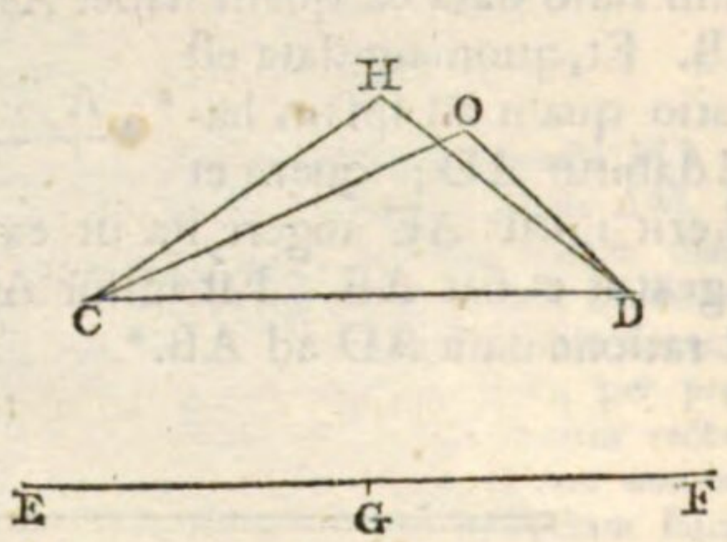
PROP. III.

" In quâcunque curvâ, vel parte curvæ, quæ verſùs eaſdem partes tota eſt con-
" cava; limes rationis quam habet arcus curvæ ad ipſius chordam, (decreſcen-
" tibus, ſcilicet, utriſque,) eſt ratio æqualis ad æquale."

Demonſtratio.

Sit recta linea AB chorda curvæ AB. Patet AB rectam ſemper minorem eſſe curvâ AB, ut Archimedes aſſumpſit. Oſtendendum verò eſt, decreſcente curvâ et rectâ, manente autem puncto A, poſſe rationem rectæ ad curvam majorem fieri quâcunque ratione datâ minoris ad majorem.





Sit ratio data ea quam habet recta CD ad ipſam EF; et, bifariàm ſectâ EF in G, conſtituatur ſupèr rectam CD triangulum iſoſceles habens latera CH, HD æqualia rectis EG, GF; quod fieri poſſe ex 22. 1. manifeſtum eſt. Contingat recta AK curvam in puncto A; ad quod fiat angulus KAL, qui major ſit angulo CHD *; occurrátque recta AL curvæ in B. Contingat recta MN, ipſi AB parallela, curvam in M; quod fieri poſſe patet, ſi concipiatur recta AB moveri ſemper ſibi parallela donec curvæ in unico puncto M occurrat. Jungatur AM; et ſupèr rectam CD conſtituatur triangulum COD æquiangulum triangulo MNA. Eſt igitùr angulus COD æqualis ipſi MNA, hoc eſt, angulo KAL, qui, ex conſtructione, major eſt angulo CHD; major igitùr eſt angulus COD angulo CHD. Si autem circà triangulum CHD deſcribatur circulus, rectæ CH, HD, quæ inter ſe ſunt æquales, ſimùl majores erunt duabus quibuſvis rectis, ad circumferentiam in quâ eſt punctum H ſuper CD deſcriptam, inflexis, per Prop. 47. libri Sereni Antiſſenſis de cono; et quoniam punctum O cadit intrà hanc circumferentiam, quia angulus COD major eſt angulo CHD, multo erunt CH, HD, majores ipſis CO, OD; recta igitùr CD

* Hoc autem fieri poſſe manifeſtum eſt ex eo quòd arcus BA ita minui poteſt, per motum puncti B verſùs punctum A, ut chorda AB quantumvis propè accedat ad tangentem AN, ſeu ut faciat angulum acutum BAN cum tangente AK, productâ ultrà punctum A, qui ſit quovis angulo acuto minor, et pro-inde ut faciat angulum BAK cum ejuſdem tangentis parte AK quovis angulo obtuſo majorem, ideóque angulo obtuſo CHD. Nota Editoris F. M.

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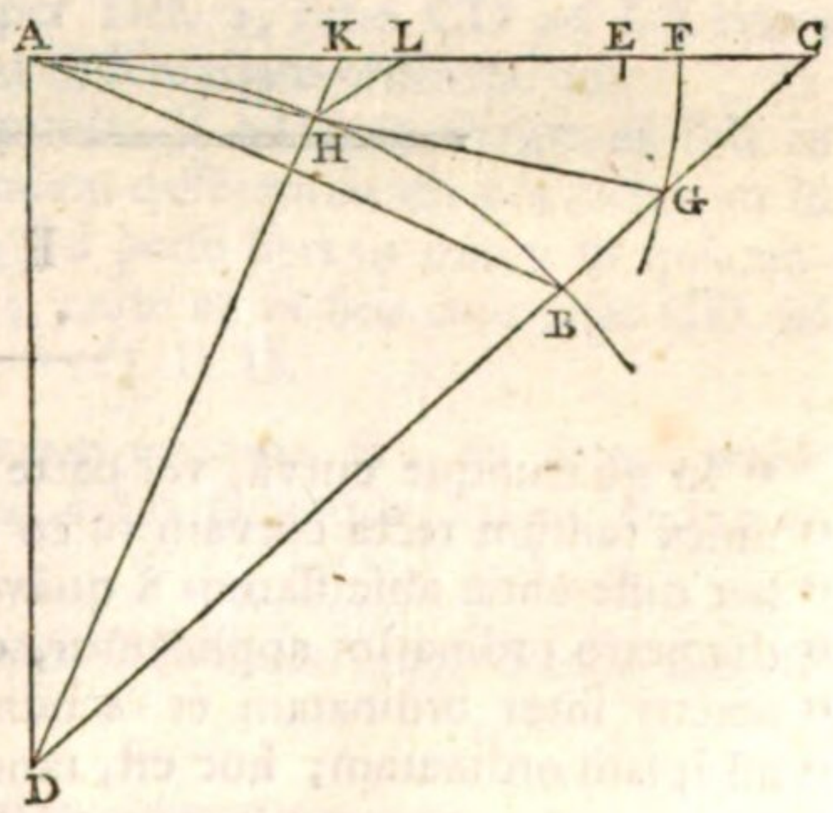
(per El. 5. 8.) majorem habet rationem ad ipsas CO, OD simùl, quàm ad ipsas CH, HD. Est verò recta AM ad ipsas MN, NA simùl, ut recta CD ad ipsas CO, OD, simùl, quoniam similia sunt MNA, COD triangula. Igitùr recta AM majorem habet rationem ad rectas MN, NA simùl, quàm CD habet ad ipsas CH, HD simùl, hoc est, ad rectam EF: recta igitùr AM ad ipsas MN, NA, et multo magìs ad curvam AM; (quæ, per aliam Archimedis assumptionem, ipsis minor est,) majorem habet rationem, ratione CD ad EF, hoc est, ratione datâ; et propterea ratio æqualis ad æquale est limes rationis rectæ AM ad curvam AM. Q. E. D.

PROP. IV.

" Sit curva, vel pars curvæ, (ut in præcedente,) AB, quam contingat recta AC
" in puncto A; et à dato puncto D in rectâ quæ perpendicularis est ad AC in A,
" ad partes concavas curvæ AB ducatur recta DC, quæ curvæ in puncto B oc-
" currat, rectæ verò AC in C, et ducatur recta AB. His positis, si recta DC con-
" cipiatur moveri circà punctum D versùs punctum A, erit ratio æqualis ad
" æquale limes rationis chordæ AB ad segmentum AC contingentis."

Casus Primus.

Primò sit chorda AB semper minor segmento AC contingentis, quando DC appropinquat punctum A; quod quidem semper fiet in casu quo semicirculus super diametrum AD cadit totus intrà curvam, quoniam in hoc casu angulus ABC semper erit major recto. Ostendendum verò est rationem AB ad AC majorem fieri posse quâcunque ratione datâ minoris ad majorem.



Demonstratio.

Sit ratio data ea quam habet AE ad AC; et, si ratio AB ad AC non major fuerit ratione AE ad AC, sumatur quodvis punctum F inter E et C. Et, quoniam AB minor est AF, (nam vel æqualis est ipsi AE, vel ipsâ minor, quia ratio AB ad AC ponitur non major ratione AE ad AC,) circulus descriptus centro A, intervallo AF, necessariò rectæ BC occurret; occurrat ipsi BC in G, et jungatur AG, quæ necessariò occurret curvæ *; occurrat in H, et juncta

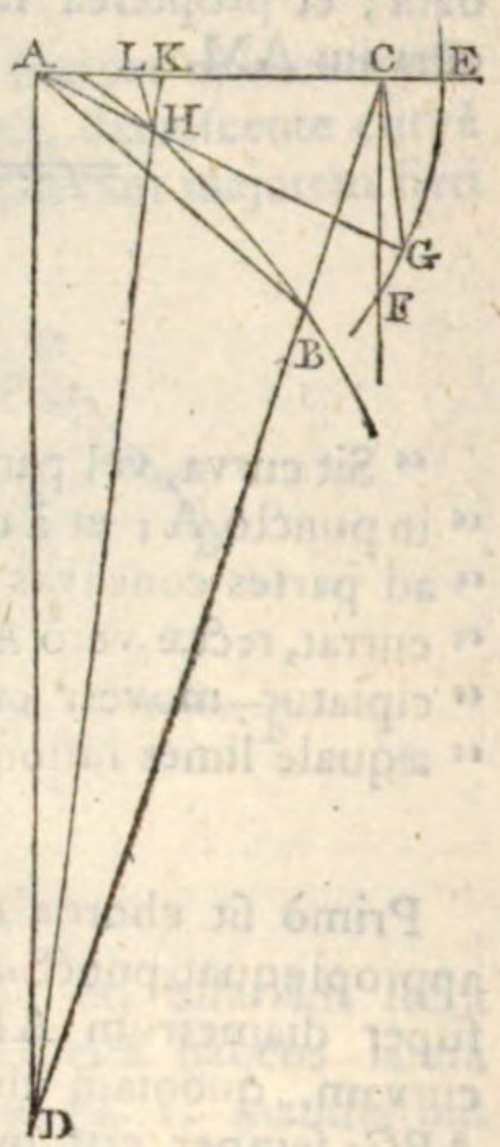
* Ponitur enim à quovis curvæ puncto unicam tantùm rectam duci posse quæ curvam in eo puncto contingat.

et

et producta DH rectæ AC occurrat in K, et ducatur ad AC recta HL parallela ipsi GC. Minor igitùr est AK rectâ AL, et propterea ratio AH ad AK major est ratione AH ad AL, hoc est, ratione AG, sive AF, ad AC; ratio igitùr AH ad AK, multo erit major ratione datâ AE ad AC. Q. E. D.

Casus Secundus.

Sit autem, quando DC appropinquat punctum A, chorda AB semper major segmento AC contingentis. Ostendendum est in hoc casu rationem AB ad AC minorem fieri posse ratione quâcunque datâ majoris ad minorem.



Demonstratio.

Sit ratio data ea quam habet AE ad AC; et centro A, intervallo AE, circulus describatur cujus circumferentiæ occurrat recta CF, ipsi AD parallela, in F; et in circumferentiâ EF sumatur quodvis punctum G inter curvam AB et contingentem AC; jungantur AG, GC, quarum AG necessariò curvæ occurret in quodam puncto H; ducatur DH quæ ipsi AC occurrat in K, et rectæ GC parallela ducatur HL. Est igitùr AK major AL, et propterea ratio AH ad AK minor est ratione AH ad AL, hoc est, ratione AG, sive AE, ad AC, hoc est, ratione datâ. Q. E. D.

PROP V.

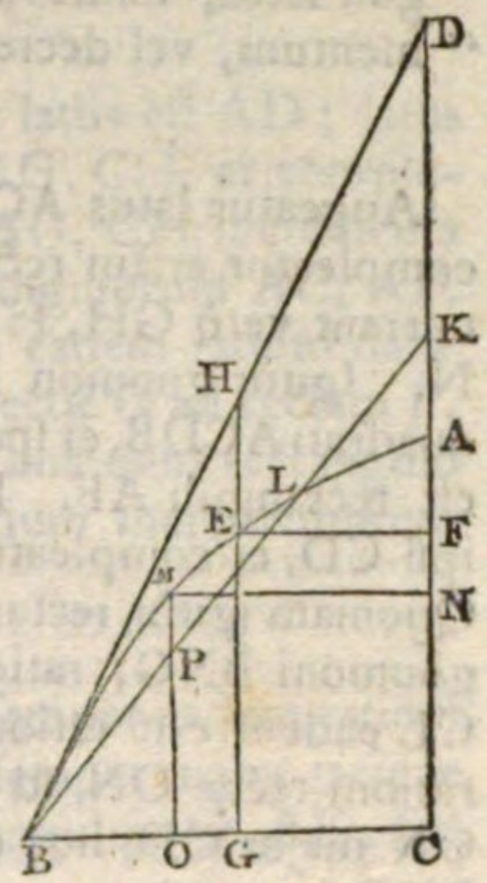
"In quâcunque curvâ, vel parte curvæ, à cujus quovis puncto duci poterit "unica tantùm recta curvam in eo puncto contingens, Limes rationis quam ha- "bet differentia abscissarum à quâvis diametro, ad differentiam rectarum quæ "diametro ordinatìm applicantur, eadem est rationi quam habet segmentum di- "ametri inter ordinatam et rectam quæ curvam in termino ordinatæ contingit, "ad ipsam ordinatam; hoc est, rationi quam habet subtangens ad ordinatam."

Demonstratio.

Cas. 1. Quando curva concava est versùs axem.

Sit enim curva AB; et à puncto in ipsâ B ducatur ordinatìm applicata BC ad diametrum

diametrum AC, ſumatur verò quodvis aliud in curvâ punctum E; et, primò, ſit E inter punctum B et diametri verticem A, ducatúrque ordinata EF ad diametrum, et compleatur parallelogrammum EFCG; contingat verò BD curvam in B, occurrátque diametro AC (productæ ultrà verticem A,) in D; contingenti autem occurrat GE in H. Eſt igitùr EG minor GH; quarè ratio EG ad GB minor erit ratione GH ad GB, hoc eſt, ratione DC ad CB. Et eſt EG differentia abſciſſarum AC, AF; GB verò differentia eſt ordinatarum BC, EF: oſtenſum igitùr eſt in hoc caſu, (ſcilicet, quando punctum E eſt inter B et A puncta,) differentiam abſciſſarum minorem habere rationem ad differentiam ordinatarum quam habet CD ad CB. Sit jam ratio quævis minor ratione CD ad CB, ſítque iſta ratio eadem rationi CK ad CB. Minor igitùr eſt CK quam CD. Jungatur KB, quæ curvæ neceſſariò occurret; occurrat ei in L, et ſumatur in curvâ punctum quodvis M inter B et L, ducatúrque ad diametrum ordinata MN, et ad BC ducatur MO parallela ipſi NC; occurrat verò MO rectæ BK in P. Eſt igitùr MO major ipſâ PO, quoniam, ſcilicet, BL eſt intrà curvam BML; quarè ratio MO ad OB major erit ratione PO ad OB, hoc eſt, ratione KC ad CB. Eſt verò MO differentia abſciſſarum AC, AN; OB verò eſt differentia ordinatarum BC, MN. Quoniam igitùr oſtenſum eſt rationem differentiæ abſciſſarum ad differentiam ordinatarum in omni caſu minorem eſſe ratione CD ad CB, ſed tamen poſſe fieri majorem quâcunque ratione CK ad CB, quæ minor ſit ratione CD ad CB; erit, per Def. 4, ratio CD ad CB limes rationis quam habet differentia abſciſſarum ad differentiam ordinatarum.



Si ſumatur punctum E ad alteras partes puncti B ad quas ſumptum fuit in præcedente caſu, ſimilitèr oſtendi poteſt rationem differentiarum prædictarum in omni caſu majorem eſſe ratione CD ad CB; ſed poſſe fieri ut minor ſit quâcunque ratione quæ major ſit ratione CD ad CB; unde et in hoc caſu ratio CD ad CB erit Limes rationis differentiarum. Q. E. D.

Cas. 2. Fiat alia figura in quâ curva ad axem convexa ſit: et, ſi pro *major* legatur *minor*, et vice versâ; oſtendetur hic caſus ſecundus iiſdem verbis ac primus. Q. E. D.

Cor. Si curva fuerit Parabola, Limes rationis differentiarum, ſeu ratio CD ad CB, eadem eſt rationi AC bis ad CB.

PROP. VI.

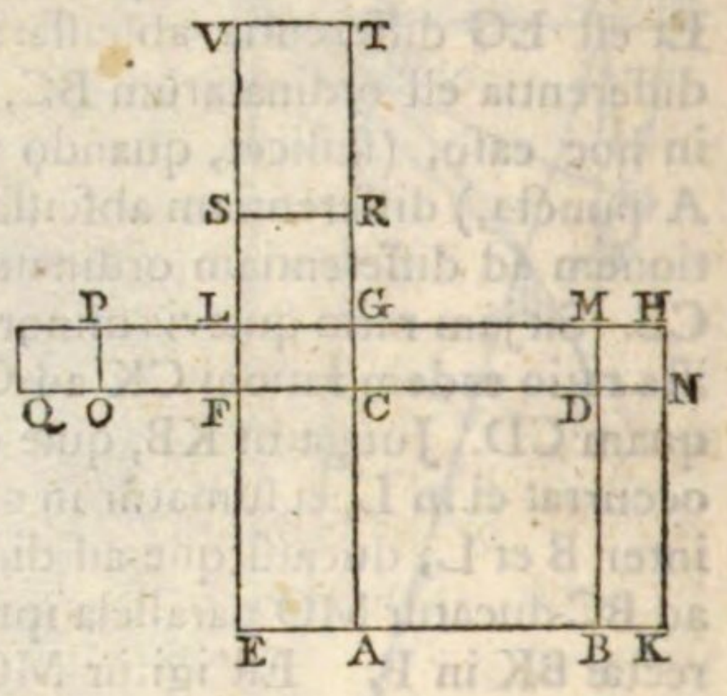
"Si quadratum ABDC et rectangulum AEFC habeant latus commune AC, "alterum verò rectanguli latus AE vel CF datum ſit; augeatur verò, vel minu-

"atur»

" atur, quomodocunque ipſorum latus commune AC, et ſimùl augeantur, vel mi-
" nuantur, ipſæ figuræ: ratio quam latus quadrati bis, habet ad datum rectan-
" guli latus, Limes erit rationis incrementi, vel decrementi, quadrati ad incre-
" mentum, vel decrementum, rectanguli."

Demonſtratio.

Augeatur latus AC et fiat AG; et, deſcripto ſuper AG quadrato AGHK, compleatur etiàm rectangulum AELG; occurrant verò GH, HK, ipſis BD, CD, in M, N. Igitùr gnomon BNG incrementum eſt quadrati ACDB, et ſpatium CL incrementum eſt rectanguli AF. Fiat autem CO æqualis ipſi CD, et compleatur rectangulum COPG. Quoniam igitùr rectangulum OH æquale eſt gnomoni BNG, ratio BNG ad rectangulum CL eadem erit rationi OH ad CL, hoc eſt, rationi rectæ ON ad ipſam FC. Eſt autem ON major OD, hoc eſt, CD bis; quarè ratio BNG, incrementi quadrati, ad CL, incrementum rectanguli, (in omni caſu quo latus commune augetur) major eſt ratione lateris quadrati bis ad FC datum rectanguli latus.



Q. E. D.

Oſtendendum reſtat, rationem incrementorum quadrati et rectanguli poſſe minorem fieri quâvis ratione quæ major eſt ratione CD bis ad FC. Sit enim quævis ratio QD ad FC major ratione CD bis, ſeu OD, ad FC; igitùr eſt QD major quàm OD. Producatur autem OD ad N, ut ON minor ſit ipsâ QD; et per N ducatur NK parallela ipſi AC; ſupèr AK deſcribatur quadratum AKHG, et compleatur rectangulum ONHP. Quoniam igitùr incrementum quadrati, ſeu gnomon BNG, æquale eſt rectangulo OH, erit BNG ad CL ut OH ad CL, hoc eſt, ut ON ad FC; ex conſtructione verò eſt ON minor QD; quarè ratio OH ad CL, hoc eſt, ratio incrementi quadrati ad incrementum rectanguli, minor eſt ratione datâ QD ad FC. Ergò, per Def. 4. ratio OD, hoc eſt, CD bis, ad CF, Limes eſt rationis dictorum incrementorum.

Eadem verò ratio Limes erit rationis decrementorum, ut patet. Q. E. D.

Cor. Hinc, ſi rectangulum ARSE, datum habens latus AE, æquale fuerit quadrato AD, et augeantur vel diminuantur latera ita ut rectangulum et quadratum ſemper inter ſe æqualia ſint; ratio lateris quadrati CD bis ad datum rectanguli latus CF Limes erit rationis incrementi lateris rectanguli ad incrementum lateris quadrati.

Nam augeatur latus quadrati AC et ipſum quadratum AD, et fiat latus AG et quadratum AH; ſit autem AV æquale ipſi AH; quare et RV æquale erit gnomoni CMK. Eſt autem, per hanc propoſitionem, ratio CD bis ad CF Limes rationis CMK ad CL, hoc eſt, rationis RV ad CL, hoc eſt, rationis RT ad CG, hoc eſt, rationis incrementi lateris rectanguli AR ad incrementum lateris quadrati AC. Q. E. D.

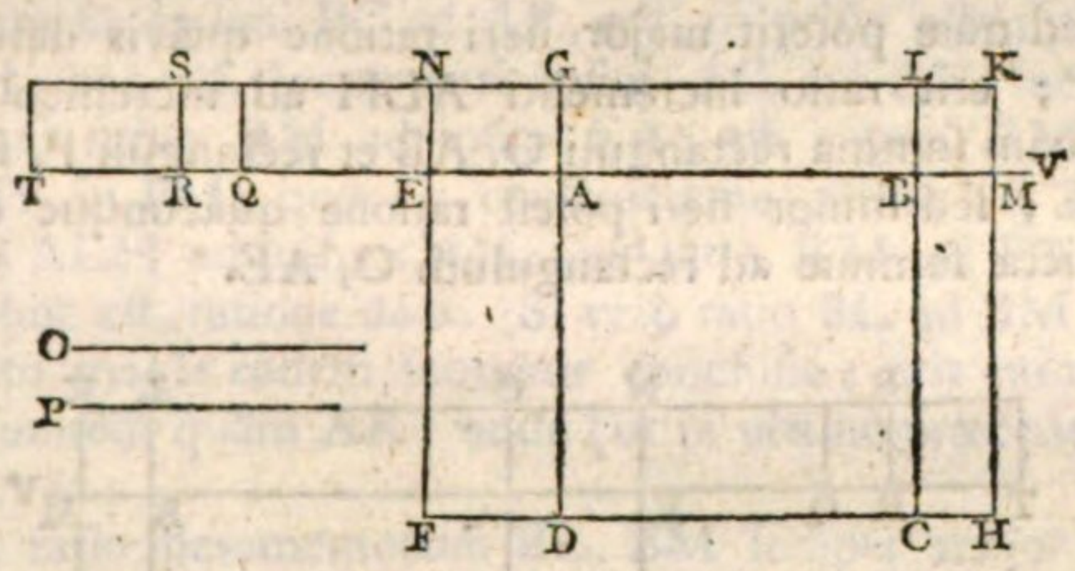
PROP.

PROP VII.

" Sint duo rectangula ABCD, AEFD, quibus commune latus est AD ; latus
" verò AE datum sit ; et augeantur latera DA, DC rectis AG, CH, et comple-
" antur rectangula GDHK, LBMK, AENG : sunt igitùr AG, CH incrementa
" laterum DA, DC ; et spatia ALH, AN incrementa rectangulorum AC, AF.
" Decrescant jàm GA, HC ; et vel sit ratio ipsarum semper eadem rationi datæ
" rectæ O ad datam P, vel Limes rationis ipsarum sit ratio rectæ O ad rectam P.
" His positis, erit ratio quam habet rectangulum, O, AB unà cum rectangulo
" P, BC ad rectangulum O, AE, Limes rationis decrescentium incrementorum
" ALH, AN."

Demonstratio.

Pars 1. Sit ratio incrementorum AG, CH, seu BL, BM, semper eadem rationi O ad P ; erit ratio incrementi ALH ad incrementum AN semper major ratione quam summa rectanguli O, AB et ipsius P, BC habet ad rectangulum O, AE ; sed



minor potest fieri ratione quâcunque quæ major est ratione rectanguli O, AB unà cum rectangulo P, BC ad rectangulum O, AE. Fiat enim ut O ad P ita BC ad AQ. Et, quoniam, ex hypothesi, est BL ad BM ut O ad P, erit BL, sive AG, ad BM ut BC ad AQ ; igitùr rectangulum QG æquale est ipsi BH ; et pro-inde (addito utrinque rectangulo AK,) erit rectangulum QK æquale spatio ALH : ratio igitùr spatii ALH ad rectangulum AN eadem est rationi rectanguli QK ad ipsum AN, hoc est, rationi rectæ QM ad ipsam EA. Quoniam verò est O ad P ut BC ad AQ, erit rectangulum O, AQ, æquale rectangulo P, BC ; additóque communi rectangulo O, AB, erit rectangulum O, QB æquale rectangulo O, AB simùl cum rectangulo P, BC ; igitùr rectangulum O, AB simùl cum ipso P, BC est ad ipsum O, EA ut O, QB ad O, EA, hoc est, ut recta QB ad ipsam EA. Ratio autem rectæ QM ad ipsam EA major est ratione rectæ QB ad EA ; igitùr ratio spatii ALH ad AN major est ratione rectanguli O, AB simùlcum ipso P, BC ad ipsum O, EA.

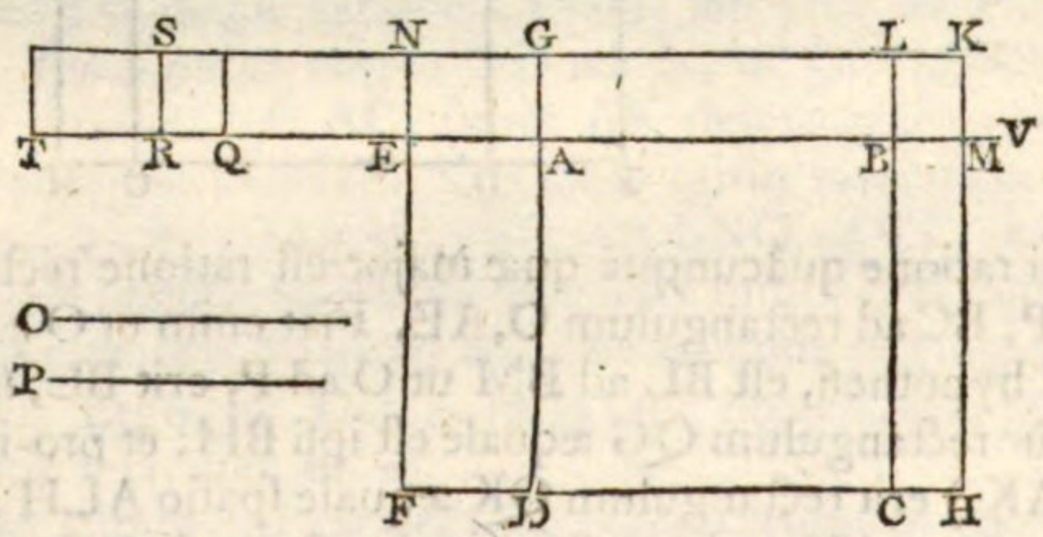
Ostendendum restat, rationem ALH ad AN posse fieri minorem ratione quâcunque quæ major est ratione rectanguli O, AB simùl cum ipso P, BC ad ipsum O, EA, hoc est, ratione rectæ QB ad ipsam EA. Sit ratio data ea quam habet

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recta QV ad ipsam EA; est igitùr recta QB minor ipsa QV. Sumatur quodvis punctum M inter B et V, et per M ducatur recta KMH parallela ipsi AD; et ut AQ ad BC, hoc est, ut P ad O, ita fiat BM ad MK, et compleantur parallelogramma DK, AN. Est igitùr spatium ALH æquale rectangulo QK, ut ostensum fuit; quarè ALH est ad rectangulum AN ut recta QM ad ipsam EA, hoc est, ratio spatii ALH ad rectangulum AN minor est ratione rectæ QV ad ipsam EA, hoc est, ratione datâ. Ratio igitùr quam habet rectangulum O, AB unà cum rectangulo P, BC ad rectangulum O, EA est Limes rationis incrementi ALH ad incrementum AN. Q. E. D.

Pars 2. Sint duo rectangula ABCD, AEFD, quibus commune latus est AD; latus verò AE datum sit: et augeantur latera DA, DC, rectis AG, CH, et compleantur rectangula GDHK, LBMK, AENG. Sunt igitùr AG, CH incrementa laterum DA, DC, et spatia ALH, AN, incrementa rectangulorum AC, AF. Decrescant jàm GA, CH, et Limes rationis ipsarum sit ratio rectæ O ad rectam P. His positis, erit ratio quam habet summa rectanguli O, AB et rectanguli P, BC ad rectangulum O, AE Limes rationis decrescentium incrementorum ALH, AN.

Primò. Sit ratio incrementorum AG, CH, seu BL, BM, semper minor ratione O ad P, sed quæ poterit major fieri ratione quâvis datâ quæ minor est ratione O ad P; erit ratio incrementi ALH ad incrementum AN semper major ratione quam summa rectanguli O, AB et rectanguli P, BC habet ad rectangulum O, AE; sed minor fieri potest ratione quâcunque datâ, quæ major est ratione prædictæ summæ ad rectangulum O, AE.



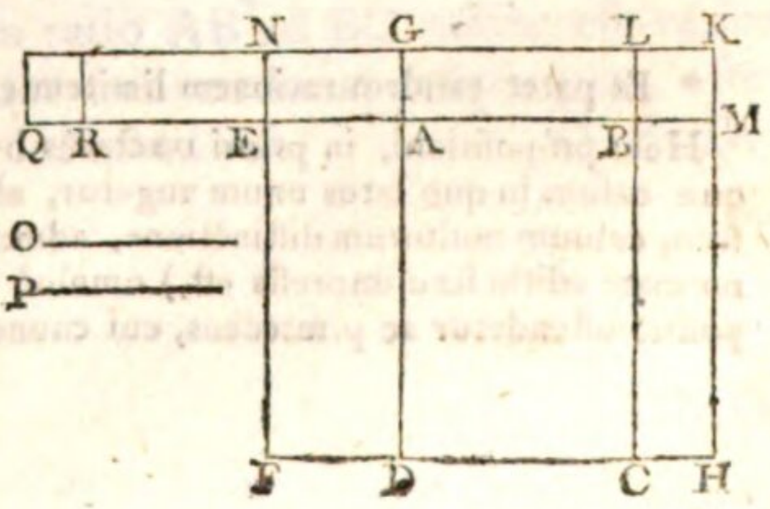
Fiat enim ut O ad P ita BC ad AQ, quæ ponatur in rectâ BA productâ. Est igitùr rectangulum P, BC æquale ipsi O, AQ, et propterea summa rectanguli O, AB et rectanguli P, BC, æquale est rectangulo O, QB. Est autem rectangulum O, QB ad rectangulum O, EA ut recta QB ad rectam EA. Sint autem AG, BM incrementa simùl facta rectarum AD, AB; et ut AG ad BM ita fiat BC ad AR. Quoniam igitùr, ex hypothesi, ratio AG ad BM minor est ratione O ad P, hoc est, ratione BC ad AQ, erit ratio BC ad AR minor ratione BC ad AQ; est igitùr recta AR major ipsâ AQ. Ducatur RS, parallela ipsi AG. Et, quoniam ut AG ad BM ita est BC ad AR, erit rectangulum AS æquale rectangulo BH; additóque communi

communi AK, erit rectangulum RK æquale ſpatio ALH: igitùr ratio quam habet ALH ad AN eadem eſt rationi RK ad AN, hoc eſt, rationi rectæ RM ad ipſam EA. Eſt autem RM major rectâ QB; quarè ratio RM ad EA major eſt ratione QB ad EA, hoc eſt, ratio incrementi ALH ad incrementum AN major eſt ratione quam habet ſumma rectanguli O, AB et ipſius P, BC ad ipſum O, AE.

Oſtendendum reſtat, rationem incrementi ALH ad ipſum AN poſſe minorem fieri ratione quâcunque datâ quæ major eſt ratione rectanguli O, AB ſimul cum ipſo P, BC ad ipſum O, AE, hoc eſt, ratione rectæ QB ad ipſam EA. Sit ratio data ea quam habet recta TB ad ipſam EA: eſt igitùr TB major quàm QB. Sumatur punctum quodvis R inter T et Q; producatur RB, et ſumatur BV recta, (incrementum, ſcilicet, ipſius AB,) minor quàm TR. Et, ſi ratio incrementi quod ſimùl fit ipſius AD ad BV non eadem fuerit rationi BC ad AR, vel major ipſâ, decreſcant incrementa ita ut ratio incrementi ipſius AD ad incrementum ſimůl factum ipſius AB eadem ſit rationi BC ad AR, vel ſit hâc ratione major; quod ex hypotheſi fieri poteſt, quoniam ratio BC ad AR minor eſt ratione BC ad AQ, hoc eſt, ratione O ad P. Sint hæc incrementa AG, BM, et compleantur rectangula, ut dictum fuit. Si igitùr ratio AG, hoc eſt, ipſius BI., ad BM eadem ſit rationi BC ad AR, erit, ut priùs oſtenſum eſt in Parte 1. ratio ſpatii ALH, hoc eſt, incrementi ipſius AC, ad AN, incrementum ipſius AF, eadem rationi rectæ RM ad ipſam EA: eſt autem RM minor ipſâ RV (decrevit enim BV in BM) quæ ex conſtructione minor eſt quàm TB: ratio igitùr incrementi ALH ad ipſum AN, ſive ratio RM ad EA, minor eſt ratione TB ad EA, hoc eſt, ratione datâ. Si verò ratio BL ad BM major ſit ratione BC ad AR, multo magìs eadem ſequetur concluſio; erit enim BL ad BM ut BC ad rectam minorem quam AR; unde (ut in ultimò præcedentibus) ſequetur concluſio fortiùs.

Secundo. Sit ratio incrementorum BL, BM ſemper major ratione O ad P, hoc eſt, ratione BC ad AQ, ſed quæ minor poterit fieri ratione quâcunque datâ quæ major eſt ratione BC ad AQ. In hoc caſu ratio incrementi rectanguli AC ad incrementum quod ſimùl fit ipſius AF, poteſt vel eadem eſſe rationi rectæ QB ad ipſam EA, vel major poteſt eſſe hâc ratione, vel eâdem minor.

Sint enim LB, BM incrementa laterum CB, AB ſimùl facta; et ut LB ad BM, ita fiat BC ad AR. Et, quoniam, ex hypotheſi, ratio LB ad BM, hoc eſt, BC ad AR, major eſt ratione BC ad AQ, erit AR minor quam AQ. Primò, ſit BM æqualis ipſi QR. Igitùr, additâ communi RB, erit RM æqualis QB, et ratio RM ad EA eadem erit rationi QB ad EA, hoc eſt, ratio ſpatii ALH, quod incrementum eſt ipſius AC, ad AN, incrementum ipſius

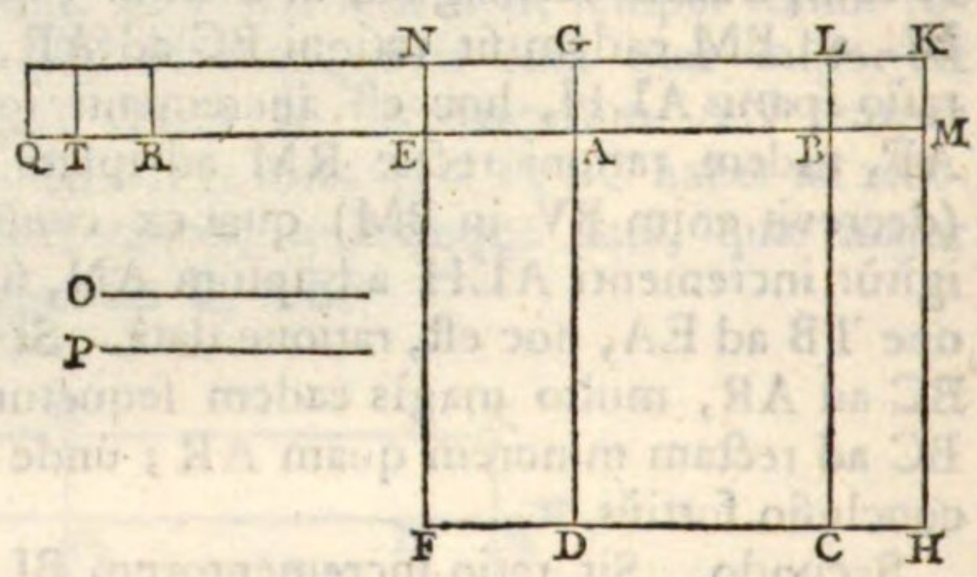


AF,

AF, eadem erit rationi QB ad EA, hoc eſt, rationi rectanguli O, AB unà cum ipſo P, BC ad rectangulum O, AE. Secundò, ſi BM major fuerit quàm QR, ſimilitèr oſtendetur ratio RM ad EA major ratione QB ad AE. Tertiò, ſi BM minor fuerit quam QR, ſimilitèr oſtendetur ratio RM ad EA minor ratione QB ad EA.

Oſtendendum verò eſt in ſecundo caſu (in quo, ſcilicet, ratio RM ad EA, hoc eſt, ratio quam habet incrementum ALH rectanguli AC ad AN, incrementum rectanguli AF, major eſt ratione QB ad AE,) poſſe rationem ALH ad ipſum AN minorem fieri quâcunque ratione datâ quæ major eſt ratione QB ad EA. Sit hæc ratio data ea quam habet recta QM ad ipſam EA. Eſt igitùr QM major quàm QB; quando verò BM eſt incrementum ipſius AB, ſit LB incrementum ipſius CP; et ut LB ad BM ita fiat CB ad AR. Eſt igitùr AR minor quàm AQ, ut priùs eſt oſtenſum: erit autem ratio ſpatii ALH ad ipſum AN eadem rationi rectæ RM ad ipſam EA, quæ minor eſt ratione rectæ QM ad EA, hoc eſt, ratione datâ.

In tertio caſu (in quo, ſcilicet, ratio RM ad EA minor eſt ratione QB ad EA,) oſtendendum eſt poſſe rationem ſpatii ALH, quod incrementum eſt ipſius AC, ad ipſum AN, majorem fieri quâcunque ratione datâ quæ minor eſt ratione QB ad EA. Sit hæc ratio data ea quam habet RB ad EA. Eſt igitùr RB minor quàm QB; quarè et RA minor eſt quàm QA, et propterea ratio BC ad RA major eſt ratione BC ad AQ. Ex hypotheſi autem poteſt ratio incrementi rectæ AD ad incrementum rectæ AB minor fieri quâcunque ratione quæ major eſt ratione BC ad AQ: ſint igitùr AG, ſive LB, et BM, earum incrementa, quæ minorem habent rationem quàm BC ad RA; et ut LB ad BM, ita fiat BC ad TA: igitùr ratio BC ad TA minor eſt ratione BC ad RA, et pro-inde erit TA major quàm RA, et TB major quàm RB; quarè TM multo major eſt quam RB. Eſt igitùr ratio TM ad AE, hoc eſt, ratio incrementi ALH rectanguli AC ad AN incrementum rectanguli AF, major ratione RB ad AE, hoc eſt, ratione datâ *. Q. E. D.



* Et patet eandem rationem limitem eſſe rationis decrementorum, quando latera diminuuntur.

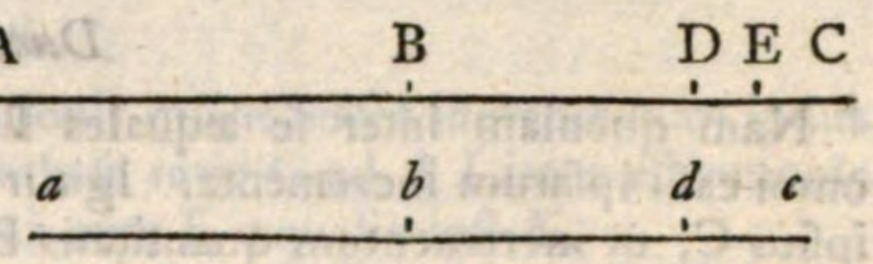
Huic propoſitioni, in priori tractatûs hujus exemplari MS. addita fuit alia etiam de rectangulo, quæ caſum in quo latus unum augetur, alterum verò diminuitur, complectebatur. Hæc verò propoſitio, caſuum multorum diſtinctione, admodùm prolixa deveniebat, et in exemplari noviſſimo (ad cujus normam editio hæc impreſſa eſt,) omninò omiſſa fuit. Eodem verò prorsùs ratiocinii modo hæc propoſitio oſtendetur ac præcedens, cui enunciatio propoſitioni conveniens eſt præfixa.

PROP.

PROP. VIII.

" Si duæ rationes variabiles ſint ſemper inter ſe eædem, erunt rationes quæ " ipſarum ſunt Limites inter ſe eædem."

Rationis AB ad BC limes ſit ratio AB ad BD, hoc eſt, ſit ratio AB ad BC ſemper minor ratione AB ad BD, ſed quæ poterit fieri major quâvis ratione quæ minor eſt ratione AB ad BD; vel, è contrà ſit ratio AB ad BC ſempèr major ratione AB ad BD, ſed quæ minor fieri poterit quâvis ratione quæ major eſt ratione AB ad BD. Et eodem modo rationis *ab* ad *bc* (quæ rationi AB ad BC eadem eſt,) limes ſit ratio *ab* ad *bd*. Erit ratio AB ad BD eadem rationi *ab* ad *bd*.



Demonſtratio.

Primò, ſit ratio AB ad BC minor ratione AB ad BD, hoc eſt, appropinquet ad limitem ſuum ratio AB ad BC augendo. Igitùr quoniam ratio *ab* ad *bc* ſemper eadem eſt rationi AB ad BC, neceſſe eſt ut etiam ratio *ab* ad *bc* appropinquet ad limitem ſuum augendo, hoc eſt, ut ratio *ab* ad *bc* minor ſit ratione *ab* ad *bd*. Si autem eadem non ſint inter ſe rationes AB ad BD, *ab* ad *bd*, una ipſarum erit alterâ major. Sit ratio AB ad BD major ratione *ab* ad *bd*; fiátque ratio AB ad BE eadem rationi *ab* ad *bd*. Quoniam igitùr ratio AB ad BE minor eſt ratione AB ad BD; poterit ratio AB ad BC major fieri ratione AB ad BE. Fiat ita major: et ratio *ab* ad *bc* (quæ ſemper eſt eadem rationi AB ad BC,) ſimùl major fiet ratione AB ad BE, hoc eſt, ratione *ab* ad *bd*. Sed ratio *ab* ad *bc* ſemper minor eſt ratione *ab* ad *bd*, ut oſtenſum fuit; quod abſurdum eſt. Eſt igitùr ratio BA ad AD eadem rationi *ab* ad *bd*. Q. E. D.

Eodem modo in altero caſu, viz. quando ratio AB ad BC major eſt ratione AB ad BD, quæ ipſius eſt limes, et ad ipſum minuendo appropinquat, oſtendetur, rationem AB ad BD eandem eſſe rationi *ab* ad *bd*. Q. E. D.

PROP.

PROP. IX.

"Sint A et B quantitates variabiles utcunque ex aliis quantitatibus com-
" positæ ; sint autem in omni casu æquales : et sit C quantitas quævis alia varia-
" bilis : et ratio α ad γ limes sit rationis quam incrementum ipsius A habet ad
" incrementum quantitatis C; ratio verò β ad γ limes sit rationis incrementorum
" ipsarum B, C ; erit α æqualis ipsi β."

Demonstratio.

Nam quoniam inter se æquales sunt quantitates A, B, æqualia erunt in omni casu ipsarum incrementa. Igitur incrementum ipsius A est ad incrementum ipsius C, ut incrementum quantitatis B ad idem ipsius C incrementum. Rationes verò quæ earundum rationum sunt Limites inter se sunt eædem, per Prop. 8. et est ratio α ad γ Limes rationis incrementi ipsius A ad incrementum ipsius C, et ratio β ad γ Limes rationis incrementi ipsius B ad incrementum ipsius C; ergo est α ad γ ut β ad γ, ideóque α æqualis est ipsi β. Q. E. D.

Exemp. 1. Sit rectangulum A, B æquale rectangulo C, D; sintque latera A, B, C, quantitates variabiles; latus verò D sit invariabilis. Et sit A, E aliud rectangulum cujus latus E est quantitas invariabilis: ratio verò α ad β sit Limes rationis incrementorum laterum A, B, et ratio α ad γ Limes rationis incrementorum laterum A, C. Erit [per Prop. 7.] ratio A, β; B, α simùl, ad E, α Limes rationis incrementi rectanguli A, B ad incrementum rectanguli A, E. Est autem D, γ ad E, α Limes rationis incrementorum rectangulorum C, D, et A, E. Ergò, per hanc Prop. est A, β unà cum B, α æquale ipsi D, γ.

Ex. 2. Sit rectangulum A, B æquale quadrato ex C; sintque A, B, C, quantitates variabiles, et rationis incrementorum ipsarum A, B Limes sit ratio α ad β, rationis verò incrementorum ipsarum A, C Limes sit ratio α ad γ. Aliud autem rectangulum sit D, E, cujus latus D est quantitas invariabilis, E verò variabilis; sitque ratio α ad ε Limes rationis incrementorum ipsarum A, E. Erit per Prop. 7. ratio $A\beta + B\alpha$ ad $D\varepsilon$ Limes rationis incrementorum ipsorum A, B, D, E: et, per Prop. 6. ratio $2C\gamma$ ad $D\varepsilon$ Limes rationis incrementorum quadrati ex C et ipsius D, E; ergò, per hanc, erit $A\beta + B\alpha = 2C\gamma$.

PROP.

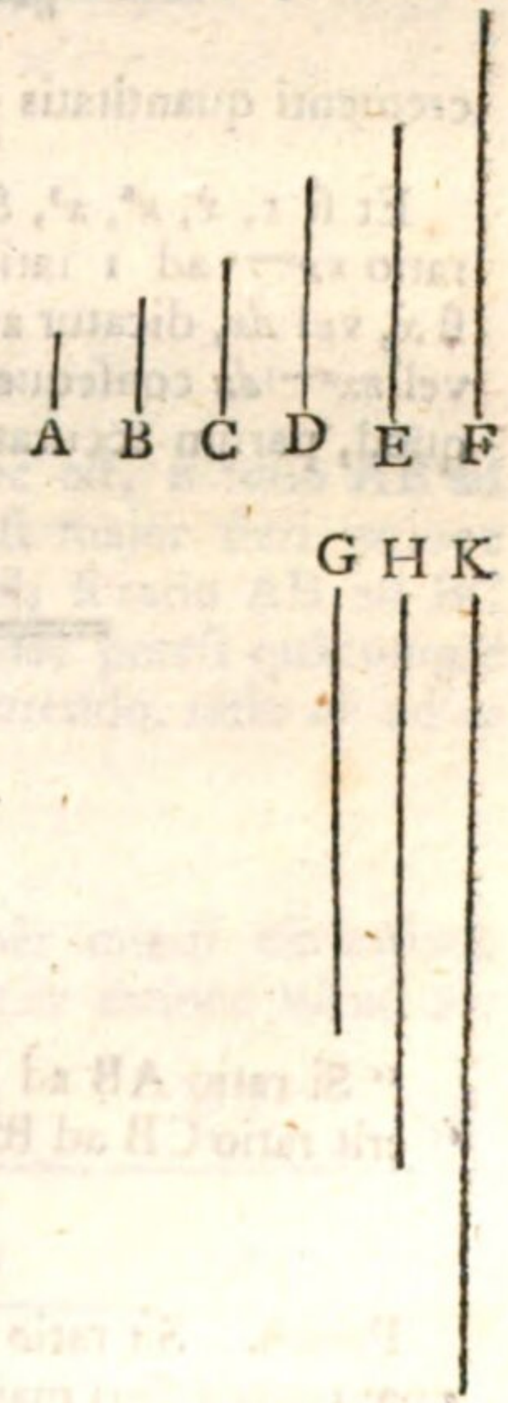
PROP. X.

" Sint rectæ quotcunque A, B, C, D, E, F, deinceps proportionales, quarum prima A data ſit; auctâ verò ſecundâ B, augeantur ſimùl reliquæ ita ut proportionales ſemper ſint; et ratio quæ Limes eſt rationis incrementorum penultimæ E et ſecundæ B eadem ſit rationi quam habet G, (multiplex, ſcilicet, antepenultimæ D,) ad ipſam A; ſíque H eadem multiplex ipſius E atque G ipſius D. Erit ratio quæ Limes eſt rationis incrementorum ultimæ F et ſecundæ B eadem rationi quam H et E ſimùl habent ad primam A."

Demonſtratio.

Sit ratio quam habet K ad A Limes rationis quam habet incrementum ipſius F ad incrementum ipſius B; et, ex hypotheſi, ratio G ad A Limes eſt rationis incrementorum ipſarum E, B. Erit H unà cum E æqualis ipſi K.

Quoniam enim ut E ad B ita eſt D ad A, et ſunt H, G ipſarum E, D æque multiplices, erit [Cor. 4. 5.] H ad B ut G ad A, et rectangulum B, G æquale erit rectangulo A, H: eſt autem ratio K ad A, hoc eſt, ratio rectanguli A, K ad quadratum ex A, Limes rationis incrementorum ipſarum F, B, hoc eſt, rationis incrementorum rectangulorum A, F et A, B. Et quoniam rectangula E, B; A, B habent latus commune B; et eſt recta A data; et, ex hypotheſi, ratio G ad A Limes eſt rationis incrementorum ipſarum E, B; et, invertendo, ratio A ad G eſt Limes rationis incrementorum ipſarum B, E.* Igitùr, per Prop. 7. ratio rectanguli A, E unà cum rectangulo G, B ad quadratum ex A Limes eſt rationis incrementi rectanguli E, B ad incrementum ipſius A, B. Quoniam igitùr duo ſunt rectangula A, F; E, B inter ſe æqualia, ratio autem quæ Limes eſt rationis incrementi ipſius A, F ad incrementum tertii rectanguli A, B oſtenſa eſt eadem rationi rectanguli A, K ad quadratum ex A; et ratio quæ Limes eſt rationis incrementorum rectangulorum E, B; A, B oſtenſa eſt eadem rationi rectanguli A, E unà cum ipſo G, B ad idem quadratum ex A; erit per Prop. 9. rectangulum A, K æquale rectangulo A, E unà cum ipſo G, B, hoc eſt, ut oſtenſum eſt, unà cum ipſo A, H. Eſt igitùr recta K æqualis ipſi E unà cum H. Q. E. D.



* Per Prop. ſequentem quæ non ab hac pendet.

COR. 1.

Cor. 1. Hinc in ſerie deinceps proportionalium, A, B, C, D, E, &c. ſi ratio *c* ad *b* Limes ſit rationis incrementorum ipſarum C, B; et ratio *d* ad *b* Limes ſit rationis incrementorum ipſarum D, B, et ita deinceps, poſitis, ſcilicet, *c*, *d*, *e*, *f*, antecedentibus rationum limitum ad communem conſequentem *b*, ſítque *n* numerus exponens locum cujuſvis termini F, ſeu ejuſdem diſtantiam à primo A incluſivè, adeóque $n-1$ numerus exponens locum termini antecedentis E; erit *f* ad *b* ut $\overline{n-1} \times$ E ad A.

Quoniam enim rectangulum A, C æquale eſt quadrato ex B, et data eſt recta A, erit (per Cor. Prop. 6.) ratio *c* ad *b* (quæ eſt Limes rationis incrementorum ipſarum C, B) eadem rationi 2B ad A: igitùr, per hanc propoſitionem, erit *d* ad *b* ut 2C + C, hoc eſt, 3C, ad A; et ſimilitèr erit *e* ad *b* ut 4D ad A; et ita deinceps: ſcilicet, ſi *n* ſit numerus exponens locum cujuſvis termini F, ſeu ejuſdem diſtantiam à primo A incluſivè, erit *f* ad *b* ut $\overline{n-1} \times$ E ad A.

Cor. 2. Et ſi a, x, $\frac{x^2}{a}$, $\frac{x^3}{a^2}$, $\frac{x^4}{a^3}$, &c. ſit ſeries deinceps proportionalium, quarum ultima ſit $\frac{x^n}{a^{n-1}}$; quoniam $n+1$ eſt numerus termini $\frac{x^n}{a^{n-1}}$, erit n numerus termini penultimi $\frac{x^{n-1}}{a^{n-2}}$, et propterèa ratio $n \times \frac{x^{n-1}}{a^{n-2}}$ ad a erit Limes rationis incrementi quantitatis $\frac{x^n}{a^{n-1}}$ ad incrementum quantitatis x.

Et ſi 1, x, x^2, x^3, &c. ſint deinceps proportionales, quarum ultima ſit x^n; erit ratio nx^{n-1} ad 1 ratio quæ Limes eſt rationis incrementorum ipſarum x^n, x. Et ſi $\dot{x}$, vel dx, dicatur antecedens rationis incrementorum ipſarum x, x^n; erit $nx^{n-1}\dot{x}$ vel $nx^{n-1}dx$ conſequens ejuſdem rationis. Et hæc $nx^{n-1}\dot{x}$ vel $nx^{n-1}dx$ eſt illud quod, parùm accurato loquendi modo, vocatur *fluxio*, vel *differentia*, ipſius x^n.

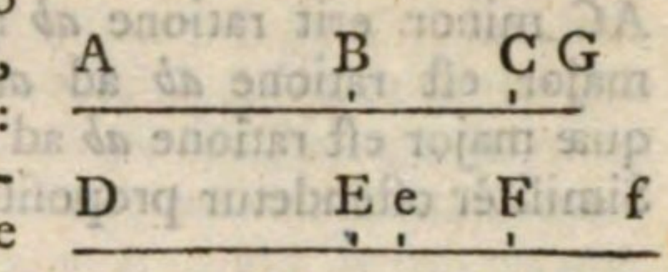
PROP. XI.

" Si ratio AB ad BC Limes ſit rationis variabilis DE ad EF; invertendo, " erit ratio CB ad BA Limes rationis FE ad ED."

Demonſtratio.

Primò. Sit ratio variabilis DE ad EF ſemper minor ratione AB ad BC, ſed quæ poterit fieri major quâvis ratione quæ minor eſt ratione AB ad BC. Invertendo igitùr erit ratio FE ad ED ſemper major ratione CB ad BA: prætereà verò

verò oſtendendum eſt poſſe rationem FE ad ED minorem fieri quavis ratione, quæ major eſt ratione CB ad BA. Sit ratio GB ad BA major ipſâ CB ad BA; et, invertendo, erit ratio AB ad BG minor ratione AB ad BC: ex hypotheſi autem poterit ratio DE ad EF major fieri quâvis ratione quæ minor eſt ipſâ ratione AB ad BC; poterit igitùr ratio DE ad EF major fieri ratione AB ad BG. Fiat ita major ratione AB ad BG; et ſit iſta major ratio ea quam habet *De* ad *ef*; ergò, invertendo, ratio *fe* ad *e*D minor erit ratione GB ad BA. Sed, ut oſtenſum fuit, eſt ratio *fe* ad *e*D ſemper major ratione CB ad BA. Quare [per Definitionem 4tam, ſcilicet, limitis rationis.] ratio CB ad BA Limes eſt rationis *fe* ad *e*D, ſeu FE ad ED. Q. E. D.



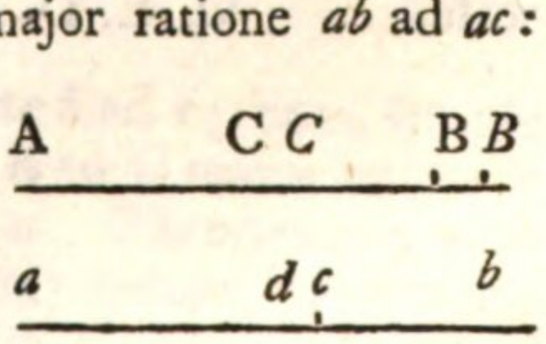
Similitèr, ſi ratio DE ad EF ſemper major fuerit ratione AB ad BC, ſed quæ minor poterit fieri quâvis ratione, quæ ratione AB ad BC major eſt; oſtendetur rationem CB ad BA limitem eſſe rationis FE ad ED.

PROP. XII.

" Si rationis AB ad BC Limes ſit ratio *ab* ad *bc*, hoc eſt, ſi ratio AB ad " BC ſemper minor ſit ratione *ab* ad *bc*, ſed quæ poteſt major fieri ratione " quâcunque datâ quæ minor eſt ratione *ab* ad *bc*; vel, ſi ratio AB ad BC " ſempèr major ſit ratione *ab* ad *bc*, ſed quæ minor fieri poteſt quâcunque " ratione datâ quæ major eſt ratione *ab* ad *bc*: erit, convertendo, ratio *ab* ad *ac* " Limes rationis AB ad AC."

Demonſtratio.

Nam in primo caſu, quoniam ratio AB ad BC ſempèr minor eſt ratione *ab* ad *bc*; convertendo, erit ratio AB ad AC ſempèr major ratione *ab* ad *ac*: oſtendendum verò reſtat, poſſe rationem AB ad AC, minorem fieri quâcunque ratione datâ, quæ major eſt ratione *ab* ad *ac*. Sit ratio *ab* ad *ad* major ratione *ab* ad *ac*: et, convertendo, erit ratio *ab* ad *bd* minor ratione *ab* ad *bc*; ex hypotheſi autem ratio AB ad BC poteſt fieri major quâcunque ratione *ab* ad *bd* quæ minor eſt rati-



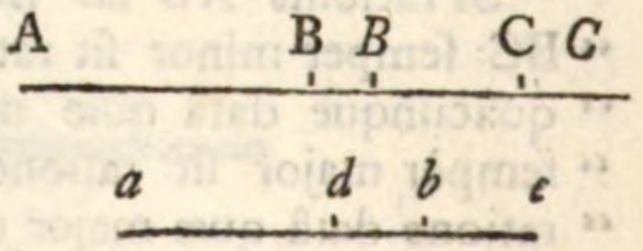
one *ab* ad *bc*; fiat, et fit ratio *AB* ad *BC*; igitùr, convertendo, ratio *AB* ad *AC* minor erit ratione *ab* ad *ad*. Quoniam igitùr ratio AB ad AC ſempèr major eſt ratione *ab* ad *ac*, ſed minor fieri poteſt quâcunque ratione datâ, quæ major eſt ratione *ab* ad *ac*; erit ratio *ab* ad *ac* Limes rationis AB ad AC. Similitèr oſtendetur propoſitio in caſu reliquo. Q. E. D.

PROP. XIII.

" Si rationis AB ad BC Limes ſit ratio *ab* ad *bc*, hoc eſt, ſi ratio AB ad " BC ſempèr minor ſit ratione *ab* ad *bc*, ſed quæ major fieri poteſt quâ- " cunque ratione datâ, quæ minor eſt ratione *ab* ad *bc*; vel, ſi ratio AB ad " BC ſempèr major ſit, &c. Componendo, rationis AC ad CB Limes erit " ratio *ac* ad *cb*."

Demonſtratio.

Nam, quoniam ratio AB ad BC ſempèr minor eſt ratione *ab* ad *bc*, componendo, ratio AC ad CB ſempèr minor erit ratione *ac* ad *cb*: oſtendendum verò reſtat, rationem AC ad CB majorem fieri poſſe quâcunque ratione datâ, quæ minor eſt ratione *ac* ad *cb*. Sit ratio data *ac* ad *cd* minor ratione *ac* ad *cb*; et, dividendo, ratio *ad* ad *dc* minor erit ratione *ab* ad *bc*; igitùr ratio AB ad BC major fieri poteſt ratione *ad* ad *dc*. Fiat ita major ratione *ad* ad *dc*, et deveniat ratio *AB* ad *BC*; igitùr, componendo, ratio *AC* ad *CB* major erit ratione *ac* ad *cd*, hoc eſt, ratione datâ; et propterèà rationis AC ad CB Limes eſt ratio *ac* ad *cb*. Et ſimilitèr in reliquo caſu. Q. E. D.



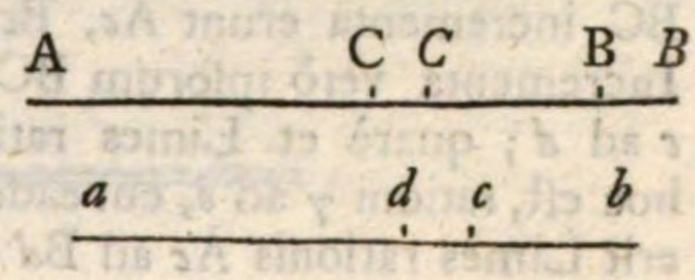
PROP.

PROP. XIV.

" Si rationis AB ad BC Limes ſit ratio *ab* ad *bc*, hoc eſt, ſi ratio AB ad " BC ſempèr minor ſit ratione *ab* ad *bc*, &c. Dividendo, rationis AC ad " CB Limes eſt ratio *ac* ad *cb*."

Demonſtratio.

Nam, quoniam ratio AB ad BC ſempèr minor eſt ratione *ab* ad *bc*, dividendo, erit ratio AC ad CB ſempèr minor ratione *ac* ad *cb*: oſtendendum reſtat, rationem AC ad CB majorem fieri poſſe quâcunque ratione datâ, quæ minor eſt ratione *ac* ad *cb*. Sit ratio data *ad* ad *db* minor ratione *ac* ad *cb*: et, componendo, ratio *ab* ad *bd* minor erit ratione *ab* ad *bc*: ex hypotheſi autem ratio AB ad BC major fieri poteſt ratione quâcunque datâ *ab* ad *bd*, quæ minor eſt ratione *ab* ad *bc*. Fiat ita major ratione *ab* ad *bd*, et deveniat ratio *AB* ad *BC*; et dividendo, ratio *AC* ad *CB* major erit ratione *ad* ad *db*, hoc eſt, ratione datâ. Quoniam igitùr ratio AC ad CB ſempèr minor eſt ratione *ac* ad *cb*, ſed major fieri poteſt ratione quâcunque datâ *ad* ad *db* quæ minor eſt ratione *ac* ad *cb*, erit rationis AC ad CB Limes ratio *ac* ad *cb*. Et ſimilitèr in reliquo caſu. Q. E. D.



PROP. XV.

" Si ratio A ad B ſempèr eadem fuerit rationi *a* ad *b*; rationis autem B " ad C Limes ſit ratio *b* ad *c*; erit ratio *a* ad *c* Limes rationis A ad C."

Demonſtratio.

Sit ratio B ad C ſempèr minor, vel ſempèr major, ratione *b* ad *c*; puta, ſempèr minor. Quoniam igitùr eſt A ad B ut *a* ad *b*, ratio verò B ad C minor ratione *b* ad *c*; erit, ex æquo, ratio A ad C minor ratione *a* ad *c*. Oſtendendum verò prætereà eſt, rationem A ad C majorem fieri poſſe quâvis ratione, quæ minor eſt ipſâ *a* ad *c*. Sit igitùr ratio *a* ad *d* minor ratione *a* ad *c*; unde erit *d*

major quàm *c*; quarè ratio *b* ad *d* minor erit ratione *b* ad *c*, quæ, ſcilicet, Limes eſt rationis B ad C; ergò, per definitionem quartam, poterit ratio B ad C major fieri ratione *b* ad *d*. Fiat itaque: et, quoniam ratio A ad B fit ſimùl eadem rationi *a* ad *b*; erit, ex æquo, ratio A ad C major ratione *a* ad *d*, hoc eſt, ratione propoſitâ. Quoniam igitùr ratio A ad C ſempèr eſt minor ratione *a* ad *c*, poterit autem fieri major quâvis ratione *a* ad *d* quæ minor eſt ratione *a* ad *c*; erit (per definitionem) ratio *a* ad *c* Limes rationis A ad C. Q. E. D.

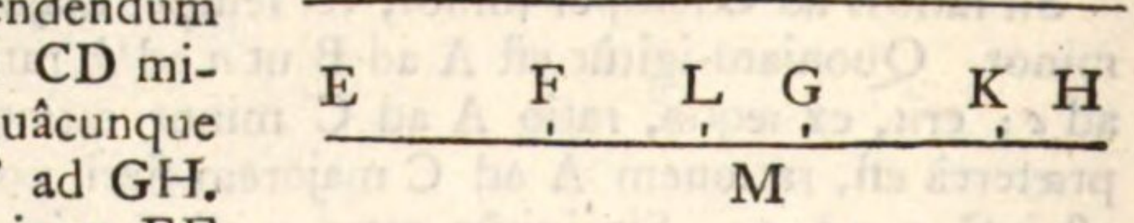
Cor. Sint AC, BD rectangula quæ data habent latera A, B; Limes autem rationis incrementorum laterum variabilium C, D, ſit ratio γ ad δ; erit Limes rationis incrementorum ipſorum AC, BD, eadem rationi quæ componitur ex rationibus A ad B et γ ad δ. Denotentur enim incrementa quævis ſimùl facta laterum C, D, rectis *c*, *d*; et ſumatur rectangulum BC. Ipſorum igitùr AC, BC incrementa erunt A*c*, B*c*, quorum ratio ſempèr eadem eſt rationi A ad B. Incrementa verò ipſorum BC, BD, ſunt B*c* B*d*; quorum ratio eſt eadem ipſi *c* ad *d*; quarè et Limes rationis B*c* ad B*d* eadem eſt Limiti ipſius *c* ad *d*, hoc eſt, rationi γ ad δ, cui eadem fiat ratio B ad E; et, per hanc Propoſitionem, erit Limes rationis A*c* ad B*d* eadem rationi A ad E, hoc eſt, ei quæ compoſita eſt ex rationibus A ad B, et (B ad E, hoc eſt) γ ad δ.

PROP. XVI.

"Si rationis AB ad BC Limes ſit ratio EF ad FG, et rationis BC ad "CD Limes ſit ratio FG ad GH, utraque verò ratio augeatur vel minuatur. "Erit ratio EF ad GH Limes rationis AB ad CD."

Demonſtratio.

Utraque enim ratio minuatur. Et, ex definitione Limitis rationis, erit ratio AB ad BC ſempèr major ratione EF ad FG, et ratio BC ad CD ſempèr major ratione FG ad GH. Igitùr, ex æquali, ratio AB ad CD ſempèr major erit ratione EF ad GH. Oſtendendum verò reſtat, rationem AB ad CD minorem fieri poſſe ratione quâcunque datâ, quæ major eſt ratione EF ad GH. Sit ratio EF ad GK major ratione EF ad GH: eſt igitùr GH major GK: fiat ut FG ad GH ita quædem FL ad GK;



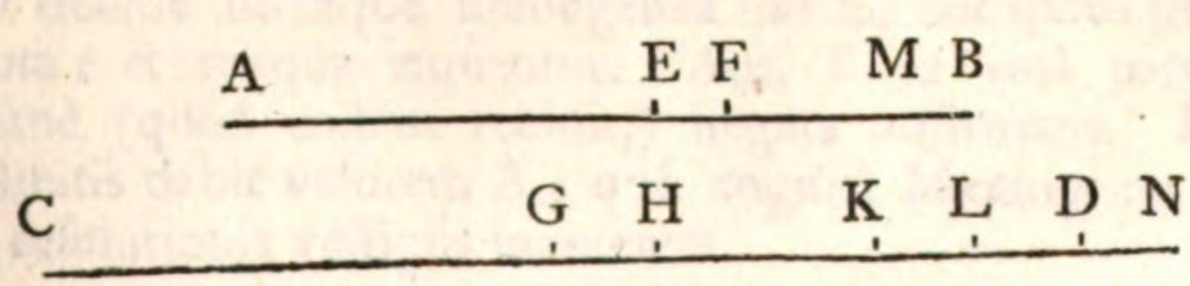
GK; et, quoniam GH major eſt GK, erit FG major FL (14. 5.) Sumatur FM, quæ minor ſit quàm FG, major autem quàm FL; erit igitùr ratio EF ad FM major ratione EF ad FG, ratio autem FM ad GK major erit ratione FL ad GK. Quoniam verò rationis decreſcentis AB ad BC Limes eſt ratio EF ad FG, poterit ratio AB ad BC minor fieri quâcunque ratione datâ, quæ major eſt ratione EF ad FG. Decreſcat igitùr ratio AB ad BC ita ut minor fiat ratione EF ad FM: et, quoniam rationis decreſcentis BC ad CD Limes eſt ratio FG ad GH, poteſt ratio BC ad CD minor fieri ratione quâcunque datâ, quæ major eſt ratione FG ad GH, hoc eſt, ratione FL ad GK: fiat igitùr ratio BC ad CD minor ratione FM ad GK. Quoniam igitùr ratio AB ad BC minor facta eſt ratione EF ad FM, ratio autem BC ad CD minor facta eſt ratione FM ad GK, erit ratio AB ad CD, ex æquali, minor ratione EF ad GK, hoc eſt, ratione datâ. Similitèr oſtendetur caſus reliquus quando ratio utraque augetur, invertendo. Q. E. D.

PROP. XVII*.

"Sint duæ magnitudines AB, CD, síntque aliæ magnitudines AE, AF, &c. "minores quidem ipſâ AB, ſed quæ majores poſſunt fieri quâvis magnitudine, "quæ ipſâ AB ſit minor; et ſint magnitudines CG, CH, &c. minores qui- "dem ipſâ CD, ſed quæ majores poſſunt fieri quâvis magnitudine quæ minor "ſit ipſâ CD; ſit autem ratio ipſius AE ad CG vel AF ad CH, &c. ſempèr "eadem rationi datæ; huic datæ rationi eadem erit ratio AB ad CD."

Demonſtratio.

Si enim non ſit ut AE ad CG ita AB ad CD; erit AE ad CG ut AB vel ad magnitudinem minorem ipſâ CD, vel ad majorem. Sit, primò, ut AB ad minorem ipſâ CD; sítque AE ad CG ut AB ad CK. Quoniam igitùr CK minor eſt ipſâ CD, poteſt quædam ipſarum CG, CH, &c. major fieri ipſâ CK. Sit hæc CL; sítque AM ea ipſarum AE, AF, &c. magnitudinum, quæ ad CL datam habet rationem quam habet AE ad CG. Ut autem AE ad CG, ita eſt AB ad CK; ergò ut AB ad CK, ita eſt AM ad CL. Ex hypotheſi au-



* Propoſitio hæc utiliſſima, etiàm de Limitibus quantitatum, præcedenti ſeriei adjungitur.

tem

tem AB major est AM ; major igitùr est CK quam CL. [14. 5.] sed et minor, quod fieri non potest. Non igitùr est AE ad CG ut AB ad aliquam magnitudinem quæ minor est ipsâ CD. Eâdem ratione ostendetur neque esse CG ad AE, ut CD ad aliquam magnitudinem minorem ipsâ AB. Ostendendum restat neque esse ut AE ad CG, ita AB ad aliquam magnitudinem quæ major est ipsâ CD. Si enim fieri potest, sit AE ad CG ut AB ad CN majorem ipsâ CD. Invertendo, igitùr, erit CG ad AE ut CN ad AB. Quoniam verò est CN major CD, erit CN ad AB, ut CD ad magnitudinem ipsâ AB minorem [14. 5.] Ergò ut CG est ad AE, ita CD ad magnitudinem minorem ipsâ AB ; quod fieri non posse ostensum est. Non igitùr ut AE ad CG ita est AB ad magnitudinem aliquam ipsâ CD majorem : ostensum autem est neque ad minorem : quarè ut AE ad CG ita est AB ad CD. Q. E. D.

Hæc est propositio generalis quæ adhibetur in Propositionibus 2, 5, 11, 12, 18, Lib. 12. Euclidis, et in aliis multis propositionibus in operibus aliorum Geometriæ Scriptorum ; est autem propositio maximæ utilitatis, quia, cum semèl demonstrata est, (ut jam factum est,) istas omnes propositiones breviores potest reddere.

Finis Tractatûs ROBERTI SIMSON *de Limitibus Quantitatum et Rationum.*

EXCERPTUM BREVE

EX OPERIBUS DOCTISSIMI VIRI,

PETRI FERMATII, GALLI,

(Galliæ Regis illuſtriſſimi, Ludovici decimi quarti, in Curiâ Parliamenti ſui apud Toloſam Conſiliarii,)

IN QUO DESCRIBITUR METHODUS AB EO INVENTA, AD INVESTIGANDAM

MAXIMAM ET MINIMAM

Magnitudinem quam Quantitas aliqua variabilis (quæ non ad infinitum creſcit vel decreſcit,) habere poſſit.

Vide FERMATII opera Toloſæ impreſſa Anno Domini 1679, pag. 63.

OMNIS de inventione Maximæ et Minimæ doctrina duabus poſitionibus ignotis innititur, et hâc unicâ præceptione.

Statuatur quilibet quæſtionis terminus eſſe A, ſive planum, ſive ſolidum, aut longitudo, pro-ùt propoſito ſatisfieri par eſt; et inventa maxima aut minima in terminis ſub A gradu utlibet involutis. Ponatur rursùs idem qui priùs eſſe terminus A + E; iterúmque inveniatur maxima aut minima in terminis ſub A et E, gradibus utlibet co-efficientibus.

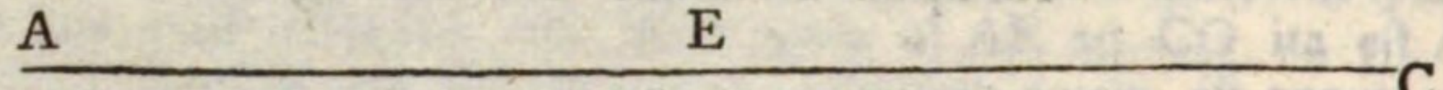
Adæquentur (ut loquitur Diophantus,) duo homogenea maximæ aut minimæ æqualia: et, demptis communibus, (quo peracto homogenea omnia ex parte alterutrâ ab E, vel ipſius gradibus, afficiuntur,) applicentur omnia ad E, vel ad elatiorem ipſius gradum, donec aliquod ex homogeneis ex parte utrâvis affectione ſub E omninò liberetur.

Elidantur deinde utrinque homogenea ſub E, aut ipſius gradibus, quomodo libet involuta; et reliqua æquentur. Aut, ſi ex unâ parte nihil ſupereſt, æquentur ſanè (quod eodem recidit,) negata adfirmatis. Reſolutio ultimæ iſtius æqualitatis dabit valorem A; quâ cognitâ Maxima aut Minima ex repetitis prioris reſolutionis veſtigiis innoteſcet.

Exemplum

Exemplum Subjicimus.

Sit recta AC ita dividenda in E, ut rectangulum AEC sit Maximum.



Recta AC dicatur B. Ponatur pars altera B esse A. Ergò [pars] reliqua erit B − A, et rectangulum sub segmentis erit $B \times A - A^2$, quod debet inveniri maximum.

Ponatur rursùs pars altera ipsius B esse A + E. Ergò reliqua erit B − A − E, et rectangulum sub segmentis erit B in A − A^2 + B in E − 2E in A − E^2; quod debet adæquari superiori rectangulo B in A − A^2. Demptis communibus, B in E adæquabitur 2E in A + E^2; et, omnibus per E divisis, B adæquabitur 2A + E. Elidatur E. [et jam] B æquabitur 2A. Igitùr B [sive data linea recta AC] bifariàm est dividenda, ad solutionem Propositi.

Nec potest Generalior dari Methodus.

Nota in hoc excerptum ex operibus Fermatii.

Hæc Enunciatio Regulæ Fermatianæ de quantitate maximâ vel minimâ inveniendâ, ab ipso Fermatio tradita, valdè obscura et intellectu difficilis mihi videtur, et demonstratione omninò caret. Sed per sequentem Diatriben celeberrimi Hugenii (qui omnes Scientiæ partes quas aggreditur, perspicuè semper tractare solet,) interpretatio simùl ipsius regulæ et ejusdem demonstratio satìs fient manifestæ.

F. M.

DIATRIBE,

ILLUSTRISSIMI VIRI,

CHRISTIANI HUGENII,

CONTINENS

DEMONSTRATIONEM PRÆCEDENTIS REGULÆ

À PETRO FERMATIO

INVENTÆ, DE DETERMINATIONE QUANTITATUM

MAXIMARUM ET MINIMARUM.

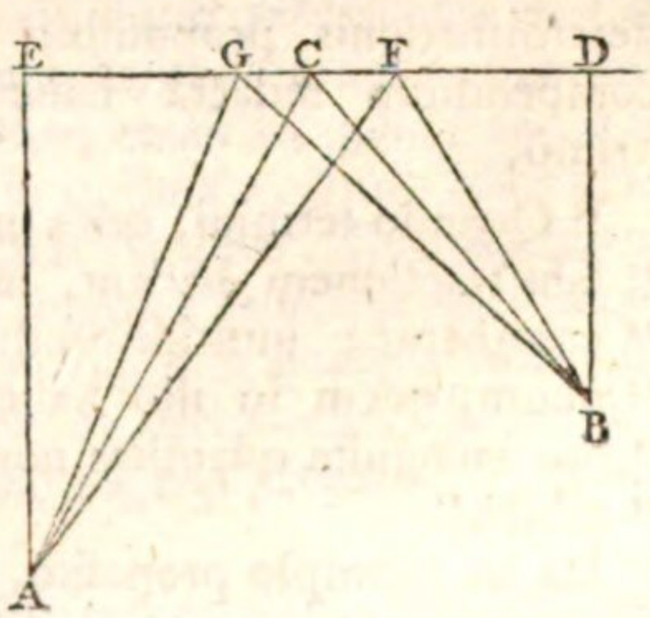
Olim edita Lugduni Batavorum Anno Domini 1724, in ſecundo Volumine Operum Hugenii, paginis 490, 491, 492, &c. - - - - 498.

AD inveſtiganda Maxima & Minima in Geometricis quæſtionibus, regulam certam primus, quod ſciam, Fermatius * adhibuit: cujus originem, ab ipſo non traditam, cum exquirerem, inveni ſimùl quo pacto ea ipſa regula ad mirabilem brevitatem perduci poſſet, útque inde eadem illa exiſteret quam poſteà vir ampliſſimus, Johannes Huddenius, dederat, tanquam partem regulæ ſuæ generalioris atque elegantiſſimæ, quæ ab alio prorſùs principio pendet. Hæc à Franciſco Schotenio edita eſt unà cum Carteſianis de Geometriâ libris. Fermatianæ autem regulæ examen quod inſtitui, eſt hujuſmodi.

* Vide Fermatii opera, Toloſæ impreſſa, Anno 1679, p. 63, et ſeq. ad p. 74.

Examen regulæ à Fermatio inventæ.

Quoties Maximum, aut Minimum, in problemate aliquo determinandum proponitur, certum eſt utrinque æqualitatis caſum exiſtere: ut, ſi data ſit poſitione recta ED & puncta A, B, oporteátque invenire in ED punctum C, unde ductis CA, CB, quadrata earum, ſimùl ſumpta, ſint minima quæ eſſe poſſint; neceſſe eſt ab utrâque parte puncti C, eſſe puncta G & F, à quibus ducendo rectas GA, GB et FA, FB, oriatur ſumma quadratorum GA, GB æqualis ſummæ quadratorum FA, FB, & utraque ſumma major quadratis CA, CB ſimùl ſumptis.



Ut igitùr inveniam punctum C, unde ductis CA, CB fiat ſumma quadratorum ab ipſis omnium minima; ductis AE, BD perpendicularibus in rectam ED, quarum AE, dicatur a; BD, b; intervallum verò E D, c: fingo primùm GF, differentiam duarum EG, EF, æqualem datæ lineæ quæ vocetur e; et quæro quanta futura ſit EG, quam appello x, ut quadrata GA, GB ſimul ſumpta æquentur quadratis FA, FB.

Itaque, quia AE $= a$, et EG $= x$, erit quadratum AG $= aa + xx$. Et quia GD $= c - x$, et DB $= b$, erit quadratum GB $= bb + cc - 2cx + xx$. Unde quadrata AG, GB ſimul ſumpta fient $= aa + bb + cc - 2cx + 2xx$, qui dicantur *termini priores*; ídque ſimiliter in quovis alio problemate intelligendum, ubi maximum aut minimum inquiritur. Rurſùs autem, quia EF $= x + e$, ſi ubique in ſummâ quadratorum inventâ ſubſtituam $x + e$ pro x, et quadratum ab $x + e$ pro xx, atque ita deinceps, ſi altior poteſtas ipſius x reperiatur, certum eſt exorituram ſummam quadratorum FA, FB; quæ quidem erit $aa + bb + cc - 2cx - 2ce + 2xx + 4ex + 2ee$, æquanda ſummæ quadratorum AG, GB; dicantur autem hi *termini poſteriores*.

Itaque erit $aa + bb + cc - 2cx + 2xx = aa + bb + cc - 2cx - 2ce + 2xx + 4ex + 2ee$. Ex quâ æquatione prodibit valor EG ſive x, quando GF, ſive e, certæ magnitudinis lineam refert.

Ponendo autem e infinitè parvam, apparebit ex eâdem æquatione quanta futura ſit EG, cum ipſi EF æqualis eſt; adeóque habebitur determinatio quæſita puncti C, unde ductæ CA, CB faciant ſummam quadratorum minimam; nempe, ſublatis primùm, ſi quæ ſunt, fractionibus, (quæ in hoc exemplo nullæ ſunt) delentur termini qui utrinque iidem habentur, quales ſunt neceſſariò omnes quibus litera e admixta non eſt; ídque facile eſt intelligere, cum dixerimus poſteriores terminos ex prioribus deſcribi, ponendo $x + e$, vel poteſtatem ejus, quoties invenitur x, vel poteſtas ejus aliqua, in prioribus. Deinde omnes termini per e dividuntur; quibúſque poſt eam diviſionem adhuc unum e aut plura ineſſe inveniuntur, ii delentur, quippe cum quantitates infinitè parvas contineant reſpectu cæterorum terminorum quibus nullum ampliùs ineſt e. Ex quibus denique ſolis invenitur quantitas x quæſita in caſu determinationis propoſito; & hæc eſt ratio methodi Fermatianæ, quâ in compendium redactâ hanc aliam inveni, cujus partes duæ ſunt. Nam primò,

Regulæ ab Hugenio inventæ Pars prior.

" Quando termini, quos maximum aut minimum deſignare volumus, nul-
" lam fractionem habent, in cujus denominatore quantitas incognita quæſita
" continetur; multiplicandus eſt terminus quiſque per numerum dimenſi-
" onum, quem in illo habet quantitas incognita, omiſſis terminis iis in qui-
" bus incognita quantitas non reperitur; omniaque illa producta ſunt æquanda
" nihilo."

Ita in exemplo propoſito, ubi termini priores inventi ſunt $aa + bb + cc$

$-2cx+2xx$, ſummam duorum quadratorum continentes, quam volo eſſe minimam; tantummodò hujuſmodi inſtituenda erit multiplicatio,

$$\frac{aa+bb+cc-2cx+2xx}{\text{æquandi nihilo } -2cx+4xx=0;}$$

Ex quâ orientur termini $\overset{1}{}\ \overset{2}{}$

Unde fit $\frac{1}{2}c=x.$

Ita quoque, ſi priores termini ſint $3ax^3-bx^3-\frac{2bba^2}{3c}x+ab^2$,

multiplicatio erit hujuſmodi $\overset{3}{}\ \overset{3}{}\ \overset{1}{}$

Unde termini æquandi nihilo $9ax^3-3bx^3-\frac{2bba^2}{3c}x=0$

Et (dividendo terminos per x,) $9axx-3bxx-\frac{2bba^2}{3c}=0.$

Explicatio hujus primæ partis regulæ ab Hugenio inventæ.

Hujus compendii ratio ut intelligatur, ſciendum primò, quoniam termini poſteriores ex prioribus deſcribuntur, ponendo tantùm ubique $x+e$ pro x, neceſſariò omnes terminos priores etiàm in poſterioribus reperiri; ideóque illos nihil opus eſſe deſcribi, (cum [ſi deſcriberentur] utrobique mox delendi forent,) atque adeò illos tantùm [eſſe] ſcribendos in quibus unum e vel plura inſunt, ut in exemplo noſtro $-2ce+4ex+2ee$; eóſque æquandos [eſſe] nihilo. Sed etiàm illos quibus plura quam unum e inerunt, ſcribi fruſtrà apparet, cum, diviſione factâ per e, delendos [eſſe] poſteà conſtet, ut paulò ante diximus. Itaque nulli prætereà ab initio deſcribendi [ſunt] inter terminos poſteriores quàm ii quibus inerit e ſimplex.

Hi autem termini [quibus inerit e ſimplex] ex terminis prioribus facilè deducuntur, cum conſtet [eos] nihil aliud eſſe quàm ſecundos terminos poteſtatum ab $x+e$, quia cæteri omnes [termini iſtarum poteſtatum] plura quam unum e vel nullum habent. Adeò ut, ubicunque in prioribus terminis habetur x, ſcribendum ſit in poſterioribus e; et ubi habetur xx in prioribus, ponendum $2ex$ in poſterioribus; & ubi x^3 in prioribus, in poſterioribus $3exx$, atque ita deinceps. Dicti autem termini ſecundi cujuſque poteſtatis $x+e$ ex ipſâ poteſtate x facilè deſcribuntur mutando unum x in e, et præponendo numerum dimenſionum ipſius x: ita enim ab xx fit $2ex$, et ab x^3, $3exx$; atque in cæteris pari modo. Itaque ex terminis prioribus in quibus x, (quos ſolos conſiderandos eſſe patuit,) facilè etiàm termini poſteriores, (ii quos nihilo adæquandos diximus,) deſcribuntur multiplicando tantùm ſingulos in numerum dimenſionum quas in ipſis habet x. Nam mutare unum x in e ne quidem opus eſt, cum eodem redeat, ſive omnes poſteà per e, ſive per x, dividantur. Et ex his quidem aperta eſt ratio compendii ad primam partem regulæ pertinentis. Nunc ad alteram veniamus, quæ eſt hujuſmodi.

Regulæ ab Hugenio inventæ Pars ſecunda.

" Si termini quos maximum, aut minimum, deſignare volumus, fractiones ha-
" beant in quarum denominatoribus occurrat quantitas incognita, delendæ primùm

" mùm ſunt quantitates cognitæ, ſi quæ adſint; deinde, ſi reliquæ quantitates " non habeant eundem denominatorem, eò reducendæ ſunt. Tunc termini " ſinguli numeratorem fractionis conſtituentes, ducendi ſunt in terminos ſingulos " denominatoris, productáque ſingula multipla ſumenda ſunt ſecundùm numerum " quo dimenſiones quantitatis incognitæ in termino numeratoris differunt à di- " menſionibus ejuſdem incognitæ quantitatis in termino denominatoris. Signa " autem affectionis [nempe + et −] productis ſingulis præponenda [ſunt] " qualia lex multiplicationis exigit, quoties dimenſiones quantitatis incognitæ " plures ſunt in termino numeratoris quam in termino denominatoris; at quoties " contrà evenit, contraria quoque ſigna productis [ſunt] præponenda; quæ " denique omnia [ſunt] æquanda nihilo."

Sint, exempli gratiâ, inventi termini priores, quos maximum deſignare velimus, iſti $\frac{bx^3 - ccxx - 2bccx}{bcc + x^3}$, ubi nulla eſt quantitas cognita. Hîc ergò, ſecundùm regulam, multiplico terminos omnes numeratoris primùm per bcc; prioríſque producti ex bx^3 in bcc, ſcribo triplum, quia bx^3 habet tres dimenſiones quantitatis incognitæ x, bcc verò nullam. Secundi producti ex $-ccxx$ in bcc ſcribo duplum, propterea quod in $-ccxx$ duæ ſunt dimenſiones x, et in bcc nulla. Tertium verò productum ex $-2bccx$ in bcc ſcribo ſimplex, quia in $-2bccx$ et bcc differentia dimenſionum x eſt unitas. Tribus autem hiſce productis vera ſigna affectionis adſcribo, quoniam dimenſiones x in terminis numeratoris excedunt eas quæ in termino bcc, quippe quæ nullæ ſunt; ita ut tria hæc producta ſint

$$3bbccx^3 - 2bc^4xx - 2bbc^4x.$$

Jam porrò terminos omnes eoſdem numeratoris duco in x^3, terminum alterum denominatoris, primúmque productum ex bx^3 in x^3 ſcribere omitto, ſive per o multiplico, quoniam eædem dimenſiones utrobique ſunt ipſius x, ideóque differentia nulla. Secundum autem productum ex $-ccxx$ in x^3 ſcribo ſimplex, quia in his terminis differentia dimenſionum x eſt unitas. At tertium productum ex $-2bccx$ in x^3 ſcribo duplum, quia differentia dimenſionum x in his eſt 2. Signa verò affectionis productis hiſce duobus adſcribo contraria iis quæ requireret lex multiplicationis, eo quod dimenſiones x pauciores ſunt utrobique in terminis numeratoris quàm in x^3, termino denominatoris.

Itaque producta bina erunt hæc $+ ccx^5 + 4bccx^4$; quæ, addita tribus præcedentibus productis

$$+ 3bbccx^3 - 2bc^4xx - 2bbc^4x,$$

faciunt ſummam æquandam nihilo

$$ccx^5 + 4bccx^4 + 3bbccx^3 - 2bc^4xx - 2bbc^4x = o;$$

quâ æquatione diviſâ per $ccbx + ccxx$, fit $x^3 + 3bxx - 2bcc = o$.

Quomodo

Quomodo autem ad hæc perventum ſit, uno exemplo rurſùs explicabimus; ex quo eandem in omnibus cæteris rationem eſſe intelligetur. Videamus igitùr priores terminos quos modò propoſueram, nempe $\frac{bx^3 - ccxx - 2bccx}{bcc + x^3}$; ex quibus ſi alios, quibuſcum eos comparem, (ut initio factum eſt,) deſcribere velim, ponendo ubique $x + e$ ubi eſt x; video quidem primò omnes illos in poſterioribus terminis poſſe negligi in quibus plura quàm unum e inerit, quia ſempèr ex iis quantitates orientur in quibus plura uno e inerunt, quaéque proinde delendæ tandem erunt, ob cauſam in ſuperioribus traditam. Explicatio hujus ſecundæ partis regulæ ab Hugenio inventæ.

Itaque erunt termini priores	æquandi poſterioribus
$\frac{bx^3 - ccxx - 2bccx}{bcc + x^3}$	$= \frac{+ 3bexx - 2ccex - 2bcce,}{bcc + x^3 + 3exx}$
	$bx^3 - ccxx - 2bccx,$

qui nempe ex prioribus hâc lege deſcripti ſunt, ut ubicunque eſt x, vel aliqua poteſtas ejus, in prioribus, ibi ponatur $x + e$, vel ejuſdem poteſtatis $x + e$ duo priores termini; quoniam ſcimus in cæteris plura quam unum e contineri.

Jam verò porrò, quia termini in quibus nullum e in numeratore ac denominatore priorum ac poſteriorum terminorum, iidem planè reperiuntur, patet multiplicationes alternas eorum terminorum denominatoris in terminos numeratoris partis alterius e carentes, omitti poſſe, cum quantitates inde ortæ eædem utrinque eſſent futuræ, ideóque delendæ *. Quarè in terminis poſterioribus ii tantùm ab initio ſcribendi erant in quibus unum e, omiſſis omnibus reliquis, ut æquatio hîc futura ſit iſta.

* Ponatur enim $A = bx^3 - ccxx - 2bccx$, et $B = bcc + x^3$. Et erit $\frac{A}{B} = \frac{A + 3bexx - 2c^2ex - 2bc^2e}{B + 3exx}$, ideóque $\frac{AB + A \times 3exx}{B \times \overline{B + 3exx}}$ erit $= \frac{AB + B \times \overline{3bexx - 2c^2ex - 2bc^2e}}{B \times \overline{B + 3exx}}$ et pro-inde $AB + A \times 3exx$ erit $= AB + B \times \overline{3bexx - 2c^2ex - 2bc^2e}$, et (auferendo AB utrinque,) $A \times 3exx$ erit $= B \times \overline{3bexx - 2c^2ex - 2b^2ce}$, ſive (ſubſtituendo pro quantitatibus A et B eorum valores $bx^3 - ccxx - 2bccx$ et $bcc + x^3$,) $\overline{bx^3 - ccxx - 2bccx} \times 3exx$ erit $= \overline{bcc + x^3} \times \overline{3bexx - 2c^2ex - 2bc^2e}$, hoc eſt, $3bex^5 - 3ccex^4 - 6bccex^3$ erit $= 3bbccexx - 2bc^4ex - 2bbc^4e + 3bex^5 - 2c^2ex^4 - 2bc^2ex^3$. Ergò (addendo utrinque $3ccex^4 + 6bccex^3$, erit $3bex^5 = 3bbccexx - 2bc^4ex - 2bbc^4e + 3bex^5 + ccex^4 + 4bccex^3$, et (auferendo utrinque $3bex^5$) erit $o = 3bbccexx - 2bc^4ex - 2bbc^4e + ccex^4 + 4bccex^3$, et (terminos ordinando ſecundùm poteſtates quantitatis x,) erit $ccex^4 + 4bccex^3 + 3bbccexx - 2bc^4ex - 2bbc^4e = o$, et (dividendo terminos per cce) erit $x^4 + 4bx^3 + 3bbxx - 2bccx - 2bbcc = o$, et (dividendo totam quantitatem quinquinomiam $x^4 + 4bx^3 + 3bbxx - 2bccx - 2bbcc$ per quantitatem binomiam $x + b$,) erit $x^3 + 3bx^2 - 2bcc = o$, ſive $x^3 + 3bx^2 = 2bcc$; quæ eſt eadem æquatio quam Hugenius exhibuit ſecundùm regulam ſuam ad finem paginæ ſuperioris, ſeu 116.

F. M.

$$\frac{bx^3 - ccxx - 2bccx}{bcc + x^3} = \frac{3bexx - 2ccex - 2bcce}{3exx}$$

Hîc jam multiplicationes alternæ per denominatores inſtituendæ eſſent ad tollendas fractiones. Verùm examinando diligentiùs quænam futura ſint harum multiplicationum producta, aliud adhuc compendium inveniemus, et nec ſcribendos quidem omninò eſſe terminos poſteriores: quia enim deſcribuntur ex prioribus mutato x in e, præpoſitóque numero dimenſionum ipſius x, non difficile eſt colligere ex ſolis terminis prioribus quænam futura ſint iſta omnia producta.

Ita, quoniam propter $-ccxx$ in prioribus, habetur $-2ccex$ in poſterioribus; et propter x^3 in denominatore priorum, in poſteriorum denominatore eſt $3exx$; facilè perſpicitur utraque producta ex $-ccxx$ in $3exx$ et ex $-2ccex$ in x^3, quæ ſunt $-3cce x^4$ et $-2cce x^4$, eaſdem literas habitura, ſed diverſos numeros præpoſitos 3 et 2, ídque inde fieri quòd in termino $ccxx$ unam dimenſionem minùs habeat x quàm in termino x^3. Itaque et auferendo poſteà ex utrâque parte æquationis, $-2cce x^4$, apparet ſuperfuturum $-cce x^4$ à parte terminorum priorum. Quare ab initio hoc ſciri poteſt, multiplicando tantùm in terminis prioribus $-ccxx$ numeratoris in x^3 denominatoris, unúmque x in e mutando, ac productum ſimplex ſcribendo; quia differentia dimenſionum x in iſtis duobus terminis eſt unitas.

Eâdem ratione producta ex $-2bccx$ in $3exx$, et ex $-2bcce$ in x^3, quæ eaſdem literas habent, (ſunt enim $-6bcce x^3$ et $-2bcce x^3$,) habebunt numeros præpoſitos diverſos, propterea quod in $-2becx$ una tantùm eſt dimenſio x, at in x^3 tres; unde, ablato ex utrâque parte æquationis $-2bcce x^3$, ſcio ſuperfuturum à parte terminorum priorum $-4bcce x^3$: quod rurſùs ab initio cognoſci potuit, quia eadem quantitas oritur, multiplicando $-2bccx$ numeratoris terminorum priorum, in x^3 denominatoris, mutandóque unum x in e, et productum multiplicando per 2, quæ eſt differentia dimenſionum x in terminis $-2bccx$ & x^3.

At quoniam in bx^3 et in x^3 eadem eſt dimenſio x, ſequetur producta ex bx^3 in $3exx$, et ex $3bexx$ in x^3, tum literas eaſdem, tum eoſdem numeros præpoſitos habitura, ideóque ſeſe mutuò ſublatura, ut proinde multiplicatio illa omitti poſſit.

Atque hujuſmodi animadverſionibus inventum eſt quod in regulâ præcipitur, terminos ſingulos numeratoris in ſingulos denominatoris terminos eſſe ducendos, productáque quælibet multipla eſſe ſumenda ſecundùm differentiam dimenſionum quantitatis incognitæ in terminis binis qui in ſe mutuò ducuntur. Nam quòd non præcipitur unum x in e mutandum, id hanc rationem habet, quòd non referat utrum poſteà per e an per x omnes termini dividantur.

"Quòd verò, ſi ſigna affectionis vera productis ſingulis præponenda dicuntur, quoties dimenſiones x plures ſunt in numeratore quam in denominatore,"

id

id quoque ex jam dictis intelligetur; uti consequentèr etiàm hoc, " quòd contraria signa sunt adponenda, quoties dimensionum numerus contrà se habet." Velùt hîc, productum ex bx^3 in bcc scribendum est cum signo —, præposito numero 3, ut fiat $-3bbccx^3$, quia nempe, propter bx^3, scimus in posterioribus terminis fore $3bexx$; quod ductum in bcc faciet $+3bbccexx$, sed translatum in partem priorem æquationis, fiet $-3bbccexx$; sive, non mutato x in e, $-3bbccx^3$.

Quod denique in regulâ habetur, quoties in prioribus terminis priusquam ad eundem denominatorem reducantur, quantitates cognitæ occurrunt, eas primùm omnium delendas, id ex hoc sequenti exemplo intelligetur rectè præcipi. Sint enim reperti termini priores, quos maximum aut minimum designare oporteat, isti $\frac{x^3}{2a-x} - 2vx + xx + vv$; ubi vv quantitatem cognitam significet: id igitùr delendum esse ut appareat, videamus quid futurum sit si non deleatur. Nempe, ut ad eundem denominatorem cum cæteris omnibus reducatur, ducendum erit vv in $2a-x$; fiétque inde $\frac{2avv - xvv}{2a-x}$ in terminis prioribus. Propter quos in terminis posterioribus, secundùm superiùs explicata, scribetur $\frac{-evv}{-e}$, adeóque, multiplicatione alternatìm utrinque per denominatores institutâ, ducendum erit hinc $2a-x$ in $-evv$; inde $-e$ in $2avv - xvv$. Ex quibus multiplicationibus eosdem utrinque terminos oriri necesse est, cum utrobique eadem hæc tria in se mutuò ducantur $2a-x$ in $-e$ in vv, qui proinde termini sese mutuò sublaturi essent, ideóque frustrà scriberentur; ac proinde liquet tutò deleri posse ab initio quantitatem vv. Idémque quod in hoc exemplo accidit, necessariò quoque in quibuslibet aliis contingere, diligentèr intuenti manifestum erit *.

* Hæc explicatio secundæ partis regulæ ab Hugenio inventæ de determinatione quantitatum Maximarum et Minimarum mihi videtur esse minùs perspicua et intellectu facilis quàm explicatio primæ partis hujus regulæ in priore parte hujus diatribes in paginâ 115. Itaque, ut veritas hujus secundæ partis regulæ ab Hugenio inventæ faciliùs et plenius intelligi possit, investigationem maximæ, vel minimæ, magnitudinis quam fractio prædicta $\frac{bx^3 - ccxx - 2bccx}{bcc + x^3}$ (quam Hugenius adduxit pro exemplo suæ regulæ in hâc secundâ ejus parte,) habere possit, hîc deducere conabor à magno Principio quod Hugenius, in initio hujus diatribes, in paginis 113 et 114, pro fundamento Methodi suæ adoptavit. Hoc verò Principium ita enuntiari poterit.

" Si sint duo quantitates variabiles x et y, quarum prima x continuò crescit, sive à nihilo ad infinitum, sive â magnitudine quâdam finitâ ad aliam majorem magnitudinem finitam, secunda verò y sit quantitas genita ex primâ x per operationes arithmeticas, hoc est, per additionem et subtractionem, et multiplicationem, et divisionem, ut est fractio prædicta $\frac{bx^3 - ccxx - 2bccx}{bcc + x^3}$; et

" et ſi ſecunda quantitas y, ſeu $\frac{bx^3 - ccxx - 2bccx}{bcc + x^3}$, non continuò creſcit dum x creſcit, ſed " primò creſcit ad certam quandam magnitudinem et tunc ab iſtâ magnitudine decreſcit; vel, è " contrà, primò decreſcit ad certam quandam magnitudinem, et tunc ab iſtâ magnitudine creſcit; " et, ſi hæc ejus maxima, aut minima, magnitudo vocetur M; et magnitudo quam habet prima quan- " titas x cum ſecunda quantitas y, ſeu $\frac{bx^3 - ccxx - 2bccx}{bcc + x^3}$, eſt æqualis M, vocetur m;—His po- " ſitis, manifeſtum eſt quòd erunt duo valores primæ quantitatis x quorum prior erit minor, et " alter erit major, quàm quantitas m, qui, ſi ſubſtituerentur pro x in terminis fractionis " $\frac{bx^3 - ccxx - 2bccx}{bcc + x^3}$, efficerent ut duo valores iſtius fractionis inde ortæ eſſent ſibi mutuò " æquales."

Igitùr, ſi minor horum valorum quantitatis x per Litteram x deſignetur, et major per quantitatem binomiam $x + e$, ita ut e ſit eorum differentia, erit fractio $\frac{bx^3 - ccxx - 2bccx}{bcc + x^3}$ æqualis fractioni $\frac{b \times \overline{x+e}^3 - cc \times \overline{x+e}^2 - 2bcc \times \overline{x+e}}{bcc + \overline{x+e}^3}$, ſeu fractioni

$$\frac{b \times \overline{x^3 + 3x^2e + 3xe^2 + e^3} - cc \times \overline{x^2 + 2xe + e^2} - 2bcc \times \overline{x+e}}{bcc + x^3 + 3x^2e + 3xe^2 + e^3},$$

ſeu (ſi per notam &c deſignemus terminos qui continent e^2 et e^3,) fractioni

$$\frac{b \times \overline{x^3 + 3x^2e + \&c} - cc \times \overline{x^2 + 2xe + \&c} - 2bcc \times \overline{x+e}}{bcc + x^3 + 3x^2e + \&c},$$ ſeu

fractioni $\frac{bx^3 + 3bx^2e + \&c - ccxx - 2ccxe - \&c - 2bccx - 2bcce}{bcc + x^3 + 3x^2e + \&c}$.

Ergò (multiplicando utraſque partes per $bcc + x^3$, denominatorem primæ fractionis $\frac{bx^3 - ccx^2 - 2bccx}{bcc + x^3}$) erit quantitas trinomia

$bx^3 - ccx^2 - 2bccx$ æqualis fractioni

$$\frac{bbccx^3 + 3bbccx^2e + \&c - bc^4xx - 2bc^4xe - \&c - 2bbc^4x - 2bbc^4e + bx^6 + 3bx^5e + \&c - ccx^5 - 2ccx^4e - \&c - 2bccx^4 - 2bccx^3e}{bcc + x^3 + 3x^2e + \&c},$$

et (multiplicando utrinque per denominatorem $bcc + x^3 + 3x^2e + \&c$, ſecundæ fractionis,) erit quantitas multinomia $bbccx^3 - bc^4x^2 - 2bbc^4x + bx^6 - ccx^5 - 2bccx^4 + 3bx^5e - 3ccx^4e - 6bccx^3e + \&c$ æqualis quantitati multinomiæ $bbccx^3 + 3bbccx^2e + \&c - bc^4xx - 2bc^4xe - \&c - 2bbc^4x - 2bbc^4e + bx^6 + 3bx^5e + \&c - ccx^5 - 2ccx^4e - \&c - 2bccx^4 - 2bccx^3e$, et (addendo utrinque $bc^4x^2 + 2bbc^4x + ccx^5 + 2bccx^4$,) erit quantitas multinomia $bbccx^3 + bx^6 + 3bx^5e - 3ccx^4e - 6bccx^3e + \&c$ æqualis quantitati multinomiæ $bbccx^3 + 3bbccx^2e + \&c. - 2bc^4xe - \&c. - 2bbc^4e + bx^6 + 3bx^5e + \&c. - 2ccx^4e - \&c - 2bccx^3e$, et (auferendo utrinque $bbccx^3 + bx^6$,) erit quantitas multinomia $3bx^5e - 3ccx^4e - 6bccx^3e + \&c =$ quantitati multinomiæ $3bbccx^2e + \&c - 2bc^4xe - 2bbc^4e + 3bx^5e + \&c - 2ccx^4e - \&c - 2bccx^3e$; in quâ æquatione omnes termini hîc ſcripti continent quantitatem e, et omnes alii termini non hîc ſcripti, ſed per notas + &c vel − &c deſignati, continent quantitates e^2 et e^3, ſeu quadratum et cubum quantitatis e. Igitùr ſi omnes termini, (tàm hîc ſcripti quàm alii per notas + &c et − &c deſignati,) per quantitatem e dividantur, habebimus æquationem $3bx^5 - 3ccx^4 - 6bccx^3 + \&c = 3bbccx^2 + \&c - 2bc^4x - 2bbc^4 + 3bx^5 + \&c - 2ccx^4 - \&c - 2bccx^3$, in quâ termini hîc ſcripti quantitatem e non continebunt, ſed termini non ſcripti, ſed per notas + &c et −

et — &c designati, eam adhuc retinebunt, quia ante hanc divisionem continebant quantitates e^2 et e^3. Ergò, si e potest diminui in infinitum, seu fieri æqualis nihilo, omnes isti termini qui per notas + &c et — &c designantur, et in se retinent quantitatem e, fient simùl in isto casu æquales nihilo, seu evanescent, et pro-inde æquatio ultima $3bx^5 - 3ccx^4 - 6bccx^3 + \&c = 3bbccx^2 + \&c - 2bc^4x - 2bbc^4 + 3bx^5 + \&c - 2ccx^4 - \&c - 2bccx^3$ convertetur in æquationem $3bx^5 - 3ccx^4 - 6bccx^3 = 3bbccx^2 - 2bc^4x - 2bbc^4 + 3bx^5 - 2ccx^4 - 2bccx^3$, sine notis + &c et — &c, seu ullâ ratione habitâ terminorum qui per istas notas designabantur. Sed æquatio prædicta $3bx^5 - 3ccx^4 - 6bccx^3 + \&c = 3bbccx^2 + \&c - 2bc^4x - 2bbc^4 + 3bx^5 + \&c - 2ccx^4 - \&c - 2bccx^3$ est semper vera, quantumvis parvam fieri supponamus quantitatem e, seu differentiam quantitatum duarum x et $x + e$; atque idcircò erit vera etiàm quando ista differentia est prorsùs nulla. Et pro-inde in isto casu, seu cum x et $x + e$ sunt inter se æquales, et ideò æquales quantitati m (quæ anteà fuerat minor quàm $x + e$, sed major quam x,) æquatio $3bx^5 - 3ccx^4 - 6bccx^3 = 3bbccx^2 - 2bc^4x - 2bbc^4 + 3bx^5 - 2ccx^4 - 2bccx^3$ erit vera. Ergò (addendo utrinque $2bbc^4$) erit $2bbc^4 + 3bx^5 - 3ccx^4 - 6bccx^3 = 3bbccx^2 - 2bc^4x + 3bx^5 - 2ccx^4 - 2bccx^3$, et (addendo utrinque $3ccx^4 + 6bccx^3$,) erit $2bbc^4 + 3bx^5 = 3bbccx^2 - 2bc^4x + 3bx^5 + ccx^4 + 4bccx^3$, et (auferendo utrinque $3bx^5$,) erit $2bbc^4 = 3bbccx^2 - 2bc^4x + ccx^4 + 4bccx^3$, et (dividendo omnes terminos per cc,) erit $2bbcc = 3bbx^2 - 2bcex + x^4 + 4bx^3$, seu $x^4 + 4bx^3 + 3bbx^2 - 2bccx = 2bbcc$, seu $x^4 + 4bx^3 + 3bbx^2 - 2bccx - 2bbcc = o$, et (divisione factâ per quantitatem binomiam $x + b$,) erit $x^3 + 3bx^2 - 2bcc = o$, et (addendo utrinque $2bcc$,) $x^3 + 3bx^2 = 2bcc$. Ergò quantitas m, seu valor primæ quantitatis variabilis x, cum secunda quantitas variabilis y, seu fractio $\frac{bx^3 - ccxx - 2bccx}{bcc + x^3}$, (quæ ex primâ x per operationes arithmeticas derivatur,) est æqualis quantitati M, seu est maxima, aut minima, quæ esse possit, erit radix æquationis cubicæ $x^3 + 3bx^2 = 2bcc$, quæ per resolutionem hujus æquationis invenietur. Inventâ autem quantitate m, quantitas M, seu maxima, vel minima, magnitudo fractionis $\frac{bx^3 - ccxx - 2bccx}{bcc + x^3}$ erit æqualis fractioni $\frac{bm^3 - ccmm - 2bccm}{bcc + m^3}$.

Q. E. I.

F. M.

Finis Dialribes HUGENII *de inventione Maximorum et Minimorum.*

ADDITAMENTUM AD PRÆCEDENTEM HUGENII DIATRIBEN

DE INVENIENDIS

MAXIMIS VEL MINIMIS QUANTITATIBUS,

CONTINENS

EXEMPLA HUJUS METHODI IN QUANTITATIBUS QUIBUSDAM

BINOMIIS ET TRINOMIIS

QUÆ OCCURRUNT IN

Reſolutione Quarundam Æquationum Quadraticarum, Cubicarum, Biquadraticarum, et altiorum Ordinum.

A FRANCISCO MASERES, ANGLO,

REGIÆ SOCIETATIS LONDINENSIS SOCIO.

Exemplum Primum.

DESIGNET p lineam aliquam datam, et x lineam aliam variabilem quæ creſcit continuò ab o, ſive nihilo, donec fiat æqualis datæ p. Et requiratur invenire quænam ſit magnitudo lineæ x, quando quantitas binomia $px - xx$, ſeu exceſſus rectanguli ſub p et x ſupèr quadratum ex x, eſt Maxima.

Solutio.

In contemplandâ hâc quantitate binomiâ $px - xx$, manifeſtum eſt eam bìs fieri æqualem nihilo, nempe, primò, cum x eſt æqualis nihilo, (et pro-inde $p \times x - xx$ eſt $= p \times o - o \times o = o - o = o$,) et, ſecundò, cum x eſt æqualis datæ p, in quo caſu px erit $= pp$, et xx erit etiàm $= pp$, et pro-inde $px - xx$ erit $= pp - pp = o$. Neceſſe eſt igitùr ut, dum x creſcit continuò à nihilo uſque ad p, quantitas binomia $px - xx$ primùm creſcat à nihilo ad certam aliquam magnitudinem, et deinde decreſcat ab iſtâ magnitudine ad o, ſive nihilum. Requiritur autèm invenire magnitudinem lineæ x quando quan-

titas binomia $px - xx$ attigit hanc ſuam maximam magnitudinem, ex quâ incipit decreſcere. Hoc verò ita inveniri poteſt.

Deſignetur hic valor ipſius x, quem quærimus, per Litteram M. Et ſit y alius valor ipſius x minor ipſo M. Erit igitùr quantitas binomia $py - yy$ minor quantitate binomiâ p M — M M. Ergò, dum quantitas x creſcit à valore M uſque ad p, et quantitas binomia $px - xx$ decreſcit à quantitate p M — MM ad o, hæc quantitas binomia $px - xx$ fiet in uno aliquo hujus temporis puncto æqualis quantitati binomiæ $py - yy$, quæ eſt minor ipſâ p M — MM. Ponatur z pro valore ipſius x in hoc temporis puncto. Et erit $pz - zz =$ $py - yy$.

Ponatur jam e pro $z - y$, ſeu exceſſu lineæ z ſupèr lineam y. Et tunc erit $z = y + e$, et pro-inde $zz\ (= \overline{y + e}^2) = y^2 + 2ye + e^2$, et $pz\ (= p \times \overline{y + e}) = py + pe$, et $pz - zz = py + pe - y^2 - 2ye - e^2$.

Sed $pz - zz$ eſt $= py - yy$.

Ergò $py + pe - yy - 2ye - ee$ erit $= py - yy$; et (addendo utrinque $2ye + ee$,) erit $py + pe - yy = py - yy + 2ye + ee$; et (auferendo utrinque $py - yy$,) erit $pe = 2ye + ee$; et (dividendo omnes terminos per e,) erit $p = 2y + e$. Hoc autèm verum eſt, utcunque parva fuerit quantitas e. Erit igitùr etiàm verum quando quantitas e eſt infinitè parva, ſeu nullius omninò magnitudinis. Sed, ubi e eſt nullius omninò magnitudinis, $2y + e$ erit $= 2y + o$, ſive $2y$. Ergò p erit $= 2y$, atque ideò y erit $= \frac{p}{2}$, ſeu dimidiæ co-efficienti p.

Sed, ob e in hoc caſu $= o$, erit z, ſeu $y + e$, $(= y + o) = y$.

Ergò z erit etiàm $= \frac{p}{2}$.

Ergò in hoc caſu duo valores ipſius x, ſcilicet, y et z (quorum prior y fuerat minor ipſo M, alter verò z fuerat major ipſo M,) fiunt æquales inter ſe, et uterque æqualis ipſi $\frac{p}{2}$. Ergò et valor M (qui inter illos fuerat mediæ cujuſdam magnitudinis,) erit etiàm æqualis ipſi $\frac{p}{2}$; hoc eſt, valor ipſius x in quantitate binomiâ $px - xx$ quando iſta quantitas eſt maxima quæ eſſe poſſit, erit æqualis quantitati $\frac{p}{2}$. Q. E. I.

Coroll. 1. Ergò p M — MM, ſeu maxima magnitudo quantitatis binomiæ $px - xx$, erit $p \times \frac{p}{2} - \frac{p}{2} \times \frac{p}{2} \left(= \frac{pp}{2} - \frac{pp}{4} = \frac{2pp}{4} - \frac{pp}{4} \right) = \frac{pp}{4}$. Et pro-inde, ſi in æquatione quadraticâ hujus formæ, $px - xx = q$, homogeneum comparationis, ſeu quantitas cognita, q ſit æqualis $\frac{pp}{4}$, ſive quadrato ipſius $\frac{p}{2}$, ſeu dimidiæ co-efficientis p, æquatio habebit tantùm unam radicem, ſcilicet,

ſcilicet, $\frac{p}{2}$; ſed, ſi quantitas cognita q ſit minor ipſâ $\frac{pp}{4}$, ſeu quadrato ipſius $\frac{p}{2}$, æquatio habebit duas radices, quarum una erit minor ipſâ $\frac{p}{2}$, altera verò erit major ipſâ $\frac{p}{2}$, ſed minor ipſâ p.

Coroll. 2. Quantitas binomia $px - xx$ eſt æqualis $\overline{p - x} \times x$, ſeu rectangulo ſub lineis $p - x$ et x. Ergò maxima magnitudo rectanguli ſub lineis $p - x$ et x, quæ ſimùl ſumptæ ſunt æqualis lineæ p, eſt $\frac{pp}{4}$, ſeu $\frac{p}{2} \times \frac{p}{2}$, ſeu quadratum dimidiæ lineæ p; quod Euclidis Elementorum Libri ſecundi Propoſitioni quintæ conſentaneum eſt.

Exemplum Secundum.

Deſignet q planam ſuperficiem aliquam datam, et x lineam quandam variabilem, quæ creſcit continuò à nihilo donec quadratum ejus, xx, fiat æquale datæ ſuperficiei planæ q. Et requiratur invenire quænam ſit magnitudo lineæ x, quando quantitas binomia $qx - x^3$, ſeu exceſſus ſolidi ſub datâ ſuperficie planâ q et lineâ x ſupèr cubum ipſius x, eſt Maxima.

Solutio.

Hæc quantitas binomia $qx - x^3$ erit bìs æqualis nihilo, nempe, primò cum x eſt æqualis nihilo, ſeu o, et pro-inde $qx - x^3$ erit $= q \times o - o \times o \times o$ $(= o - o) = o$, et ſecundò, cum xx eſt æquale ſuperficiei datæ q, et pro-inde x eſt æqualis $\surd q$, et qx eſt $= q\surd q$, et x^3 eſt $(= xx \times x) = q\surd q$, et $qx - x^3$ eſt $= q\surd q - q\surd q = o$. Ergò, dum linea x creſcit continuò à nihilo uſque ad $\surd q$, quantitas binomia $qx - x^3$ creſcet à nihilo uſque ad certam quandam magnitudinem, et deinde decreſcet ab iſtâ magnitudine ad o, ſeu nihilum. Requiritur autèm invenire magnitudinem lineæ x, quando quantitas binomia $qx - x^3$ attigit hanc ſuam maximam magnitudinem, ex quâ incipit decreſcere. Hoc verò ita inveniri poteſt.

Deſignetur hic valor, quem quærimus, ipſius x per Litteram M. Et ſit y alius valor ipſius x minor ipſâ M. Erit igitùr quantitas binomia $qy - y^3$ minor quantitate binomiâ $q\,\mathrm{M} - \mathrm{M}^3$. Ergò, dum quantitas x creſcit à valore M uſque

uſque ad $\surd q$, et quantitas binomia $qx - x^3$ decreſcit à valore $q\,M - M^3$ ad o, hæc quantitas binomia $qx - x^3$ fiet in uno aliquo hujus temporis puncto æqualis quantitati binomiæ $qy - y^3$, quæ eſt minor ipſâ $q\,M - M^3$. Ponatur z pro valore ipſius x in iſto temporis puncto. Et erit $qz - z^3 = qy - y^3$.

Ponatur jam e pro $z - y$, ſeu exceſſu lineæ z ſuper lineam y. Et tunc erit $z = y + e$, et $z^3\ (= \overline{y + e}^3) = y^3 + 3y^2e + 3ye^2 + e^3$, et $qz\ (= q \times \overline{y + e}) = qy + qe$, et pro-inde $qz - z^3$ erit $= qy + qe - y^3 - 3y^2e - 3ye^2 - e^3$.

Sed $qz - z^3$ eſt $= qy - y^3$.

Ergò $qy + qe - y^3 - 3y^2e - 3ye^2 - e^3$ erit $= qy - y^3$; et (addendo utrinque $3y^2e + 3ye^2 + e^3$,) erit $qy + qe - y^3 = qy - y^3 + 3y^2e + 3ye^2 + e^3$; et (auferendo utrique $qy - y^3$,) erit $qe = 3y^2e + 3ye^2 + e^3$; et (dividendo omnes terminos per e,) erit $q = 3y^2 + 3ye + e^2$. Hoc autèm verum eſt, utcunque parva fuerit quantitas e. Erit igitùr etiàm verum quando quantitas e eſt infinité parva, ſeu nullius omninò magnitudinis. Sed, ubi e eſt nullius omninò magnitudinis, quantitas $3y^2 + 3ye + e^2$ erit $(= 3y^2 + 3y \times o + o \times o = 3y^2 + o \times o) = 3y^2$. Ergò in hoc caſu q erit $= 3y^2$, atque ideò y^2 erit $= \frac{q}{3}$, et y erit $= \frac{\surd q}{\surd 3}$.

Sed, ob e in hoc caſu $= o$, erit z, ſeu $y + e$, $(= y + o) = y$.

Ergò z erit etiàm $= \frac{\surd q}{\surd 3}$.

Ergò in hoc caſu duo valores ipſius x, ſcilicet, y et z (quorum prior y fuerat minor ipſâ M, alter verò z fuerat major ipſâ M,) fiunt æquales inter ſe, et uterque æqualis ipſi $\frac{\surd q}{\surd 3}$. Ergò et M (quæ inter y et z fuerat mediæ cujuſdam magnitudinis,) erit etiàm æqualis ipſi $\frac{\surd q}{\surd 3}$; hoc eſt, valor ipſius x in quantitate binomiâ $qx - x^3$ quando iſta quantitas eſt maxima quæ eſſe poſſit, erit æqualis quantitati $\frac{\surd q}{\surd 3}$. Q. E. I.

Coroll. 1. Ergò $q\,M - M^3$, ſeu maxima magnitudo quantitatis binomiæ $qx - x^3$, erit $(= q \times \frac{\surd q}{\surd 3} - \frac{q}{3} \times \frac{\surd q}{\surd 3} = \frac{3q}{3} \times \frac{\surd q}{\surd 3} - \frac{q}{3} \times \frac{\surd q}{\surd 3} = \frac{2q}{3} \times \frac{\surd q}{\surd 3}) = \frac{2q\surd q}{3 \times \surd 3}$. Et pro-inde, ſi in æquatione cubicâ hujus formæ, $qx - x^3 = r$, homogeneum comparationis, ſeu quantitas cognita, r ſit æqualis quantitati $\frac{2q\surd q}{3\surd 3}$, æquatio habebit tantùm unam radicem, ſcilicet, $\frac{\surd q}{\surd 3}$; ſed, ſi quantitas cognita r ſit minor quantitate $\frac{2q\surd q}{3\surd 3}$, æquatio habebit duas radices, quarum

rum una erit minor ipſâ $\frac{\sqrt{q}}{\sqrt{3}}$, altera verò erit major ipſâ $\frac{\sqrt{q}}{\sqrt{3}}$, ſed minor ipſâ $\sqrt{q}$.

Coroll. 2. Cum quantitas cognita r in æquatione cubicâ $qx - x^3 = r$ eſt æqualis quantitati $\frac{2q\sqrt{q}}{3\sqrt{3}}$, quantitas $\frac{r}{2}$ erit æqualis quantitati $\frac{q\sqrt{q}}{3\sqrt{3}}$, et proinde $\frac{rr}{4}$, ſeu quadratum prioris quantitatis, erit æquale $\frac{q^3}{27}$, ſeu quadrato quantitatis poſterioris. Atque ideò, ſi in aliquâ æquatione cubicâ hujus formæ, $qx - x^3 = r$, quantitas $\frac{rr}{4}$ ſit æqualis quantitati $\frac{q^3}{27}$, æquatio habebit tantùm unam radicem, ſcilicet, $\frac{\sqrt{q}}{\sqrt{3}}$; ſed, ſi quantitas $\frac{rr}{4}$ ſit minor quantitate $\frac{q^3}{27}$, æquatio habebit duas radices, quarum una erit minor ipſâ $\frac{\sqrt{q}}{\sqrt{3}}$, altera verò erit major ipſâ $\frac{\sqrt{q}}{\sqrt{3}}$, ſed minor ipſâ $\sqrt{q}$.

Exemplum Tertium.

Deſignet p lineam aliquam datam, et x lineam aliam variabilem, quæ creſcit continuò à nihilo donec fiat æqualis datæ p. Et requiratur invenire quænam ſit magnitudo lineæ x, quando quantitas binomia $px^2 - x^3$, ſeu exceſſus ſolidi ſub datâ lineâ p et quadrato lineæ x ſuper cubum lineæ x eſt Maxima.

Solutio.

Quantitas binomia $px^2 - x^3$ eſt bìs æqualis nihilo, nempe, primò, cum x eſt $= o$, et iterùm cum x eſt $= p$. Igitùr, dum x creſcit à nihilo uſque ad p, quantitas binomia $px^2 - x^3$, primùm, creſcet ad certam aliquam magnitudinem, et deinde decreſcet ab iſtâ magnitudine ad nihilum. Requiritur autèm invenire magnitudinem lineæ x eo temporis puncto quo quantitas binomia $px^2 - x^3$ attingit hanc ſuam maximam magnitudinem, ex quâ incipit decreſcere. Hoc verò ita inveniri poteſt.

Deſignetur

Defignetur hic valor, quem quærimus, ipfius x per Litteram M. Et fit y alius valor ipfius x minor ipfo M. Erit igitùr quantitas binomia $py^2 - y^3$ minor quantitate binomiâ $p\,M^2 - M^3$. Ergò, dum quantitas x crefcit à valore M ad valorem p, et quantitas binomia $px^2 - x^3$ decrefcit à valore $p\,M^2 - M^3$ ad o, hæc quantitas binomia fiet in uno aliquo hujus temporis puncto æqualis quantitati binomiæ $py^2 - y^3$, quæ eft minor ipfâ $p\,M^2 - M^3$. Ponatur z pro valore ipfius x in ifto temporis puncto. Et erit $pzz - z^3 = pyy - y^3$.

Ponatur jam e pro $z - y$. Et erit $z = y + e$, et $zz\ (= \overline{y + e}^2) = yy + 2ye + ee$, et $z^3\ (= \overline{y + e}^3) = y^3 + 3y^2e + 3ye^2 + e^3$, et $pzz\ (= p \times yy + 2ye + ee) = pyy + 2pye + pee$; et pro-inde erit $pzz - z^3 = pyy + 2pye + pee - y^3 - 3y^2e - 3ye^2 - e^3$.

Sed $pzz - z^3$ eft æqualis $pyy - y^3$.

Ergò et $pyy + 2pye + pee - y^3 - 3y^2e - 3ye^2 - e^3$ erit $= pyy - y^3$; et (addendo utrinque $3y^2e + 3ye^2 + e^3$,) erit $pyy + 2pye + pee - y^3 = pyy - y^3 + 3y^2e + 3ye^2 + e^3$; et (auferendo utrinque $pyy - y^3$,) erit $2pye + pee = 3y^2e + 3ye^2 + e^3$; et (dividendo omnes terminos per e,) erit $2py + pe = 3y^2 + 3ye + e^2$. Hoc autèm verum eft, utcunque parva fuerit quantitas e. Erit igitùr etiàm verum quando quantitas e eft omninò nullius magnitudinis. Sed, ubi e eft nullius magnitudinis, $2py + pe$ erit $2py$, et $3y^2 + 3ye + e^2$ erit $= 3y^2$. Ergò $2py$ erit $= 3y^2$, et pro-inde erit $2p = 3y$, et $y = \frac{2p}{3}$.

Sed, ob e in hoc cafu $= o$, erit z, feu $y + e$, $(= y + o) = y$.

Ergò z erit etiàm $= \frac{2p}{3}$.

Ergò in hoc cafu duo valores ipfius x, fcilicet, y et z, (quorum prior y fuerat minor ipfâ M, alter verò z fuerat major ipfâ M,) fiunt æquales inter fe, et uterque æqualis ipfi $\frac{2p}{3}$. Ergò et M (quæ inter y et z fuerat mediæ cujufdam magnitudinis,) erit etiàm æqualis ipfi $\frac{2p}{3}$; hoc eft, valor ipfius x in quantitate binomiâ $px^2 - x^3$, quando ifta quantitas eft maxima quæ effe poffit, erit æqualis quantitati $\frac{2p}{3}$. Q. E. I.

Corolл. Ergò $p\,M^2 - M^3$, feu maxima magnitudo quantitatis binomiæ $px^2 - x^3$, erit æqualis $(p \times \frac{4pp}{9} - \frac{8p^3}{27} = \frac{12p^3}{27} - \frac{8p^3}{27} =) \frac{4p^3}{27}$. Et pro-inde, fi in æquatione cubicâ hujus formæ, $px^2 - x^3 = r$, homogeneum comparationis, feu quantitas cognita, r fit æqualis quantitati $\frac{4p^3}{27}$, æquatio habebit tantùm unam radicem, fcilicet, $\frac{2p}{3}$; fed fi quantitas cognita r fit minor quanti-

tate

tate $\frac{4p^3}{27}$, æquatio habebit duas radices, quarum una erit minor ipſâ $\frac{2p}{3}$, altera verò erit major ipſâ $\frac{2p}{3}$, ſed minor ipſâ p.

Exemplum Quartum.

Deſignet p quantitatem aliquam datam, et x quantitatem aliam variabilem, quæ creſcit continuò à nihilo donec fiat æqualis datæ p. Et requiratur invenire quænam ſit magnitudo quantitatis x quando quantitas binomia $px^3 - x^4$ eſt maxima quæ eſſe poſſit.

Solutio.

Quantitas binomia $px^3 - x^4$ eſt bìs æqualis nihilo, nempe, primùm, cum x eſt $= o$, et iterùm cum x eſt $= p$. Igitùr, dum x creſcit à nihilo uſque ad p, neceſſe eſt ut quantitas binomia $px^3 - x^4$ primùm creſcat à nihilo ad certam aliquam magnitudinem, et deinde decreſcat ab iſtâ magnitudine ad nihilum. Requiritur autèm invenire magnitudinem quantitatis x, in eo temporis puncto in quo quantitas binomia $px^3 - x^4$ attingit hanc ſuam maximam magnitudinem, ex quâ incipit decreſcere. Hoc verò ita inveniri poteſt.

Deſignetur hic valor, quem quærimus, ipſius x per Litteram M. Et ſit y alius valor ipſius x minor ipſo M. Erit igitùr quantitas $py^3 - y^4$ minor quantitate $p\,M^3 - M^4$. Ergò, dum quantitas x creſcit à valore M uſque ad valorem p, et quantitas binomia $px^3 - x^4$ decreſcit à quantitate $p\,M^3 - M^4$ ad o, hæc quantitas binomia fiet in uno aliquo temporis puncto æqualis quantitati binomiæ $py^3 - y^4$, quæ eſt minor ipſâ $p\,M^3 - M^4$. Ponatur z pro valore quantitatis x in iſto temporis puncto. Et erit $pz^3 - z^4 = py^3 - y^4$.

Ponatur jam e pro $z - y$. Et erit $z = y + e$, et $z^3\ (= \overline{y + e}^3) = y^3 + 3y^2e + 3ye^2 + e^3$, et $z^4\ (= \overline{y + e}^4) = y^4 + 4y^3e + 6y^2e^2 + 4ye^3 + e^4$, et $pz^3\ (= p \times \overline{y^3 + 3y^2e + 3ye^2 + e^3}) = py^3 + 3py^2e + 3pye^2 + pe^3$, et proinde erit $pz^3 - z^4 = py^3 + 3py^2e + 3pye^2 + pe^3 - y^4 - 4y^3e - 6y^2e^2 - 4ye^3 - e^4$; vel, (ſi deſignemus quantitates $3pye^2 + pe^3$ et $6y^2e^2 + 4ye^3 + e^4$, in quibus littera e bìs, vel ſæpiùs quam bìs, occurrit, per notas, &c, &c,) erit $pz^3 - z^4 = py^3 + 3py^2e + \&c - y^4 - 4y^3e - \&c$.

Sed $pz^3 - z^4$ eſt $= py^3 - y^4$.

Ergò et $py^3 + 3py^2e$ + &c $- y^4 - 4y^3e -$ &c erit $= py^3 - y^4$; et (addendo utrinque $4y^3e$ + &c,) erit $py^3 + 3py^2e$ + &c $- y^4 = py^3 - y^4 + 4y^3e$ + &c ; et (auferendo utrinque $py^3 - y^4$,) erit $3py^2e$ + &c $= 4y^3e$ + &c ; et (dividendo omnes terminos per e,) erit $3py^2$ + &c $= 4y^3$ + &c. Hoc autèm verum eſt, utcunque parva fuerit quantitas e. Erit igitùr etiàm verum quando quantitas e eſt omninò nullius magnitudinis. Sed, ubi e eſt nullius magnitudinis, omnes termini in quibus e invenitur, et qui deſignantur in æquatione $3py^2$ + &c $= 4y^3$ + &c per notas &c, fient etiàm nullius magnitudinis, et pro-inde omitti poſſunt ; atque ideò iſta æquatio fiet $3py^2 = 4y^3$. Ergò $3p$ erit $= 4y$, et y erit $= \frac{3p}{4}$.

Sed, ob e in hoc caſu $= o$, erit z, ſeu $y + e$, $(= y + o) = y$.

Ergò z erit etiàm $= \frac{3p}{4}$.

Ergò in hoc caſu duo valores ipſius x, ſcilicet, y et z, (quorum prior, y, fuerat minor ipſâ M, alter verò, z, fuerat major ipſâ M,) fiunt æquales inter ſe, et uterque æqualis ipſi $\frac{3p}{4}$. Ergò et M (quæ inter y et z fuerat mediæ cujuſdam magnitudinis,) erit etiàm æqualis ipſi $\frac{3p}{4}$; hoc eſt, valor ipſius x in quantitate binomiâ $px^3 - x^4$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit æqualis quantitati $\frac{3p}{4}$. Q. E. I.

Coroll. Ergò $p\,M^3 - M^4$, ſeu maxima magnitudo quantitatis binomiæ $px^3 - x^4$, erit $(= p \times \frac{27p^3}{64} - \frac{81p^4}{256} = \frac{4p}{4} \times \frac{27p^3}{64} - \frac{81p4}{256} = \frac{108p^4}{256} - \frac{81p^4}{256}) = \frac{27p^4}{256}$. Et pro-inde, ſi in æquatione biquadraticâ hujus formæ, $px^3 - x^4 = s$, homogeneum comparationis, ſive quantitas cognita, s, ſit æqualis quantitati $\frac{27p^4}{256}$, æquatio habebit tantùm unam radicem, ſcilicet, $\frac{3p}{4}$; ſed, ſi quantitas cognita s ſit minor quantitate $\frac{27p^4}{256}$, æquatio habebit duas radices, quarum una erit minor ipſâ $\frac{3p}{4}$, altera verò erit major ipſâ $\frac{3p}{4}$, ſed minor ipſâ p.

Exemplum

Exemplum Quintum.

Defignet r quantitatem aliquam datam, et x quantitatem aliam variabilem, quæ crefcit continuò à nihilo ufque ad $\sqrt[3]{r}$, feu $r^{\frac{1}{3}}$, feu donec cubus ejus x^3 fiat æqualis datæ quantitati r. Et requiratur invenire quænam fit magnitudo quantitatis x, quando quantitas binomia $rx - x^4$ eft maxima quæ effe poffit.

Solutio.

Quantitas binomia $rx - x^4$ erit bis æqualis nihilo, nempe, primùm, cum x eft $= o$, et iterùm cum x eft $= \sqrt[3]{r}$, et pro-inde x^3 eft $= r$. Igitùr, dum x crefcit à nihilo ufque ad $\sqrt[3]{r}$, et x^3 crefcit à nihilo ufque ad r, neceffe eft ut quantitas binomia $rx - x^4$, primùm, crefcat à nihilo ad certam aliquam magnitudinem, et deinde decrefcat ab iftâ magnitudine ad nihilum. Requiritur autèm invenire magnitudinem quantitatis x in eo temporis puncto in quo quantitas binomia $rx - x^4$ attingit hanc fuam maximam magnitudinem, ex quâ incipit decrefcere. Hoc verò ita inveniri poteft.

Defignetur hic valor, quem quærimus, ipfius x per Litteram M. Et fit y alius valor ipfius x minor valore M. Erit igitùr quantitas $ry - y^4$ minor quantitate r M $-$ M^4. Ergò, dum quantitas x crefcit à valore M ufque ad valorem $\sqrt[3]{r}$, feu $r^{\frac{1}{3}}$, et quantitas x^3 crefcit à valore M^3 ad valorem r, et quantitas binomia $rx - x^4$ decrefcit à quantitate r M $-$ M^4 ad o, hæc quantitas binomia $rx - x^4$ fiet in uno aliquo hujus temporis puncto æqualis quantitati binomiæ $ry - y^4$, quæ eft minor ipfâ r M $-$ M^4. Ponatur z pro valore quantitatis x in ifto temporis puncto. Et erit $rz - z^4 = ry - y^4$.

Ponatur jam e pro $z - y$. Et erit $z = y + e$, et z^4 $(= \overline{y + e}^4) = y^4 + 4y^3e +$ &c, et rz $(= r \times y + e) = ry + re$; et pro-inde erit $rz - z^4 = ry + re - y^4 - 4y^3e -$ &c.

Sed $rz - z^4$ eft $= ry - y^4$.

Ergò et $ry + re - y^4 - 4y^3e -$ &c erit $= ry - y^4$; et (addendo utrinque $4y^3e +$ &c,) erit $ry + re - y^4 = ry - y^4 + 4y^3e +$ &c; et (auferendo utrinque $ry - y^4$,) erit $re = 4y^3e +$ &c; et (dividendo omnes terminos per e,) erit $r = 4y^3 +$ &c, in quâ æquatione omnes termini qui per notam &c defignentur continent in fe quantitatem e. Hæc autèm æquatio vera eft, quan-

tumvis parva fuerit quantitas *e*. Erit igitùr vera etiàm quando quantitas *e* eſt omninò nullius magnitudinis. Sed, ubi *e* eſt omninò nullius magnitudinis, omnes termini in quibus *e* invenitur, et qui in æquatione $r = 4y^3 +$ &c per notam &c deſignantur, erunt etiàm nullius magnitudinis, et pro-inde omitti poſſunt; atque ideò iſta æquatio fiet in hoc caſu $r = 4y^3$. Ergò y^3 erit $= \frac{r}{4}$, et y erit $= \frac{\sqrt[3]{r}}{\sqrt[3]{4}}$, ſeu $\sqrt[3]{\frac{r}{4}}$.

Sed, ob *e* in hoc caſu $= o$, erit z, ſeu $y + e$, $(= y + o) = y$.

Ergò et z erit $= \frac{\sqrt[3]{r}}{\sqrt[3]{4}}$, ſeu $\sqrt[3]{\frac{r}{4}}$.

Ergò in hoc caſu duo valores ipſius x, ſcilicet, y et z (quorum prior y fuerat minor ipſâ M, alter verò z fuerat major ipſâ M,) fiunt æquales inter ſe, et uterque æqualis ipſi $\frac{\sqrt[3]{r}}{\sqrt[3]{4}}$, ſive $\sqrt[3]{\frac{r}{4}}$. Ergò et M (quæ inter y et z fuerat mediæ cujuſdam magnitudinis,) erit etiàm æqualis ipſi $\frac{\sqrt[3]{r}}{\sqrt[3]{4}}$, ſeu $\sqrt[3]{\frac{r}{4}}$; hoc eſt, valor ipſius x in quantitate binomiâ $rx - x^4$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit æqualis quantitati $\frac{\sqrt[3]{r}}{\sqrt[3]{4}}$, ſeu $\sqrt[3]{\frac{r}{4}}$, ſeu radici cubicæ fractionis $\frac{r}{4}$. Q. E. I.

Coroll. Ergò $r\,M - M^4$, ſeu maxima magnitudo quantitatis binomiæ $rx - x^4$, erit $= r \times \frac{\sqrt[3]{r}}{\sqrt[3]{4}} - \overline{\frac{\sqrt[3]{r}}{\sqrt[3]{4}}}^4$.

Ponatur jam $c = \frac{\sqrt[3]{r}}{\sqrt[3]{4}}$, ſive $\sqrt[3]{\frac{r}{4}}$. Et c^3 erit $= \frac{r}{4}$, et $4c^3$ erit $= r$, et $4c^3 \times c$ erit $= r \times \frac{\sqrt[3]{r}}{\sqrt[3]{4}}$, hoc eſt, $4c^4$ erit $= r \times \frac{\sqrt[3]{r}}{\sqrt[3]{4}}$; et c^4 erit $= \overline{\frac{\sqrt[3]{r}}{\sqrt[3]{4}}}^4$. Ergò $4c^4 - c^4$ erit $= r \times \frac{\sqrt[3]{r}}{\sqrt[3]{4}} - \overline{\frac{\sqrt[3]{r}}{\sqrt[3]{4}}}^4$; hoc eſt, $3c^4$ erit $= r \times \frac{\sqrt[3]{r}}{\sqrt[3]{4}} - \overline{\frac{\sqrt[3]{r}}{\sqrt[3]{4}}}^4$. Ergò $r\,M - M^4$, ſeu maxima magnitudo quantitatis binomia $rx - x^4$, erit æqualis quantitati $3c^4$, in quâ litera *c* deſignat $\frac{\sqrt[3]{r}}{\sqrt[3]{4}}$, ſeu $\sqrt[3]{\frac{r}{4}}$, ſeu radicem cubicam quartæ partis datæ quantitatis *r*, quæ eſt co-efficiens quantitatis x in quantitate binomiâ $rx - x^4$.

Atque

Atque hinc ſequitur quòd, ſi in aliquâ æquatione biquadraticâ hujus formæ, $rx - x^4 = s$, homogeneum comparationis, ſive quantitas cognita, s ſit æqualis quantitati $r \times \sqrt[3]{\frac{r}{4}} - \sqrt[3]{\frac{r}{4}}^4$, ſive, (ponendo $c = \sqrt[3]{\frac{r}{4}}$), æqualis quantitati $3c^4$, æquatio habebit tantùm unam radicem, ſcilicet, c, ſeu $\sqrt[3]{\frac{r}{4}}$; ſed, ſi quantitas cognita s ſit minor quantitate $3c^4$, æquatio habebit duas radices, quarum una erit minor ipſâ c, ſeu $\sqrt[3]{\frac{r}{4}}$, altera verò erit major ipſâ c, ſive $\sqrt[3]{\frac{r}{4}}$, ſed minor $\sqrt[3]{r}$.

Exemplum Sextum.

Deſignet P quantitatem aliquam datam, et m et n duos numeros integros quoſcunque; et ſit x quantitas variabilis, quæ creſcit continuò à nihilo uſque ad $\sqrt[n]{\text{P}}$, ſeu $\text{P}^{\frac{1}{n}}$, ſeu donec x^n fiat æqualis P, et pro-inde x^{m+n} fiat æqualis $\text{P}x^m$. Et requiratur invenire quænam ſit magnitudo quantitatis x, quando quantitas binomia $\text{P}x^m - x^{m+n}$ eſt maxima quæ eſſe poſſit.

Solutio.

Quantitas binomia $\text{P}x^m - x^{m+n}$ erit bis æqualis nihilo, nempe, primùm, cum x eſt $= o$, et iterùm cum x eſt $= \sqrt[n]{\text{P}}$, ſeu $\text{P}^{\frac{1}{n}}$, et x^n eſt $= \text{P}$, et x^{m+n} eſt $= \text{P}x^m$. Igitùr, dum x creſcit à nihilo uſque ad $\text{P}^{\frac{1}{n}}$, et x^n creſcit à nihilo uſque ad P, neceſſe eſt ut quantitas binomia $\text{P}x^m - x^{m+n}$, primùm, creſcat à nihilo uſque ad certam aliquam magnitudinem, et deinde decreſcat ab iſtâ magnitudine ad nihilum. Requiritur autèm invenire magnitudinem quantitatis x in eo temporis puncto in quo quantitas binomia

$\text{P}x$

$Px^m - x^{m+n}$ attingit hanc ſuam maximam magnitudinem, ex quâ incipit decreſcere. Hoc verò ita inveniri poteſt.

Deſignetur hic valor, quem quærimus, ipſius x per Litteram M. Et ſit y alius valor ipſius x minor valore M. Erit igitùr quantitas $Py^m - y^{m+n}$ minor quantitate $PM^m - M^{m+n}$. Ergò, dum quantitas x creſcit à valore M uſque ad valorem $P\frac{1}{n}$, et quantitas x^n creſcit à valore M^n uſque ad valorem P, et quantitas binomia $Px^m - x^{m+n}$ decreſcit à valore $PM^m - M^{m+n}$ ad o, hæc quantitas binomia $Px^m - x^{m+n}$ fiet in uno aliquo hujus temporis puncto æqualis quantitati binomiæ $Py^m - y^{m+n}$, quæ eſt minor ipſâ PM^m. Ponatur z pro valore quantitatis x in iſto temporis puncto. Et erit $Pz^m - z^{m+n} = Py^m - y^{m+n}$.

Ponatur jam $e = z - y$. Et erit $z = y + e$, et pro-inde $z^m = \overline{y+e}^m$ = (per theorema binomium,) ſeriei $y^m + m \times y^{m-1} \times e + m \times \frac{m-1}{2} \times y^{m-2} \times e^2 +$ &c, et Pz^m erit = ſeriei $Py^m + m\,P \times y^{m-1} \times e + m\,P \times \frac{m-1}{2} \times {}^{m-2} \times e^2 +$ &c; et z^{m+n} erit $= \overline{y+e}^{m+n}$ = (per theorema binomium,) $y^{m+n} + \overline{m+n} \times y^{m+n-1} \times e + \overline{m+n} \times \frac{m+n-1}{2} \times y^{m+n-2} \times e^2 +$ &c; atque ideò $Pz^m - z^{m+n}$ erit = Seriei $Py^m + m\,P \times y^{m-1} \times e + m\,P \times \frac{m-1}{2} \times y^{m-2} \times e^2 +$ &c $- y^{m+n} - \overline{m+n} \times y^{m+n-1} \times e - \overline{m+n} \times \frac{m+n-1}{2} \times y^{m+n-2} \times e^2 -$ &c.

Sed $Pz^m - z^{m+n}$ eſt $= Py^m - y^{m+n}$.

Ergò et Series $Py^m + m\,P \times y^{m-1} \times e + m\,P \times \frac{m-1}{2} \times y^{m-2} \times e^2 + $ &c $- y^{m+n} - \overline{m+n} \times y^{m+n-1} \times e - \overline{m+n} \times \frac{m+n-1}{2} \times y^{m+n-2} \times e^2 -$ &c erit $= Py^m - y^{m+n}$; et (addendo utrinque $\overline{m+n} \times y^{m+n-1} \times e + \overline{m+n} \times \frac{m+n-1}{2} \times y^{m+n-2} \times e^2 +$ &c,) Series $Py^m + m\,P \times y^{m-1} \times e + m\,P \times \frac{m-1}{2} \times y^{m-2} \times e^2 +$ &c $- y^{m+n}$ erit = Seriei $Py^m - y^{m+n}$

$- y^{m+n} + \overline{m+n} \times y^{m+n-1} \times e + \overline{\overline{m+n} \times \frac{m+n-1}{2}} \times y^{m+n-2} \times e^2 +$ &c; et (auferendo utrinque $Py^m - y^{m+n}$,) erit Series $m\,P \times y^{m-1} \times e + m\,P \times \frac{m-1}{2} \times y^{m-2} \times e^2 +$ &c $=$ Seriei $\overline{m+n} \times y^{m+n-1} \times e + \overline{m+n} \times \frac{m+n-1}{2} \times y^{m+n-2} \times e^2 +$ &c; et (dividendo omnes terminos per e,) erit Series $m\,P \times y^{m-1} + m\,P \times \frac{m-1}{2} \times y^{m-2} \times e +$ &c $=$ Seriei $\overline{m+n} \times y^{m+n-1} + \overline{m+n} \times \frac{m+n-1}{2} \times y^{m+n-2} \times e +$ &c, in quâ æquatione omnes termini qui per duas notas &c designantur continent in se quadratum et cubum, et alias altiores potestates quantitatis e. Hæc autem æquatio vera est, quantumvis parva fuerit quantitas e. Erit igitùr vera etiàm quando quantitas e est omninò nullius magnitudinis. Sed, ubi e est omninò nullius magnitudinis, omnes termini in quibus e invenitur erunt etiàm nullius magnitudinis, et pro-inde omitti possunt; atque ideò hæc æquatio fiet in hoc casu $m\,P \times y^{m-1} = \overline{m+n} \times y^{m+n-1}$. Ergò (dividendo hos terminos per y^{m-1},) erit $m\,P = \overline{m+n} \times y^n$, atque ideò y^n erit $= \frac{m\,P}{m+n}$, seu $\frac{m}{m+n} \times P$, et y erit $= \sqrt[n]{\frac{m\,P}{m+n}}$.

Sed, ob e in hoc casu $= o$, erit z seu $y + e$, $(= y + o) = y$.

Ergò et z erit $= \sqrt[n]{\frac{m\,P}{m+n}}$.

Ergò in hoc casu duo valores ipsius x, scilicet, y et z (quorum prior y fuerat minor ipsâ M, alter verò z fuerat major ipsâ M,) fiunt æquales inter se, et uterque æqualis ipsi $\sqrt[n]{\frac{m\,P}{m+n}}$. Ergò et M(quæ inter y et z fuerat mediæ cujusdam magnitudinis,) erit etiàm æqualis quantitati $\sqrt[n]{\frac{m\,P}{m+n}}$; hoc est, valor ipsius x in quantitate binomiâ $Px^m - x^{m+n}$, quando ista quantitas est maxima quæ esse possit, erit æqualis quantitati $\sqrt[n]{\frac{m\,P}{m+n}}$, seu $\sqrt[n]{\frac{m\,P}{m+n}}$. Q. E. I.

Coroll. Ergò $PM^m - M^{m+n}$, seu maxima magnitudo quantitatis binomiæ $Px^m - x^{m+n}$, erit $= P \times \sqrt[n]{\frac{m\,P}{m+n}} - \sqrt[n]{\frac{m\,P}{m+n}}^{m+n} = P \times$

mP

$\sqrt[n]{\frac{mP}{m+n}}^{\frac{1}{n}} - \sqrt[n]{\frac{mP}{m+n}}^{\frac{m+n}{n}}$. Et pro-inde, ſi in æquatione aliquâ binomiali hujus formæ, $Px^m - x^{m+n} = Q$, homogeneum comparationis, ſeu quantitas cognita, Q fuerit æqualis quantitati $P \times \sqrt[n]{\frac{mP}{m+n}}^{\frac{1}{n}} - \sqrt[n]{\frac{mP}{m+n}}^{\frac{m+n}{n}}$, æquatio habebit tantùm unam radicem, ſcilicet, $\sqrt[n]{\frac{mP}{m+n}}^{\frac{1}{n}}$; ſed, ſi quantitas cognita Q fuerit minor quantitate $P \times \sqrt[n]{\frac{mP}{m+n}}^{\frac{1}{n}} - \sqrt[n]{\frac{mP}{m+n}}^{\frac{m+n}{n}}$, æquatio habebit duas radices, quarum una erit minor ipſâ $\sqrt[n]{\frac{mP}{m+n}}^{\frac{1}{n}}$, altera verò erit major ipſâ $\sqrt[n]{\frac{mP}{m+n}}^{\frac{1}{n}}$, ſed minor ipſâ $P^{\frac{1}{n}}$, ſeu $\sqrt[n]{P}$.

Solutio omnium exemplorum præcedentium per expreſſionem generalem $\sqrt[n]{\frac{mP}{m+n}}^{\frac{1}{n}}$, *hîc inventam pro Solutione hujus exempli ſexti.*

Hoc exemplum ſextum comprehendit omnia exempla præcedentia de valoribus quantitatis x in quantitatibus binomiis $px - xx$, $qx - x^3$, $px^2 - x^3$, $px^3 - x^4$, et $rx - x^4$, quando iſtæ quantitates binomiæ ſunt maximæ quæ eſſe poſſint; adeò ut ſolutiones diverſæ iſtorum exemplorum derivari poſſunt ex ſolutione hujus exempli ſexti, tanquam caſus ejuſdem particulares, ſubſtituendo in quantite generali binomiâ $Px^m - x^{m+n}$ pro P, (co-efficiente quantitatis x^m in, primo hujus quantitatis termino,) co-efficientes p, q, p, p, et r, et pro m, ſeu indice poteſtatis ipſius x in hoc primo termino, indices 1, 1, 2, 3, et 1, et pro $m + n$, indice poteſtatis ipſius x in ſecundo hujus quantitatis termino, indices $1 + 1$, $1 + 2$, $2 + 1$, $3 + 1$, et $1 + 3$, ſeu 2, 3, 3, 4, et 4; quod quidem fieri poteſt hoc modo.

Primò

Primò, ſit $P = p$, et $m = 1$, et $n = 1$. Et quantitas binomia generalis $Px^m - x^{m+n}$ fiet quantitas binomia particularis ($px^1 - x^1 \times x^1$, ſeu) $px - x^2$: et $\left.\frac{mP}{m+n}\right|^{\frac{1}{n}}$, ſeu valor quantitatis M, erit $\left(= \left.\frac{1 \times p}{1+1}\right|^{\frac{1}{1}} = \left.\frac{p}{2}\right|^{1}\right) = \frac{p}{2}$; hoc eſt, quantitas x in quantitate binomiâ $px - xx$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit $\frac{p}{2}$, ſeu dimidium datæ quantitatis p; ut in exemplo primo determinatum eſt.

Secundò, ſit $P = q$, et $m = 1$, et $n = 2$. Et quantitas binomia $Px^m - x^{m+n}$ fiet $= (qx^1 - x^1 \times x^2$, ſeu) $qx - x^3$; et $\left.\frac{mP}{m+n}\right|^{\frac{1}{n}}$, ſeu M, erit $(= \left.\frac{1 \times q}{1+2}\right|^{\frac{1}{2}} = \left.\frac{q}{3}\right|^{\frac{1}{2}} = \sqrt{\left.\frac{q}{3}\right)} = \frac{\sqrt{q}}{\sqrt{3}}$; hoc eſt, quantitas x in quantitate binomiâ $qx - x^3$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit $\frac{\sqrt{q}}{\sqrt{3}}$; ut in exemplo ſecundo determinatum eſt.

Tertio, ſit $P = p$, et $m = 2$, et $n = 1$. Et quantitas binomia $Px^m - x^{m+n}$ fiet $= (= px^2 - x^{2+1}$, ſeu) $px^2 - x^3$; et $\left.\frac{mP}{m+n}\right|^{\frac{1}{n}}$, ſeu M, erit $(= \left.\frac{2p}{2+1}\right|^{\frac{1}{1}} = \left.\frac{2p}{3}\right|^{1}) = \frac{2p}{3}$; hoc eſt, quantitas x in quantitate binomiâ $px^2 - x^3$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit $= \frac{2p}{3}$; ut in exemplo tertio determinatum eſt.

Quarto, ſit $P = p$, et $m = 3$, et $n = 1$. Et quantitas $Px^m - x^{m+n}$ fiet $= (px^3 - x^{3+1}$, ſeu) $px^3 - x^4$; et $\left.\frac{mP}{m+n}\right|^{\frac{1}{n}}$, ſeu M, erit $(= \left.\frac{3p}{3+1}\right|^{\frac{1}{1}} = \left.\frac{3p}{4}\right|^{1}) = \frac{3p}{4}$; hoc eſt, quantitas x in quantitate binomiâ $px^3 - x^4$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit $\frac{3p}{4}$; ut in exemplo quarto determinatum eſt.

Quinto, fit $P = r$, et $m = 1$, et $n = 3$. Et quantitas binomia $Px^m - x^{m+n}$ fiet $= (rx^1 - x^1 +$, ſeu) $rx - x^4$; et $\sqrt[n]{\frac{mP}{m+n}}$ erit $(= \sqrt[3]{\frac{1 \times r}{1+3}}) = \sqrt[3]{\frac{r}{4}}$, ſeu $\sqrt[3]{\frac{r}{4}}$; hoc eſt, quantitas x in quantitate binomiâ $rx - x^4$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit $= \sqrt[3]{\frac{r}{4}}$, ſeu $\sqrt[3]{\frac{r}{4}}$; ut in exemplo quinto determinatum eſt.

Hæc ſecunda inveſtigatio valorum quantitatis M, ſeu x, in quantitatibus binomiis $px - xx$, $qx - x^3$, $px^2 - x^3$, $px^3 - x^4$, et $rx - x^4$, quando iſtæ quantitates fiunt maximæ quæ eſſe poſſint, per applicationem ſucceſſivam expreſſionis $\sqrt[n]{\frac{mP}{m+n}}$ (quæ deſignat valorem ipſius M, ſeu x, in quantitate binomiâ generali $Px^m - x^{m+n}$ quando iſta quantitas fit maxima quæ eſſe poſſit,) ad illorum inventionem, cum (ut jàm oſtenſum eſt,) eoſdem omninò valores quantitatis M nobis indicet quos anteà inveneramus in exemplis quinque præcedentibus, magnoperè confirmat inveſtigationes iſtorum valorum factas in iſtis exemplis, et novo argumento probat iſtos valores fuiſſe ibidem rectè determinatos. Et hæc exempla ſufficere poterunt pro quantitatibus binomiis. Itaque nunc pergimus ad quantitates trinomias.

Exemplum Septimum, in quantitate trinomiâ $qx + px^2 - x^3$.

Deſignent p et q quantitates aliquas datas, et x deſignet aliam quantitatem variabilem quæ creſcit continuò à nihilo donec xx fiat æqualis quantitati binomiæ $q + px$, et pro-inde x^3 fiat æqualis quantitati binomiæ $qx + px^2$, et quantitas trinomia $qx + px^2 - x^3$ fiat æqualis nihilo. Et requiratur, primò, invenire quænam ſit magnitudo quantitatis x, quando quantitas trinomia $qx + px^2 - x^3$ eſt æqualis nihilo, et, ſecundò, invenire quænam ſit magnitudo quantitatis x quando quantitas trinomia $qx + px^2 - x^3$ eſt maxima quæ eſſe poſſit.

Solutio.

Solutio.

Quando quantitas trinomia $qx + px^2 - x^3$ eſt æqualis nihilo, $qx + px^2$ erit æqualis x^3, et $q + px = xx$, et $xx - px = q$. Ergò (addendo utrinque $\frac{pp}{4}$,) $xx - px + \frac{pp}{4}$ erit $= q + \frac{pp}{4} = \frac{pp + 4q}{4}$, et (extrahendo utrinque radices quadraticas,) $x - \frac{p}{2}$ erit $= \frac{\sqrt{pp + 4q}}{2}$, et (addendo utrinque $\frac{p}{2}$,) quantitas x erit $= \frac{p + \sqrt{pp + 4q}}{2}$. Q. E. I.

Secundò, quoniam quando quantitas trinomia $qx + px^2 - x^3$ eſt æqualis nihilo, quantitas x eſt æqualis quantitati $\frac{p + \sqrt{pp + 4q}}{2}$, ſequitur quòd, dum x creſcit continuò à nihilo uſque ad $\frac{p + \sqrt{pp + 4q}}{2}$, quantitas trinomia $qx + px^2 - x^3$ erit bìs æqualis nihilo, nempe, primò, quando x eſt $= o$, et, ſecundò, quando x eſt $= \frac{p + \sqrt{pp + 4q}}{2}$. Igitùr, dum x creſcit continuò à nihilo uſque ad $\frac{p + \sqrt{pp + 4q}}{2}$, neceſſe eſt ut quantitas trinomia $qx + px^2 - x^3$, primùm, creſcat à nihilo uſque ad certam aliquam magnitudinem, et deinde decreſcat ab iſtâ magnitudine ad nihilum. Requiritur autèm invenire magnitudinem quantitatis x in eo temporis puncto in quo quantitas trinomia $qx + px^2 - x^3$ attingit hanc ſuam maximam magnitudinem, ex quâ incipit decreſcere. Hoc verò ita inveniri poteſt.

Deſignetur hic valor, quem quærimus, ipſius x per Litteram M. Et ſit y alius valor ipſius x minor valore M. Erit igitùr quantitas trinomia $qy + py^2 - y^3$ minor quantitate $q\,M + p\,M^2 - M^3$. Ergò, dum quantitas x creſcit à valore M uſque ad valorem $\frac{p + \sqrt{pp + 4q}}{2}$, et quantitas trinomia $qx + px^2 - x^3$ decreſcit à quantitate $q\,M + p\,M^2 - M^3$ ad o, hæc quantitas trinomia $qx + px^2 - x^3$ fiet in aliquo hujus temporis puncto æqualis quantitati trinomiæ $qy + py^2 - y^3$, quæ eſt minor ipſâ $q\,M + p\,M^2 - M^3$. Ponatur z pro valore quantitatis x in iſto temporis puncto. Et erit $qz + pz^2 - z^3 = qy + py^2 - y^3$.

Ponatur jam e pro $z - y$. Et erit $z = y + e$, et z^2 $(= \overline{y + e}^2) = y^2 + 2ye + e^2$, et z^3 $(= \overline{y + e}^3) = y^3 + 3y^2e + 3ye^2 + e^3$, et qz $(= q \times \overline{y + e}) = qy + qe$, et pz^2 $(= p \times \overline{y^2 + 2ye + e^2}) = py^2 + 2pye + pe^2$; et pro-inde quantitas trinomia $qz + pz^2 - z^3$ erit $= qy + qe + py^2 + 2pye + pe^2 - y^3 - 3y^2e - 3ye^2 - e^3$.

Sed quantitas trinomia $qz + pz^2 - z^3$ eſt $=$ quantitati trinomiæ $qy + py^2 - y^3$.

Ergò et quantitas multinomia $qy + qe + py^2 + 2pye + pe^2 - y^3 - 3y^2e - 3ye^2 - e^3$ erit $=$ quantitati trinomiæ $qy + py^2 - y^3$; et (addendo utrinque $3y^2e + 3ye^2 + e^3$,) erit $qy + qe + py^2 + 2pye + pe^2 - y^3 = qy + py^2 - y^3 + 3y^2e + 3ye^2 + e^3$; et (auferendo utrinque $qy + py^2 - y^3$) erit $qe + 2pye + pe^2 = 3y^2e + 3ye^2 + e^3$; et (dividendo omnes terminos per e,) erit $q + 2py + pe = 3y^2 + 3ye + e^2$. Hoc autèm verum eſt, quantumvis parva fuerit quantitas e. Erit igitùr verum etiàm quando quantitas e eſt nullius omninò magnitudinis. Sed, ubi e eſt omninò nullius magnitudinis, omnes termini in quibus e invenitur erunt etiàm nullius magnitudinis, et pro-inde omitti poſſunt. Atque ideò æquatio prædicta $q + 2py + pe = 3y^2 + 3ye + e^2$ fiet in hoc caſu $q + 2py = 3y^2$, et pro-inde $\frac{q}{3} + \frac{2py}{3}$ erit $= y^2$, et $y^2 - \frac{2py}{3}$ erit $= \frac{q}{3}$, et $y^2 - \frac{2py}{3} + \frac{pp}{9}$ erit $(= \frac{q}{3} + \frac{pp}{9} = \frac{3q}{9} + \frac{pp}{9}) = \frac{pp+3q}{9}$, et pro-inde $y - \frac{p}{3}$ erit $= \frac{\sqrt{pp + 3q}}{3}$, et y erit $= \frac{p + \sqrt{pp + 3q}}{3}$.

Sed, ob e in hoc caſu $= o$, erit z, ſeu $y + e$ $(= y + o) = y$.

Ergò et z erit $= \frac{p + \sqrt{pp + 3q}}{3}$.

Ergò in hoc caſu duo valores ipſius x, ſcilicet, y et z, (quorum prior y fuerat minor ipſâ M, alter verò z fuerat major ipſâ M,) fiunt æquales inter ſe, et uterque æqualis ipſi $\frac{p + \sqrt{pp + 3q}}{3}$. Ergò et M (quæ inter y et z fuerat mediæ cujuſdam magnitudinis,) erit etiàm æqualis ipſi $\frac{p + \sqrt{pp + 3q}}{3}$; hoc eſt, valor ipſius x in quantitate trinomiâ $qx + px^2 - x^3$, quando

quando iſta quantitas eſt maxima quæ eſſe poſſit, erit æqualis quantitati $\frac{p+\sqrt{pp+3q}}{3}$. Q. E. I.

Coroll. Ergò $qM + pM^2 - M^3$, ſeu maxima magnitudo quantitatis trinomiæ $qx + px^2 - x^3$, erit ea quæ oritur ſubſtituendo pro x in hâc quantitate trinomiâ quantitatem $\frac{p+\sqrt{pp+3q}}{3}$. Et pro-inde, ſi in aliquâ æquatione cubicâ hujus formæ, $qx + px^2 - x^3 = r$, quantitas $\frac{p+\sqrt{pp+3q}}{3}$ vocetur M, et valor quantitatis $qM + pM^2 - M^3$ inde derivatæ vocetur D, et homogeneum comparationis in hâc æquatione, ſeu quantitas cognita, r ſit æqualis quantitati D, æquatio habebit tantùm unam radicem, ſcilicet, M, ſeu $\frac{p+\sqrt{pp+3q}}{3}$; ſed, ſi quantitas cognita r ſit minor quantitate D, ſeu $qM + pM^2 - M^3$, æquatio habebit duas radices, quarum una erit minor ipſâ M, ſeu $\frac{p+\sqrt{pp+3q}}{3}$, altera verò erit major ipſâ M, ſeu $\frac{p+\sqrt{pp+3q}}{3}$, ſed minor ipſâ $\frac{p+\sqrt{pp+4q}}{2}$.

Exemplum Octavum, in quantitate trinomiâ $qx - px^2 - x^3$.

Deſignent p et q quantitates aliquas datas, et x deſignet aliam quantitatem variabilem, quæ creſcit continuò à nihilo donec quantitas binomia $px + xx$ fiat æqualis quantitati datæ q, et pro-inde quantitas binomia $px^2 + x^3$ fiat æqualis quantitati qx, et quantitas trinomia $qx - px^2 - x^3$ fiat æqualis nihilo: Et requiratur invenire, primò, quænam ſit magnitudo quantitatis x quando quantitas trinomia $qx - px^2 - x^3$ eſt æqualis nihilo; et, ſecundò, invenire quænam ſit magnitudo ejuſdem quantitatis x quando quantitas trinomia $qx - px^2 - x^3$ eſt maxima quæ eſſe poſſit.

Solutio.

Solutio.

Quando quantitas trinomia $qx - px^2 - x^3$ eſt æqualis nihilo, qx erit æqualis $px^2 + x^3$, et q æqualis $px + xx$. Ergò (addendo utrinque $\frac{pp}{4}$,) erit $q + \frac{pp}{4}$, ſeu $\frac{4q+pp}{4} = xx + px + \frac{pp}{4}$, et (extrahendo utrinque radices quadraticas,) erit $\frac{\sqrt{4q+pp}}{2} = x + \frac{p}{2}$, ſeu $x + \frac{p}{2} = \frac{\sqrt{pp+4q}}{2}$, et (auferendo utrinque $\frac{p}{2}$) erit $x = \frac{\sqrt{pp+4q}-p}{2}$. Q. E. I.

Secundò, quoniam, quando quantitas trinomia $qx - px^2 - x^3$ eſt æqualis nihilo, quantitas x eſt æqualis $\frac{\sqrt{pp+4q}-p}{2}$, ſequitur quòd, dum x creſcit continuò à nihilo uſque ad $\frac{p+\sqrt{pp+4q}}{2}$, quantitas trinomia $qx - px^2 - x^3$ erit bìs æqualis nihilo, nempe, primò, quando x eſt $= o$, et, ſecundò, quando x eſt $= \frac{\sqrt{pp+4q}-p}{2}$. Igitùr, dum x creſcit continuò à nihilo uſque ad $\frac{\sqrt{pp+4q}-p}{2}$, neceſſe eſt ut quantitas trinomia $qx - px^2 - x^3$, primùm, creſcat à nihilo uſque ad certam aliquam magnitudinem, et deinde decreſcat ab iſtâ magnitudine ad nihilum. Requiritur autèm invenire magnitudinem quantitatis x in eo temporis puncto in quo quantitas trinomia $qx - px^2 - x^3$ attingit hanc ſuam maximam magnitudinem, ex quâ incipit decreſcere. Hoc autèm ita inveniri poteſt.

Deſignetur hic valor, quem quærimus, ipſius x per Litteram M. Et ſit y alius valor ipſius x minor valore M. Erit igitùr quantitas trinomia $qy - py^2 - y^3$ minor quantitate $q\,M - p\,M^2 - M^3$. Ergò, dum quantitas x creſcit à valore M ad valorem $\frac{\sqrt{pp+4q}-p}{2}$, et quantitas trinomia $qx - px^2 - x^3$ decreſcit à quantitate $q\,M - p\,M^2 - M^3$ ad o, hæc quantitas trinomia $qx - px^2 - x^3$ fiet in uno aliquo hujus temporis puncto æqualis quantitati trinomiæ $qy - py^2 - y^3$, quæ eſt minor ipſâ $q\,M - p\,M^2 - M^3$. Ponatur z pro valore quantitatis x in iſto temporis puncto. Et erit $qz - pz^2 - z^3 = qy - py^2 - y^3$.

Ponatur

Ponatur jam $e = z - y$. Et erit $z = y + e$, et $z^2 (= \overline{y+e}^2) = y^2 + 2ye + e^2$, et $y^3 (= \overline{y+e}^3) = y^3 + 3y^2e + 3ye^2 + e^3$, et pro-inde $qz (= q \times \overline{y+e}) = qy + qe$, et $pz^2 (= p \times \overline{y^2 + 2ye + e^2}) = py^2 + 2pye + pe^2$; et quantitas trinomia $qz - pz^2 - z^3$ erit = quantitati multinomiæ $qy + qe - py^2 - 2pye - pe^2 - y^3 - 3y^2e - 3ye^2 - e^3$.

Sed quantitas trinomia $qz - pz^2 - z^3$ eſt æqualis quantitati trinomiæ $qy - py^2 - y^3$.

Ergò et quantitas multinomia $qy + qe - py^2 - 2pye - pe^2 - y^3 - 3y^2e - 3ye^2 - e^3$ erit = quantitati trinomiæ $qy - py^2 - y^3$. Et (addendo utrinque $2pye + pe^2 + 3y^2e + 3ye^2 + e^3$) erit $qy + qe - py^2 - y^3 = qy - py^2 - y^3 + 2pye + pe^2 + 3y^2e + 3ye^2 + e^3$; et (auferendo utrinque $qy - py^2 - y^3$,) erit $qe = 2pye + pe^2 + 3y^2e + 3ye^2 + e^3$; et (dividendo omnes terminos per e,) erit $q = 2py + pe + 3y^2 + 3ye + e^2$.

Hoc autèm verum eſt, quantumvis parva fuerit quantitas e. Erit igitùr verum etiàm quando quantitas e eſt omninò nullius magnitudinis. Sed, ubi e eſt omninò nullius magnitudinis, omnes termini in quibus e invenitur erunt etiàm nullius magnitudinis, et pro-inde omitti poſſunt. Atque ideò æquatio prædicta $q = 2py + pe + 3y^2 + 3ye + e^2$ fiet in hoc caſu $q = 2py + 3y^2$. Et pro-inde $\frac{q}{3}$ erit $= \frac{2py}{3} + yy$, et $yy + \frac{2py}{3} + \frac{pp}{9}$ erit $(= \frac{q}{3} + \frac{pp}{9} = \frac{pp}{9} + \frac{3q}{9}) = \frac{pp+3q}{9}$, et $y + \frac{p}{3}$ erit $= \frac{\sqrt{pp+3q}}{3}$, et y erit $= \frac{\sqrt{pp+3q} - p}{3}$.

Sed ob e, in hoc caſu, $= o$, erit z, ſeu $y + e$, $(= y + o) = y$.

Ergò et z erit $= \frac{\sqrt{pp+3q} - p}{3}$.

Ergò in hoc caſu duo valores ipſius x, ſcilicet, y et z, (quorum prior y fuerat minor ipſâ M, alter verò z fuerat major ipſâ M,) fiunt æquales inter ſe, et uterque æqualis ipſi $\frac{\sqrt{pp+3q} - p}{3}$. Ergò et M (quæ inter y et z eſt mediæ cujuſdam magnitudinis,) erit etiàm æqualis ipſi $\frac{\sqrt{pp+3q} - p}{3}$; hoc eſt, valor ipſius x in quantitate trinomiâ $qx - px^2 - x^3$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit æqualis quantitati $\frac{\sqrt{pp+3q} - p}{3}$. Q. E. I.

Coroll. Ergò $qM - pM^2 - M^3$, ſeu maxima magnitudo quantitatis trinomiæ $qx - px^2 - x^3$, erit ea quæ oritur ſubſtituendo pro x in ejus terminis quantitatem

quantitatem $\frac{\sqrt{pp+3q}-p}{3}$. Et pro-inde, ſi in aliquâ æquatione cubicâ hujus formæ, $qx - px^2 - x^3 = r$, quantitas $\frac{\sqrt{pp+3q}-p}{3}$ vocetur M, et valor quantitatis $q\mathrm{M} - p\,\mathrm{M}^2 - \mathrm{M}^3$ inde derivatæ vocetur D, et homogeneum comparationis, ſeu quantitas cognita, r in hâc æquatione ſit æqualis quantitati D, æquatio habebit tantùm unam radicem, ſcilicet, M, ſeu $\frac{\sqrt{pp+3q}-p}{3}$; ſed, ſi quantitas cognita r ſit minor quantitate D, ſeu $q\,\mathrm{M} - p\,\mathrm{M}^2 - \mathrm{M}^3$, æquatio habebit duas radices, quarum una erit minor ipſâ M, ſeu $\frac{\sqrt{pp+3q}-p}{3}$, altera verò erit major ipſâ M, ſeu $\frac{\sqrt{pp+3q}-p}{3}$, ſed minor ipſâ $\frac{\sqrt{pp+4q}-p}{2}$.

Exemplum Nonum, in quantitate trinomiâ $rx - qx^2 - x^4$.

Deſignent q et r quantitates aliquas datas, et x deſignet aliam quantitatem variabilem, quæ creſcit continuò à nihilo donec quantitas binomia $qx + x^3$ fiat æqualis quantitati datæ r, et pro-inde quantitas binomia $qx^2 + x^4$ fiat æqualis quantitati rx, et quantitas trinomia $rx - qx^2 - x^4$ fiat æqualis nihilo. Et requiratur invenire, primò, quænam ſit magnitudo quantitatis x, quando quantitas trinomia $rx - qx^2 - x^4$ eſt æqualis nihilo; et, ſecundò, invenire quænam ſit magnitudo ejuſdem quantitatis x, quando quantitas trinomia $rx - qx^2 - x^4$ eſt maxima quæ eſſe poſſit.

Solutio.

Quando quantitas trinomia $rx - qx^2 - x^4$ eſt æqualis nihilo, rx eſt æqualis $qx^2 + x^4$, et r æqualis $qx + x^3$, ſeu $x^3 + qx$. Ergò, reſolvendo æquationem cubicam $x^3 + qx = r$, (quæ unam ſolummodò habet radicem,) invenietur magnitudo quantitatis x in quantitate trinomiâ $rx - qx^2 - x^4$ quando quantitas iſta eſt æqualis nihilo. Q. E. I.

Reſolutio

Reſolutio æquationis cubicæ $x^3 + qx = r$ fieri poteſt vel per Cardani regulam primam, (à Scipione Ferreo, cive Bononienſi, olìm inventam,) vel per Raphſoni methodum reſolvendi æquationes per approximationem, et faciliùs quidem per hanc ſecundam methodum quàm per Cardani regulam. Inventâ autem radice hujus æquationis quocunque modo, deſignemus eam per Litteram *a*. Erit igitùr quantitas *a* magnitudo quantitatis variabilis *x*, quando quantitas trinomia $rx - qx^2 - x^4$ eſt æqualis nihilo. Q. E. I.

Secundò, quoniam, quando quantitas trinomia $rx - qx^2 - x^4$ eſt æqualis nihilo, quantitas *x* eſt æqualis ipſi *a*, ſequitur quòd, dum *x* creſcit continuò à nihilo uſque ad *a*, quantitas trinomia $rx - qx^2 - x^4$ erit bìs æqualis nihilo, nempe, primùm, quando *x* eſt = *o*, et iterùm, quando *x* eſt æqualis ipſi *a*. Igitùr, dum *x* creſcit continuò à nihilo uſque ad *a*, neceſſe eſt ut quantitas trinomia $rx - qx^2 - x^4$, primùm, creſcat à nihilo uſque ad certam aliquam magnitudinem, et deinde decreſcat ab iſtâ magnitudine ad nihilum. Requiritur autèm invenire magnitudinem quantitatis *x* in eo temporis puncto in quo quantitas trinomia $rx - qx^2 - x^4$ attingit hanc ſuam maximam magnitudinem, ex quâ decreſcere incipit. Hoc verò ita inveniri poteſt.

Deſignetur hic valor, quem quærimus, ipſius *x* per Litteram M. Et ſit *y* alius valor ipſius *x* minor valore M. Erit igitùr quantitas trinomia $ry - qy^2 - y^4$ minor quantitate $r\,\mathrm{M} - q\mathrm{M}^2 - \mathrm{M}^4$. Ergò, dum quantitas *x* creſcit à valore M ad valorem *a*, et quantitas trinomia $rx - qx^2 - x^4$ decreſcit à quantitate $r\,\mathrm{M} - q\mathrm{M}^2 - \mathrm{M}^4$ ad *o*, fiet in uno aliquo hujus temporis puncto æqualis quantitati trinomiæ $ry - qy^2 - y^4$, quæ eſt minor ipſâ $r\,\mathrm{M} - q\mathrm{M}^2 - \mathrm{M}^4$. Ponatur *z* pro valore quantitatis *x* in iſto temporis puncto. Et erit $rz - qz^2 - z^4 = ry - qy^2 - y^4$.

Ponatur jam $e = z - y$. Et erit $z = y + e$, et $zz\,(= \overline{y+e}^2) = yy + 2ye + ee$, et $z^4\,(= \overline{y+e}^4) = y^4 + 4y^3e + 6y^2e^2 + 4ye^3 + e^4$, et $rz\,(= r \times \overline{y+e}) = ry + re$, et $qzz\,(= q \times \overline{yy + 2ye + ee}) = qyy + 2qye + qee$; et pro-inde quantitas trinomia $rz - qzz - z^4$ erit = quantitati multinomiæ $ry + re - qyy - 2qye - qee - y^4 - 4y^3e - 6y^2e^2 - 4ye^3 - e^4$.

Sed quantitas trinomia $rz - qzz - z^4$ eſt = quantitati trinomiæ $ry - qy^2 - y^4$.

Ergò et quantitas multinomia $ry + re - qyy - 2qye - qee - y^4 - 4y^3e - 6y^2e^2 - 4ye^3 - e^4$ erit = quantitati trinomiæ $ry - qy^2 - y^4$. Et (addendo utrinque $2qye + qee + 4y^3e + 6y^2e^2 + 4ye^3 + e^4$,) erit $ry + re - qyy - y^4 = ry - qy^2 - y^4 + 2qye + qee + 4y^3e + 6y^2e^2 + 4ye^3 + e^4$; et (auferendo utrinque $ry - qy^2 - y^4$,) erit $re = 2qye + qee + 4y^3e + 6y^2e^2 + 4ye^3 + e^4$; et (dividendo omnes terminos per *e*,) erit $r = 2qy + qe + 4y^3 + 6y^2e + 4ye^2 + e^3$. Hoc autèm verum eſt, quantumvis parva fuerit quantitas *e*.

Erit igitùr verum etiàm quando quantitas e eſt omninò nullius magnitudinis. Sed, ubi quantitas e eſt omninò nullius magnitudinis, omnes termini in quibus e invenitur erunt etiàm nullius magnitudinis, et pro-inde omitti poſſunt. Atque ideò æquatio $r = 2qy + qe + 4y^3 + 6y^2e + 4ye^2 + e^3$ fiet in hoc caſu $x = 2qy + 4y^3$. Et pro-inde $\frac{r}{4}$ erit $= \frac{qy}{2} + y^3$, ſeu $y^3 + \frac{qy}{2}$, et y erit æqualis radici æquationis cubicæ $y^3 + \frac{q}{2} \times y = \frac{r}{4}$, quæ eſt unica, et per Cardani regulam primam, ſeu (ſi magìs libet,) per Raphſoni methodum reſolvendi æquationes per approximationem, inveniri poteſt. Deſignetur hæc radix per Litteram b. Et erit y, in hoc caſu, $= b$.

Sed, ob e in hoc caſu, $= o$, erit z, ſeu $y + e$, $(= y + o) = y$.

Ergò et z erit $=$ ipſi b.

Ergò in hoc caſu duo valores quantitatis x, ſcilicet, y et z, (quorum prior y fuerat minor ipſâ M, alter verò z fuerat major ipſâ M,) fiunt æquales inter ſe, et uterque æqualis ipſi b, ſeu radici æquationis cubicæ $y^3 + \frac{q}{2} \times y = \frac{r}{4}$, ſeu $x^3 + \frac{q}{2} \times x = \frac{r}{4}$. Ergò et M (quæ inter y et z fuerat mediæ cujuſdam magnitudinis,) erit etiàm æqualis ipſi b; hoc eſt, valor quantitas x in quantitate trinomiâ $rx - qx^2 - x^4$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit æqualis ipſi b, ſeu radici æquationis cubicæ $x^3 + \frac{q}{2} \times x = \frac{r}{4}$.

Q. E. I.

Coroll. Ergò $r\,\mathrm{M} - q\,\mathrm{M}^2 - \mathrm{M}^4$, ſeu maxima magnitudo quantitatis trinomiæ $rx - qx^2 - x^4$, erit ea quæ oritur ſubſtituendo pro x in ejus terminis quantitatem b, ſeu radicem æquationis cubicæ $x^3 + \frac{q}{2} \times x = \frac{r}{4}$. Et pro-inde, ſi in aliquâ æquatione biquadraticâ hujus formæ, $rx - qx^2 - x^4 = s$, quantitas b (quæ eſt radix æquationis cubicæ $x^3 + \frac{q}{2} \times x = \frac{r}{4}$,) vocetur M, et valor quantitatis trinomiæ $r\,\mathrm{M} - q\,\mathrm{M}^2 - \mathrm{M}^4$ inde derivatæ vocetur D, et homogeneum comparationis, ſeu quantitas cognita, s, in hâc æquatione ſit æqualis quantitati D, æquatio habebit tantùm unam radicem, ſcilicet, M, ſeu b, ſeu radicem æquationis cubicæ $x^3 + \frac{q}{2} \times x = \frac{r}{4}$; ſed, ſi quantitas cognita s ſit minor quantitate D, ſeu $r\,\mathrm{M} - q\,\mathrm{M}^2 - \mathrm{M}^4$, æquatio habebit duas radices, quarum una erit minor ipſâ M, ſeu b, ſeu radice æquationis cubicæ $x^3 + \frac{q}{2} + x = \frac{r}{4}$, altera verò erit major ipſâ M, ſeu b, ſed minor ipſâ a, ſeu radice æquationis cubicæ $x^3 + qx = r$.

Exemplum

Exemplum Decimum, in quantitate trinomiâ $rx + qx^2 - x^4$.

Defignent q et r quantitates aliquas datas, et x defignet quantitatem variabilem, quæ creſcit continuò à nihilo donec quantitas x^3 fiat æqualis quantitati binomiæ $r + qx$, et pro-inde quantitas x^4 fiat æqualis quantitati binomiæ $rx + qx^2$, et quantitas trinomia $rx + qx^2 - x^4$ fiat æqualis nihilo. Et requiratur invenire, primò, quænam ſit magnitudo quantitatis x quando quantitas trinomia $rx + qx^2 - x^4$ eſt æqualis nihilo; et, ſecundò, invenire, quænam ſit magnitudo ejuſdem quantitatis x, quando quantitas trinomia $rx + qx^2 - x^4$ eſt maxima quæ eſſe poſſit.

Solutio.

Quando quantitas trinomia $rx + qx^2 - x^4$ eſt æqualis nihilo, $rx + qx^2$ eſt æqualis x^4, et rx eſt æqualis $x^4 - qx^2$, et r eſt æqualis $x^3 - qx$. Ergò, reſolvendo æquationem cubicam $r = x^3 - qx$, ſeu $x^3 - qx = r$, (quæ unam ſolummodò habet radicem,) invenietur magnitudo quantitatis x in quantitate trinomiâ $rx + qx^2 - x^4$ quando quantitas iſta eſt æqualis nihilo.

Q. E. I.

Reſolutio æquationis cubicæ $x^3 - qx = r$ obtineri poteſt per Cardani regulam ſecundam (à Tartaleâ olìm inventam,) ſi quantitas $\frac{rr}{4}$ fuerit major quantitate $\frac{q^3}{27}$; et, ſi $\frac{rr}{4}$ fuerit minor ipſâ $\frac{q^3}{27}$, obtineri poteſt, facili ſatìs calculo, per Raphſoni methodum reſolvendi æquationes per approximationem. Addo etiàm, quòd in priore caſu, in quo Cardani regula ſecunda locum habet, æquatio $x^3 - qx = r$ faciliùs reſolvetur per Raphſoni approximationes quàm per hanc Cardani regulam. Inventâ autèm radice hujus æquationis cubicæ $x^3 - qx = r$, ſive per Cardani regulam ſecundam, ſive per Raphſoni methodum approximandi, ſive alio quocunque modo, deſignemus eam per Litteram a. Erit igitùr quantitas a magnitudo quantitatis x quando quantitas trinomia $rx + qx^2 - x^4$ eſt æqualis nihilo.

Secundò, quoniam, quando quantitas trinomia $rx + qx^2 - x^4$ eſt æqualis nihilo, quantitas x eſt æqualis ipſi a, ſeu radici æquationis cubicæ $x^3 - qx = r$, ſequitur quòd, dum x creſcit continuò à nihilo uſque ad a, quantitas trinomia $rx + qx^2 - x^4$, erit bìs æqualis nihilo, nempe, primùm, quando x eſt æqualis nihilo, et iterùm, quando x eſt æqualis ipſi a. Igitùr, dum x creſcit continuò à nihilo uſque ad a, neceſſe eſt ut quantitas trinomia $rx + qx^2 -$

x^4, primùm, crescat à nihilo usque ad certam aliquam magnitudinem, et deinde decrescat ab istâ magnitudine ad nihilum. Requiritur autèm invenire magnitudinem quantitatis x in eo temporis puncto in quo quantitas trinomia $rx + qx^2 - x^4$ attingit hanc suam maximam magnitudinem, ex quâ decrescere incipit. Hoc verò ita inveniri potest.

Designetur hic valor, quem quærimus, ipsius x per Litteram M. Et sit y alius valor ipsius x minor valore M. Erit igitur quantitas trinomia $ry + qy^2 - y^4$ minor quantitate $r\,M + qM^2 - M^4$. Ergò, dum quantitas x crescit à valore M ad valorem a, et quantitas trinomia $rx + qx^2 - x^4$ decrescit à quantitate $r\,M + q\,M^2 - M^4$ ad o, fiet in uno aliquo hujus temporis puncto æqualis quantitati trinomiæ $ry + qy^2 - y^4$, quæ est minor ipsâ $r\,M + qM^2 - M^4$. Ponatur z pro valore quantitatis x in isto temporis puncto. Et erit $rz + qz^2 - z^4 = ry + qy^2 - y^4$.

Ponatur jam $e = z - y$. Et erit $z = y + e$, et $zz\ (= \overline{y + e}^2) = yy + 2ye + ee$, et $z^4\ (= y + e^4) = y^4 + 4y^3e + 6y^2e^2 + 4ye^3 + e^4$, et $rz\ (= r \times \overline{y + e}) = ry + re$, et $qzz\ (= q \times \overline{yy + 2ye + ee},) = qyy + 2qye + qee$; et pro-inde quantitas trinomia $rz + qzz - z^4$ erit = quantitati multinomiæ $ry + re + qyy + 2qye + qee - y^4 - 4y^3e - 6y^2e^2 - 4ye^3 - e^4$.

Sed quantitas trinomia $rz + qzz - z^4$ est = quantitati trinomiæ $ry + qyy - y^4$.

Ergò et quantitas multinomia $ry + re + qyy + 2qye + qee - y^4 - 4y^3e - 6y^2e^2 - 4ye^3 - e^4$ erit = quantitati trinomiæ $ry + qyy - y^4$. Et (addendo utrinque $4y^3e + 6y^2e^2 + 4ye^3 + e^4$,) erit $ry + re + qyy + 2qye + qee - y^4 = ry + qyy - y^4 + 4y^3e + 6y^2e^2 + 4ye^3 + e^4$; et (auferendo utrinque $ry + qyy - y^4$,) erit $re + 2qye + qee = 4y^3e + 6y^2e^2 + 4ye^3 + e^4$; et (dividendo omnes terminos per e,) erit $r + 2qy + qe = 4y^3 + 6y^2e + 4ye^2 + e^3$. Hoc autèm verum est, quantumvis parva fuerit quantitas e. Erit igitùr verum etiàm quando quantitas e est omninò nullius magnitudinis. Sed, ubi quantitas e est omninò nullius magnitudinis, omnes termini in quibus e invenitur erunt etiàm nullius magnitudinis, et pro-inde omitti possunt. Atque ideò æquatio $r + 2qy + qe = 4y^3 + 6y^2e + 4ye^2 + e^3$ fiet in hoc casu $r + 2qy = 4y^3$. Et pro-inde r erit $= 4y^3 - 2qy$, et $\frac{r}{4}$ erit $= y^3 - \frac{q}{2} \times y$, seu $y^3 - \frac{q}{2} \times y$ erit $= \frac{r}{4}$, et y erit æqualis radici æquationis cubicæ $y^3 - \frac{q}{2} \times y = \frac{r}{4}$, seu (quod perinde est,) radici æquationis cubicæ $x^3 - \frac{q}{2} \times x = \frac{r}{4}$; quæ radix est unica, et per Cardani regulam secundam in uno casu, et per Raphsoni Methodum approximationum in omni casu satìs facilè inveniri

niri poteſt. Deſignetur hæc radix per Litteram *b*. Et erit *y*, in hoc caſu, = ipſi *b*.

Sed, ob *e* in hoc caſu = *o*, erit *z*, ſeu $y + e$, $(= y + o) = y$.

Ergò et *z* erit = ipſi *b*.

Ergò in hoc caſu duo valores quantitatis *x*, ſcilicet, *y* et *z*, (quorum prior *y* fuerat minor ipſâ M, alter verò *z* fuerat major ipſâ M,) fiunt æquales inter ſe, et uterque æqualis ipſi *b*, ſeu radici æquationis cubicæ $y^3 - \frac{q}{2} \times y = \frac{r}{4}$, vel $x^3 - \frac{q}{2} + x = \frac{r}{4}$. Ergò et M (quæ inter *y* et *z* fuerat mediæ cujuſdam magnitudinis,) erit etiàm æqualis ipſi *b*; hoc eſt, valor quantitatis *x* in quantitate trinomiâ $rx + qx^2 - x^4$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit æqualis ipſi *b*, ſeu radici æquationis cubicæ $x^3 - \frac{q}{2} \times x = \frac{r}{4}$. Q. E. I.

Corolл. Ergò $r\,M + q\,M^2 - M^4$, ſeu maxima magnitudo quantitatis trinomiæ $rx + qx^2 - x^4$, erit ea quæ oritur ſubſtituendo in ejus terminis pro *x* quantitatem *b*, ſeu radicem æquationis cubicæ $x^3 - \frac{q}{2} \times x = \frac{r}{4}$. Et proinde, ſi in aliquâ æquatione biquadraticâ hujus formæ, $rx + qx^2 - x^4 = s$, quantitas *b*, ſeu radix æquationis cubicæ $x^3 - \frac{q}{2} \times x = \frac{r}{4}$, vocetur M, et valor quantitatis trinomiæ $r\,M + q\,M^2 - M^4$ inde derivatæ vocetur D, et homogeneum comparationis, ſeu quantitas cognita, *s* in hâc æquatione ſit æqualis quantitati D, æquatio habebit tantùm unam radicem, ſcilicet, M, ſeu *b*, ſeu radicem æquationis cubicæ $x^3 - \frac{q}{2} \times x = \frac{r}{4}$; ſed, ſi quantitas cognita *s* ſit minor quantitate D, ſeu $r\,M + q\,M^2 - M^4$, æquatio habebit duas radices, quarum una erit minor ipſâ M, ſeu *b*, ſeu radice æquationis cubicæ $x^3 - \frac{q}{2} \times x = \frac{r}{4}$, altera verò erit major ipſâ M, ſeu *b*, ſeu radice æquationis cubicæ $x^3 - \frac{q}{2} \times x = \frac{r}{4}$, ſed minor ipſâ *a*, ſeu radice æquationis cubicæ $x^3 - qx = r$.

Hæc decèm exempla ſufficere poſſe videntur ad illuſtrandam Methodum ab Hugenio, in Diatribe ſuperiore, traditam et explicatam, inveniendi maximos valores quantitatum compoſitarum quæ non continuò creſcunt, ſed primùm creſcunt à nihilo ad certas quaſdam magnitudines, et deinde ex iſtis magnitudinibus

dinibus decreſcunt. Et limites, inter quos jacent radices æquationum $px - xx = q$, $qx - x^3 = r$, $px^2 - x^3 = r$, $px^3 - x^4 = s$, et aliarum omnium in Corollariis horum exemplorum memoratarum, et quos in iiſdem Corollariis determinavimus, magnæ erunt utilitatis in reſolutione æquationum iſtarum per Raphſoni Methodum approximationum ; per quam omnes æquationes ultrà quadraticas meliùs et faciliùs reſolvi poſſunt quam aliâ quâvis methodo quæ mihi nota ſit.

ALIÁ

ALIA METHODUS

INVENIENDI

MAXIMOS VALORES

Quantitatum quarundam compoſitarum ex duabus, vel pluribus, quantitatibus ſimplicibus, quarum una, vel plures, à reliquis ſubtrahuntur.

A FRANCISCO MASERES, ANGLO,

REGIÆ SOCIETATIS LONDINENSIS SOCIO.

ARTICULUS I.

IN tractatu ſuperiore exhibuimus Methodum à Fermatio olìm inventam, et à ſolertiſſimo Hugenio explicatam et illuſtratam, inveniendi maximos valores quantitatum quæ primò creſcunt à nihilo ad finitam quandam magnitudinem, et deinde ab iſtâ magnitudine decreſcunt, dum alia quantitas variabilis, ex quâ pendent et quæ plerumque deſignatur per Litteram x, creſcit continuò à nihilo uſque ad certam quandam magnitudinem. Et, præter exempla hujus Methodi ab Hugenio allata, adduximus ejuſdem exempla decèm, in quibus determinavimus valores quantitatis x, qui, ſi ſubſtituantur pro x in quantitatibus binomiis, $px - xx$, $qx - x^3$, $px^2 - x^3$, $px^3 - x^4$, $rx - x^4$, et $Px^m - x^{m+n}$, et in quantitatibus trinomiis $qx + px^2 - x^3$, $qx - px^2 - x^3$, $rx - qx^2 - x^4$, et $rx + qx^2 - x^4$, efficient ut valores iſtarum quantitatum binomialium et trinomialium ſint maximi qui eſſe poſſint. Hi autem valores quantitatis x (ut in corollariis ad dicta decèm exempla obſervatum eſt,) erunt Limites radicum duarum quas habent æquationes binomiales $px - xx = q$, $qx - x^3 = r$, $px^2 - x^3 = r$, $px^3 - x^4 = s$, $rx - x^4 = s$, et, generalitèr, $Px^m - x^{m+n} = Q$, et æquationes trinomiales $qx + px^2 - x^3 = r$, $qx - px^2 - x^3 = r$, et $rx - qx^2 - x^4 = s$, et $rx + qx^2 - x^4 = s$. Sed et alia eſt Methodus determinandi hos valores quantitatis x in prædictis quantitatibus binomiis et trinomiis, quæ mihi videtur magìs obvia et directa quàm methodus in ſuperiore tractatu explicata, quæ à Fermatio et Hugenio nobis tradita eſt; immò etiàm videtur et brevior et facilior iſtâ ſuperiore Methodo, et ad determinandos Limites inter quos jacent radices æquationum, binomialium, trinomialium,

mialium, quadrinomialium, et aliarum multinomialium, quæ duas, vel tres, vel quatuòr, vel plures, radices habent, magìs accommodata. Hanc igitùr Methodum, ob insignem ejus utilitatem in determinandis his Limitibus quos habent radices æquationum multarum multinomiarum, describere et explicare nunc conabor.

Art. II. Designet x quantitatem aliquam variabilem, quæ crescit continuò à nihilo ad infinitum. Et sint A, B, C, D, E, &c. quantitates quotlibet datæ, quæ multiplicentur in singulos terminos seriei x, x^2, x^3, x^4, x^5, &c. ad tot terminos continuatæ quot sunt quantitates datæ A, B, C, D, E, &c, ita ut efficiant Seriem Ax, Bx^2, Cx^3, Dx^4, Ex^5, &c, cujus termini omnes post primum Ax, notentur signis + vel —, hoc est, aut ipsi primo Ax addantur, aut ab ipso subtrahantur.

Tales sunt series $Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5$, &c, et $Ax + Bx^2 - Cx^3 - Dx^4 - Ex^5$, &c, et $Ax + Bx^2 - Cx^3 + Dx^4 + Ex^5$, &c, et $Ax - Bx^2 + Cx^3 - Dx^4 + Ex^5$, &c.

His positis, manifestum est, quòd, dum quantitas x crescit continuò à nihilo ad infinitum, unusquisque terminus hujus seriei crescet paritèr à nihilo in infinitum; et pro-inde, si omnes ejus termini inter se per signum + conjunguntur, seu omnes termini post primum Ax huic primo termino adduntur, tota series inde orta, nempe, series $Ax + Bx^2 + Cx^3 + Dx^4 + Ex^5$, + &c, crescet continuò, eodem modo quo quantitas ipsa x, à nihilo ad infinitum. Sed, si aliqui hujus Seriei termini signo — notentur, seu ab aliis subtrahantur, fieri poterit ut tota Series non continuò crescat à nihilo ad infinitum, sed ut primò crescat à nihilo ad certam quandam magnitudinem, et deinde decrescat ad nihilum, aut, in quibusdam casibus, decrescat, ab istâ magnitudine ad quam creverat à nihilo, non omninò ad nihilum, sed tantùm ad minoram quandam magnitudinem, et deinde ab istâ secundâ et minore magnitudine iterùm crescat. In primâ suppositione, in quâ omnes seriei termini post primum Ax signo + notantur, series inde orta $Ax + Bx^2 Cx^3 + Dx^4 + Ex^5 +$ &c nullum habet maximum valorem, quoniam, augendo x, series ista fieri potest major quantitate quâvis datâ, ut cunque magnâ. Sed in secundâ suppositione, in quâ quidam seriei termini signo — notantur, seu ab aliis subtrahuntur, fieri poterit ut Series nunquàm crescat ultrà certam quandam magnitudinem finitam, quem ubi attigit, inde statìm decrescet; quæ pro-inde magnitudo erit Seriei maximus valor.

Art. III. Jam, si series sit $Ax - Bx^2 - Cx^3 - Dx^4 - Ex^5 -$ &c, in quâ omnes termini post primum Ax signo — notantur, seu à primo Ax subtrahantur, ponatur y pro primo termino Ax, et z prò summâ omnium aliorum terminorum $Bx^2 + Cx^3 + Dx^4 + Ex^5 +$ &c, qui in istâ serie signo — notantur, seu à termino Ax subtrahuntur: Et, si series sit $Ax + Bx^2 - Cx^3 - Dx^4 - Ex^5$

$— Ex^5 —$ &c, ponatur y pro ſummâ terminorum Ax et Bx^2, qui ſigno +, notantur, ſeu per additionèm inter ſe conjunguntur, et z prò ſummâ omnium aliorum terminorum $Cx^3 + Dx^4 + Ex^5$ &c, qui ſigno — notantur, ſeu à ſummâ terminorum duorum $Ax + Bx^2$ ſubtrahuntur. Et, ſi Series ſit alia quævis, (ut $Ax + Bx^2 + Cx^3 — Dx^4 — Ex^5$ &c, ſeu $Ax — Bx^2 + Cx^3 — Dx^4 + Ex^5 —$ &c,) in quâ quidam termini poſt primum Ax notantur ſigno +, ſeu primo termino adduntur, et reliqui notantur ſigno —, ſeu à primo termino et aliis ei conjunctis ſubtrahuntur, ponatur y prò ſummâ primi termini Ax et aliorum omnium qui ſigno + notantur, ſeu primo termino adduntur, et z prò ſummâ reliquorum omnium terminorum, qui ſigno — notantur et à prioribus ſubtrahuntur. Et erit in omnibus hiſce caſibus tota ſeries, ex terminis Ax, Bx^2, Cx^3, Dx^4, Ex^5, &c confecta, æqualis quantitati binomiæ $y — z$. Agitur ergò de inveniendo maximo valore quantitatis binomæ $y — z$, ubi quantitas iſta non creſcit continuò à nihilo ad infinitum, ſed creſcit à nihilo ad certam quandam magnitudinem, et inde decreſcit.

Principium, ſeu Fundamentum, hujus Methodi inveniendi maximum valorem quantitatis binomiæ $y — z$.

Art. IV. Ut autèm inveniamus maximum valorem quantitatis binomiæ $y — z$, neceſſe erit conferre inter ſe incrementa quantitatum y et z iiſdem temporibus acquiſita. Nam, ſi incrementa quantitatis y in temporibus utcunque parvis acquiſita ſint ſemper majora quàm incrementa quantitatis z iiſdem temporibus acquiſita, manifeſtum eſt quòd quantitas binomia $y — z$, ſeu exceſſus quantitatis y ſuper quantitatem z, continuò creſcet: ſed, ſi incrementa y in temporibus utcunque parvis acquiſita ſunt primò majora quàm incrementa quantitatis z iiſdem temporibus acquiſita, dum quantitas x (à quâ pendent et derivantur quantitates y et z,) creſcit continuò à nihilo uſque ad datam quandam magnitudinem quæ vocetur a, et tunc fiunt minora incrementis contemporaneis quantitatis z, dum quantitas x creſcit à magnitudine a ad aliam magnitudinem quæ vocetur b, manifeſtum eſt quòd, dum x creſcit à nihilo uſque ad primam magnitudinem a, quantitas binomia $y — z$ continuò creſcet à nihilo ad certam quandam magnitudinem finitam, magnitudini a quantitatis x reſpondentem, et tunc decreſcet ab iſtâ magnitudine ad aliam minorem, dum x ulteriùs creſcit à primâ magnitudine a ad ſecundam magnitudinem b. Inquirendum eſt igitùr, primò, an incrementa quantitatis y unquam fiant minora incrementis contemporaneis quantitatis z; et, ſecundò, ſi aliquando incrementa prioris quantitatis y fiunt minora incrementis contemporaneis quantitatis poſterioris z, quænam erit magnitudo a quantitatis primitivæ x (à quâ quantitates y et z pendent et derivantur,) in eo temporis puncto in quo incrementum quantitatis

y in tempore quantumvis parvo acquiſitum fit æquale incremento contemporaneo quantitatis z, ſeu ceſſat eſſe majus, et incipit eſſe minus, hoc poſteriore incremento. Hæc omnia ſunt per ſe ſatìs manifeſta, et nullâ demonſtratione indigere mihi videntur. Ex his ergò principiis determinari poterunt valores quantitatis x in quantitatibus binomiis $px - xx$, $qx - x^3$, $px^2 - x^3$, $px^3 - x^4$, $rx - x^4$, et, generalitèr, $Px^m - x^{m+n}$, et in quantitatibus trinomiis $qx + px^2 - x^3$, $qx - px^2 - x^3$, $rx - qx^2 - x^4$, et $rx + qx^2 - x^4$, quorum ſubſtitutione in dictis quantitatibus binomiis et trinomiis iſtæ quantitates fient maximæ quæ eſſe poſſint. Inveſtigationes autèm horum valorum quantitatis x in his quantitatibus compoſitis in ſequentibus exemplis nunc exhibere pergimus.

Exemplum Primum, in quantitate binomiâ $px - xx$.

Art. V. Deſignet p lineam quandam datam, et x lineam aliam variabilem, quæ creſcit continuò à nihilo donec fiat æqualis datæ p. Et *inquiratur* primò, an quantitas binomia $px - xx$ creſcet continuò dum x creſcit à nihilo uſque ad p, an creſcet continuò tantùm per partem temporis in quo x creſcit ad p, ſeu dum x creſcit ad magnitudinem quandam ipſâ p minorem, et poſteà decreſcet; et, ſi inveniatur quòd quantitas binomia $px - xx$ non creſcet continuò per totum tempus per quod x creſcit à nihilo uſque ad p, ſed tantùm per partem illius temporis, ſeu dum x creſcit à nihilo uſque ad magnitudinem quandam ipſâ p minorem, et deinde decreſcet, tunc inquiratur, ſecundo loco, quænam erit magnitudo lineæ x in eo temporis puncto in quo quantitas binomia $px - xx$ creſcere deſinet et decreſcere incipiet, et pro-inde in quo valor iſtius quantitatis binomiæ erit maximus qui eſſe poſſit.

Solutio.

Concipiamus lineam datam p, (cui variabilis linea x, quæ creſcit à nihilo, fit ultimò æqualis,) in partes valdè parvas et æquales dividi, quorum numerus ſit permagnus, putà, decèm millia millionum, ſeu 10,000,000,000; et unaquæque harum partium lineæ p valdè parvarum deſignetur notâ $\dot{x}$, ſeu litterâ x cum puncto ſupèr ipſam poſito.

Ergò, dum quivis valor lineæ x, deſignatus per litteram x, creſcit ab x ad $x + \dot{x}$, ſeu augetur incremento parvo $\dot{x}$, rectangulum px creſcet à px ad $p \times \overline{x + \dot{x}}$, ſeu ad $px + p\dot{x}$, et quadratum xx creſcet ab xx ad $\overline{x + \dot{x}}^2$, ſeu ad $xx + 2x\dot{x} + \dot{x}\dot{x}$; adeò ut incrementum rectanguli px hoc valdè parvo tempore acquiſitum erit $p\dot{x}$, et incrementum contemporaneum quadrati xx erit $2x\dot{x} + \dot{x}\dot{x}$.

$\dot{x}\dot{x}$. Sed, ob extremam parvitatem quantitatis $\dot{x}\dot{x}$ præ quantitate $2x\dot{x}$, (ad quam eſt in ratione quantitatis $\dot{x}$ ad quantitatem $2x$) incrementum $2x\dot{x} + \dot{x}\dot{x}$ poteſt cenſeri tantùm æquale eſſe primo ejus termino $2x\dot{x}$; et pro inde quantitas $2x\dot{x}$ erit incrementum quadrati xx contemporaneum incremento $p\dot{x}$ rectanguli px. Eſt verò $p\dot{x}$ ad $2x\dot{x}$ ut p ad $2x$, ſeu ut $\frac{p}{2}$ ad x. Ergò incrementum rectanguli px in tempore valdè parvo in quo quivis valor lineæ x acquirit incrementum $\dot{x}$, ſeu $\frac{p}{10,000,000,000}$, erit ad incrementum contemporaneum quadrati xx ut $\frac{p}{2}$ ad x. Igitùr, dum x creſcit à nihilo uſque ad $\frac{p}{2}$, incrementum rectanguli px erit majus incremento contemporaneo quadrati xx; ſed poſteà, dum x creſcit à $\frac{p}{2}$ ad p, incrementum rectanguli px erit minus incremento contemporaneo quadrati xx. Ergò, dum x creſcit à nihilo ad $\frac{p}{2}$, quantitas binomia $px - xx$ creſcet continuò à nihilo uſque ad valorem $p \times \frac{p}{2} - \frac{p}{2} \times \frac{p}{2}$, (ſeu $\frac{pp}{2} - \frac{pp}{4}$, ſeu $\frac{2pp}{4} - \frac{pp}{4}$), ſeu $\frac{pp}{4}$; et, dum x creſcit ulteriùs à $\frac{p}{2}$ ad p, quantitas binomia $px - xx$ decreſcet continuò à $\frac{pp}{4}$ ad $p \times p - pp$, ſive ad nihilum. Ergò $\frac{p}{2}$ eſt magnitudo lineæ variabilis x in eo temporis puncto in quo valor quantitatis binomiæ $px - xx$ fit maximus qui eſſe poſſit. Q. E. I.

Coroll. 1. Ergò quantitatis binomiæ $px - xx$ maximus valor qui eſſe poſſit eſt $\frac{pp}{4}$, ſeu quadratum ex $\frac{p}{2}$, ſeu dimidio lineæ datæ p.

Coroll. 2. Ergò, ſi in æquatione quadraticâ hujus formæ, $px - xx = q$, homogeneum comparationis, ſeu quantitas cognita, q ſit æqualis $\frac{pp}{4}$, ſive quadrato ipſius $\frac{p}{2}$, ſeu dimidiæ quantitatis p, quæ eſt co-efficiens quantitatis x in primo termino px, æquatio habebit tantùm unam radicem, nempe $\frac{p}{2}$; ſed ſi quantitas cognita q ſit minor quantitate $\frac{pp}{4}$, ſeu quadrato quantitatis $\frac{p}{2}$,

æquatio habebit duas radices, quarum una erit minor ipſâ $\frac{p}{2}$, altera verò erit major ipſâ $\frac{p}{2}$, ſed minor ipſâ p.

Coroll. 3. Quantitas binomia $px - xx$ eſt æqualis $\overline{p - x} \times x$, ſeu rectangulo ſub lineis $p - x$ et x. Ergò maxima magnitudo rectanguli ſub lineis $p - x$ et x, quæ ſimùl ſumptæ ſunt æquales lineæ p, eſt $\frac{pp}{4}$, ſeu $\frac{p}{2} \times \frac{p}{2}$, ſeu quadratum dimidiæ lineæ p; quod Euclidis Elementorum Libri ſecundi propoſitioni quinto conſentaneum eſt.

Exemplum Secundum, in quantitate binomiâ $qx - x^3$.

Art. VI. Deſignet q planam ſuperficiem aliquam datam, et x lineam quandam variabilem, quæ creſcit continuò à nihilo donec quadratum ejus, xx, fiat æquale datæ ſuperficiei planæ q, et pro-inde donec ejus cubus, x^3, fiat æqualis ſolido qx. Et inquiratur, primò, an quantitas binomia $qx - x^3$ creſcet continuò dum xx creſcit à nihilo ad datam ſuperficiem q, ſeu dum x creſcit à nihilo ad $\surd q$, an in hoc tempore dicta quantitas binomia primùm creſcet à nihilo ad certam quandam magnitudinem, et deinde ab iſtâ magnitudine decreſcet; et, ſi inveniatur quòd dicta quantitas binomia in hoc tempore primùm creſcet, et deinde decreſcet, tunc inquiratur, ſecundo loco, quænam erit magnitudo lineæ x in eo temporis puncto in quo dicta quantitas binomia $qx - x^3$ creſcere deſinet et incipiet decreſcere, et pro-inde in quo valor iſtius quantitatis binomiæ erit maximus qui eſſe poſſit.

Solutio.

Concipiamus lineam $\surd q$, (cui variabilis linea x, quæ creſcit à nihilo, fit ultimò æqualis,) dividi in partes æquales valdè parvas, quarum numerus ſit decèm millia millionum, ſeu 10,000,000,000; quarum unaquæque deſignetur per $\dot{x}$.

Ergò, dum quivis valor lineæ x, designatus per Litteram x, crescit ab x ad $x + \dot{x}$, seu augetur incremento valdè parvo $\dot{x}$, solidum qx crescet à qx ad $q \times \overline{x + \dot{x}}$, seu ad $qx + q\dot{x}$, et cubus x^3 crescet ab x^3 ad $\overline{x + \dot{x}}^3$, seu ad $x^3 + 3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$; adeò ut incrementum solidi qx hoc valdè parvo tempore acquisitum erit $q\dot{x}$, et incrementum contemporaneum cubi x^3 erit $3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$, seu (negligendo terminos $3x\dot{x}^2$ et $\dot{x}^3$ ob extremam eorum parvitatem præ primo termino $3x^2\dot{x}$,) erit $3x^2\dot{x}$. Est verò $q\dot{x}$ ad $3x^2\dot{x}$ ut q ad $3x^2$, seu ut $\frac{q}{3}$ ad x^2. Ergò incrementum solidi qx acquisitum in tempore valdè parvo in quo linea x acquirit incrementum $\dot{x}$, seu $\frac{\surd q}{10,000,000,000}$, erit ad incrementum contemporaneum cubi x^3 ut $\frac{q}{3}$ ad x^2. Igitùr incrementum solidi qx non erit majus incremento contemporaneo cubi x^3 per totum tempus in quo x crescit à nihilo ad $\surd q$, seu xx crescit à nihilo ad q, sed tantùm in primâ parte istius temporis, scilicet, dum x crescit à nihilo ad $\frac{\surd q}{\surd 3}$, seu xx crescit à nihilo ad $\frac{q}{3}$. Et postea, scilicet, dum x crescit ulteriùs à $\frac{\surd q}{\surd 3}$ ad $\surd q$, seu xx crescit à $\frac{q}{3}$ ad q, incrementum parvum solidi qx erit minus incremento contemporaneo cubi x^3. Et pro-inde, quantitas binomia $qx - x^3$, seu excessus Solidi qx super cubum x^3, non crescet continuò per totum tempus in quo linea x crescit à nihilo usque ad $\surd q$, sed tantùm dum x crescit à nihilo ad $\frac{\surd q}{\surd 3}$; et postea decrescet, dum x crescit à $\frac{\surd q}{\surd 3}$ ad $\surd q$, ita ut fiat ultimò æqualis nihilo, quando x est æqualis $\surd q$, et xx est æqualis q. Atque ideò maximus valor quantitatis binomiæ $qx - x^3$ erit ille quem habet quando x est æqualis $\frac{\surd q}{\surd 3}$, qui est $q \times \frac{\surd q}{\surd 3} - \frac{q \times \surd q}{3\surd 3}$, (seu $\frac{3q\surd q}{3\surd 3} - \frac{q\surd q}{3\surd 3}$,) seu $\frac{2q\surd q}{3\surd 3}$.

Q. E. I.

Corroll. 1. Ergò, si in æquatione cubicâ hujus formæ, $qx - x^3 = r$, homogeneum comparationis, seu quantitas cognita, r, sit æqualis quantitati $\frac{2q\surd q}{3\surd 3}$, æquatio habebit tantùm unam radicem, scilicet, $\frac{\surd q}{\surd 3}$; sed, si quantitas cognita r sit minor quantitate $\frac{2q\surd q}{3\surd 3}$, æquatio habebit duas radices, quarum

rum una erit minor ipsâ $\frac{\sqrt{q}}{\sqrt{3}}$, et altera erit major ipsâ $\frac{\sqrt{q}}{\sqrt{3}}$, sed minor ipsâ $\sqrt{q}$.

Coroll. 2. Quando quantitas cognita r in æquatione cubicâ $qx - x^3 = r$ est æqualis quantitati $\frac{2q\sqrt{q}}{3\sqrt{3}}$, (quâ major esse non potest,) quantitas $\frac{r}{2}$ erit æqualis quantitati $\frac{q\sqrt{q}}{3\sqrt{3}}$, et pro-inde $\frac{rr}{4}$, seu quadratum prioris quantitatis, erit æquale $\frac{q^3}{27}$, seu quadrato quantitatis posterioris. Atque ideò, si in aliquâ æquatione cubicâ hujus formæ, $qx - x^3 = r$, quantitas $\frac{rr}{4}$ sit æqualis quantitate $\frac{q^3}{27}$, æquatio habebit tantùm unam radicem, scilicet, $\frac{\sqrt{q}}{\sqrt{3}}$; sed, si quantitas $\frac{rr}{4}$ sit minor quantitate $\frac{q^3}{27}$, æquatio habebit duas radices, quarum una erit minor ipsâ $\frac{\sqrt{q}}{\sqrt{3}}$, et altera erit major ipsâ $\frac{\sqrt{q}}{\sqrt{3}}$, sed minor ipsâ $\sqrt{q}$.

Exemplum Tertium, in quantitate binomiâ $px^2 - x^3$.

Art. VII. Designet p lineam aliquam datam, et x lineam aliam variabilem, quæ crescit continuò à nihilo, donec fiat æqualis datæ p. Et inquiratur, primò, an dum linea x crescit continuò à nihilo ad magnitudinem datæ lineæ p, quantitas binomia $px^2 - x^3$ crescet paritèr continuò, an primùm crescet continuò dum x crescit ad aliquam magnitudinem quæ sit minor datâ lineâ p, et tunc decrescet dum x crescit ulteriùs ab istâ magnitudine ad magnitudinem datæ lineæ p; et, si dicta quantitas binomia ita crescit et decrescit, tunc inquiratur, secundo loco, quænam erit magnitudo lineæ x in eo temporis puncto, in quo dicta quantitas binomia $px^2 - x^3$ crescere desinet, et decrescere incipiet, et pro-inde in quo valor istius quantitatis binomiæ $px^2 - x^3$ erit maximus qui esse possit.

Solutio.

Solutio.

Dividatur data linea p, (cui variabilis linea x fit ultimò æqualis,) in partes æquales valdè parvas, quarum numerus fit decèm millia millionum, seu 10,000,000,000, quarum unaquæque designetur per $\dot{x}$.

Ergò, dum x crescit ab x ad $x + \dot{x}$, quantitas xx crescet ad $\overline{x + \dot{x}}^2$, seu $xx + 2x\dot{x} + \dot{x}\dot{x}$, et quantitas x^3 ad $\overline{x + \dot{x}}^3$, seu $x^3 + 3x^2\dot{x} + 3x\dot{x}^2 + \dot{x}^3$, et quantitas px^2 crescet ad $p \times \overline{x + \dot{x}}^2$, (seu ad $p \times \overline{xx + 2x\dot{x} + \dot{x}\dot{x}}$,) seu ad $px^2 + 2px\dot{x} + p\dot{x}^2$. Ergò, dum x augetur incremento valdè parvo $\dot{x}$, quantitas px^2 augebitur incremento $2px\dot{x} + p\dot{x}^2$, seu (negligendo $p\dot{x}^2$ ob extremam ejus parvitatem præ primo termino $2px\dot{x}$) incremento $2px\dot{x}$, et quantitas x^3 augebitur incremento $3x\dot{x}^2 + 3x\dot{x}^2 + \dot{x}^3$, seu (negligendo duos terminos $3x\dot{x}^2 + \dot{x}^3$ ob extremam eorum parvitatem præ primo termino $3x^2\dot{x}$) incremento $3x^2\dot{x}$. Sed incrementum $2px\dot{x}$ est ad incrementum $3x^2\dot{x}$ ut $2px$ ad $3x^2$, seu ut $2p$ ad $3x$, seu ut $\frac{2p}{3}$ ad x. Ergò, tamdiù quam quantitas $\frac{2p}{3}$ erit major lineâ x, incrementum quantitatis px^2 erit majus incremento contemporaneo quantitatis x^3; sed, cum x fit major ipsâ $\frac{2p}{3}$, licèt minor ipsâ p, incrementum quantitatis px^2 erit minus incremento contemporaneo quantitatis x^3. Ergò quantitas binomia $px^2 - x^3$ non crescet continuò per totum tempus in quo linea x crescit à nihilo ad p, sed tantùm per primam partem istius temporis, seu dum x crescit ad $\frac{2p}{3}$, et tunc decrescet dum x crescit ulteriùs à $\frac{2p}{3}$ ad p, et tunc fiet æqualis $p \times p^2 - p^3$, seu nihilo. Ergò valor quantitatis binomiæ $px^2 - x^3$ erit maximus qui esse possit cum linea x est æqualis $\frac{2p}{3}$. Q. E. I.

Coroll. 1. Cum x est æqualis $\frac{2p}{3}$, xx erit $= \frac{4pp}{9}$, et x^3 erit $= \frac{8p^3}{27}$, et pro-inde $px^2 - x^3$ erit $= p \times \frac{4pp}{9} - \frac{8p^3}{27} \left(= \frac{4p^3}{9} - \frac{8p^3}{27} = \frac{12p^3}{27} - \frac{8p^3}{27}\right) = \frac{4p^3}{27}$.

Ergò maximus valor quantitatis binomiæ $px^2 - x^3$ qui esse possit erit $\frac{4p^3}{27}$.

Coroll.

Coroll. 2. Ergò, ſi in æquatione cubicâ hujus formæ, $px^2 - x^3 = r$, homogeneum comparationis, ſeu quantitas cognita r, ſit æqualis quantitati $\frac{4p^3}{27}$, æquatio habebit tantùm unam radicem, ſcilicet, $\frac{2p}{3}$; ſed, ſi quantitas cognita r ſit minor quantitate $\frac{4p^3}{27}$, æquatio habebit duas radices, quarum una erit minor ipſâ $\frac{2p}{3}$, et altera erit major ipſâ $\frac{2p}{3}$, ſed minor ipſâ p.

Exemplum Quartum, in quantitate binomiâ $px^3 - x^4$.

Art. VIII. Deſignet p quantitatem aliquam datam, et x quantitatem aliam variabilem, quæ creſcit continuò à nihilo donec fiat æqualis datæ p. Et inquiratur quænam erit magnitudo quantitatis x quando quantitas binomia $px^3 - x^4$ eſt maxima quæ eſſe poſſit.

Solutio.

Dividatur data quantitas p (cui quantitas variabilis x fit ultimò æqualis,) in partes æquales et valdè parvas, quarum numerus ſit 10,000,000,000 ſeu decèm millia millionum; et unaquæque harum partium vocetur $\dot{x}$.

Ergò, dum x creſcit ab x ad $x + \dot{x}$, quantitas px^3 creſcet à px^3 ad $p \times \overline{x + \dot{x}}^3$, (ſeu ad $p \times \overline{x^3 + 3x^2\dot{x} + \&c,}$) ſeu ad $px^3 + 3px^2\dot{x} + \&c$, et quantitas x^4 creſcet ab x^4 ad $\overline{x + \dot{x}}^4$, ſeu $x^4 + 4x^3\dot{x} + \&c$; et pro-inde, dum x acquirit incrementum $\dot{x}$, quantitas px^3 acquiret incrementum $3px^3\dot{x}$ &c, et quantitas x^4 acquiret incrementum $4x^3\dot{x}$. Eſt verò incrementum $3px^2\dot{x}$ ad incrementum $4x^3\dot{x}$ ut $3px^2$ ad $4x^3$, ſeu ut $3p$ ad $4x$, ſeu ut $\frac{3p}{4}$ ad x. Ergò, tàm diù quàm $\frac{3p}{4}$ eſt major quantitate x, ſeu dum x creſcit à nihilo ad $\frac{3p}{4}$, incrementum quantitatis px^3 erit majus incremento quantitas x^4, et poſteà, cum $\frac{3p}{4}$ eſt minor quantitate x, ſeu dum x creſcit à $\frac{3p}{4}$ ad p, incrementum quantitatis

titatis px^3 erit minus incremento quantitatis x^4. Atque ideò, dum x creſcit à nihilo ad $\frac{3p}{4}$, quantitas binomia $px^3 - x^4$ continuò creſcet; et dum x creſcit ulteriùs à $\frac{3p}{4}$ ad p, quantitas iſta continuò decreſcet, et tunc fiet æqualis $p \times p^3 - p^4$, ſeu nihilo. Ergò valor quantitatis binomiæ $px^3 - x^4$ erit maximus qui eſſe poſſit cum quantitas x eſt æqualis $\frac{3p}{4}$. Q. E. I.

Coroll. 1. Cum x eſt æqualis $\frac{3p}{4}$, x^3 erit $= \frac{27p^3}{64}$, et x^4 erit $= \frac{81p^4}{256}$, et px^3 erit $= p \times \frac{27p^3}{64}$ $\left(= \frac{27p^4}{64}\right) = \frac{108p^4}{256}$, et pro-inde $px^3 - x^4$ erit $\left(= \frac{108p^4}{256} - \frac{81p^4}{256}\right) = \frac{27p^4}{256}$. Ergò $\frac{27p^4}{256}$ erit maximus valor quantitatis binomiæ $px^3 - x^4$ qui eſſe poſſit.

Coroll. 2. Ergò, ſi in æquatione biquadraticâ hujus formæ, $px^3 - x^4 = s$, homogeneum comparationis, ſeu quantitas cognita, s ſit æqualis quantitati $\frac{27p^4}{256}$, æquatio habebit tantùm unam radicem, ſcilicet, $\frac{3p}{4}$; ſed, ſi quantitas cognita s ſit minor quantitate $\frac{27p^4}{256}$, æquatio habebit duas radices, quarum una erit minor ipſâ $\frac{3p}{4}$, et altera erit major ipſâ $\frac{3p}{4}$, ſed minor ipſâ p.

Exemplum Quintum, in quantitate binomiâ $rx - x^4$.

Art. IX. Deſignet r quantitatem aliquam datam; et ſit x quantitas alia variabilis, quæ creſcit continuò à nihilo donec fiat æqualis $\sqrt[3]{r}$, ſeu $r^{\frac{1}{3}}$, ſeu donec cubus ejus x^3 fiat æqualis datæ quantitati r. Et *inquiratur*, quænam erit magnitudo quantitatis x quando quantitas binomia $rx - x^4$ eſt maxima quæ eſſe poſſit.

Solutio.

Dividatur quantitas $\sqrt[3]{r}$, seu $r^{\frac{1}{3}}$, (cui quantitas x fit ultimò æqualis,) in partes æquales, quarum numerus sit 10,000,000,000, seu decèm millia millionum; et unaquæque harum partium vocetur $\dot{x}$.

Ergò, dum x crescit ab x ad $x + \dot{x}$, quantitas rx crescet ab rx ad $r \times \overline{x + \dot{x}}$, seu ad $rx + r\dot{x}$, et quantitas x^4 crescet ab x^4 ad $\overline{x + \dot{x}}^4$, seu ad $x^4 + 4x^3\dot{x} +$ &c; et pro-inde, dum x acquirit incrementum $\dot{x}$, quantitas rx acquiret incrementum $r\dot{x}$, et quantitas x^4 acquiret incrementum $4x^3\dot{x} +$ &c, seu (retinendo tantùm primum hujus incrementi terminum $4x^3\dot{x}$, præ quo sequentes termini, sub notâ &c comprehensi, ferè pro nihilo sunt habendi,) $4x^3\dot{x}$. Est verò incrementum $r\dot{x}$ ad incrementum $4x^3\dot{x}$ ut r ad $4x^3$, seu ut $\frac{r}{4}$ ad x^3. Ergò, tam diù quàm $\frac{r}{4}$ erit major ipsâ x^3, seu $\sqrt[3]{\frac{r}{4}}$ erit major ipsâ x, incrementum quantitatis rx erit majus incremento contemporaneo quantitatis x^4, et pro-inde, dum x crescit à nihilo ad $\sqrt[3]{\frac{r}{4}}$, quantitas binomia $rx - x^4$ continuò crescet; sed, cum $\frac{r}{4}$ fit minor ipsâ x^3, seu $\sqrt[3]{\frac{r}{4}}$ fit minor ipsâ x, incrementum quantitatis rx erit minus incremento contemporaneo quantitatis x^4, et pro-inde, dum x crescit à $\sqrt[3]{\frac{r}{4}}$ ad $\sqrt[3]{r}$, quantitas binomia $rx - x^4$ decrescet continuò, et tunc fiet æqualis $r \times r^{\frac{1}{3}} - \overline{r^{\frac{1}{3}}}^4$, seu $r^{\frac{3}{3}} \times r^{\frac{1}{3}} - r^{\frac{4}{3}}$,) seu $r^{\frac{4}{3}} - r^{\frac{4}{3}}$, seu nihilo. Ergò valor quantitatis binomiæ $rx - x^4$ erit maximus qui esse possit cum x^3 est æqualis $\frac{r}{4}$, seu cum x est æqualis $\sqrt[3]{\frac{r}{4}}$.

Q. E. I.

Coroll. 1. Quando x est æqualis $\sqrt[3]{\frac{r}{4}}$, seu $\overline{\frac{r}{4}}^{\frac{1}{3}}$, x^4 erit $= \overline{\frac{r}{4}}^{\frac{4}{3}}$, et $rx = r \times \overline{\frac{r}{4}}^{\frac{1}{3}}$, et pro-inde $rx - x^4$ erit $= r \times \overline{\frac{r}{4}}^{\frac{1}{3}} - \overline{\frac{r}{4}}^{\frac{4}{3}}$. Ponatur jam $c = \sqrt[3]{\frac{r}{4}}$, seu $\overline{\frac{r}{4}}^{\frac{1}{3}}$. Et c^3 erit $= \frac{r}{4}$, et $4c^3$ erit $= r$, et $4c^3 \times c$, seu $4c^4$,

erit $= r \times \sqrt[3]{\frac{r}{4}}$, et c^4 erit $(= \frac{r}{4} \times \sqrt[3]{\frac{r}{4}} = \sqrt[3]{\frac{r}{4}}^2 \times \sqrt[3]{\frac{r}{4}}) = \sqrt[3]{\frac{r}{4}}^4$; et pro-inde $r \times \sqrt[3]{\frac{r}{4}} - \sqrt[3]{\frac{r}{4}}^4$ erit $(= 4c^3 \times c - c^4 = 4c^4 - c^4) = 3c^4$. Ergò $rx - x^4$ erit in hoc casu $= 3c^4$. Ergò maximus valor quantitatis binomiæ $rx - x^4$ qui esse possit erit æqualis quantitati $3c^4$, in quâ Littera c designat $\sqrt[3]{\frac{r}{4}}$, seu radicem cubicam quartæ partis datæ quantitatis r, quæ est coefficiens quantitatis x in quantitate binomiâ $rx - x^4$.

Coroll. 2. Ergò, si in æquatione biquadraticâ hujus formæ, $rx - x^4 = s$, homogeneum comparationis, seu quantitas cognita, s, sit æqualis quantitati $r \times \sqrt[3]{\frac{r}{4}} - \sqrt[3]{\frac{r}{4}}^4$, seu (si c ponatur $= \sqrt[3]{\frac{r}{4}}$, seu $\sqrt[3]{\frac{r}{4}}$,) æqualis quantitati $3c^4$, æquatio habebit tantùm unam radicem, scilicet, c, seu $\sqrt[3]{\frac{r}{4}}$; sed, si quantitas cognita s sit minor quantitate $3c^4$, æquatio habebit duas radices, quarum una erit minor ipsâ c, seu $\sqrt[3]{\frac{r}{4}}$, et altera erit major ipsâ c, seu $\sqrt[3]{\frac{r}{4}}$, sed minor ipsâ $\sqrt[3]{r}$.

Exemplum Sextum, in quantitate generali binomiâ $Px^m - x^{m+n}$.

Art. X. Designet P quantitatem aliquam datam, et m et n duos numeros integros quoscunque. Et sit x quantitas variabilis, quæ crescit continuò à nihilo usque ad $\sqrt[n]{P}$, seu $P^{\frac{1}{n}}$, seu donec x^n fiat æqualis datæ P, et pro-inde x^{m+n} (seu $x^m \times x^n$, seu $x^n \times x^m$,) fiat æqualis Px^m. Et inquiratur, quænam erit magnitudo quantitatis x, quando quantitas binomia $Px^m - x^{m+n}$ est maxima quæ esse possit.

Solutio.

Dividatur quantitas $\sqrt[n]{P}$, ſeu $P^{\frac{1}{n}}$, (cui quantitas x fit ultimò æqualis,) in partes æquales valdè parvas, quarum numerus ſit 10,000,000,000, ſeu decèm millia millionum; et unaquæque harum partium vocetur $\dot{x}$.

Ergò, dum x creſcit ab x ad $x + \dot{x}$, quantitas x^m creſcet ab x^m ad $\overline{x + \dot{x}}^m$, hoc eſt, (per Newtoni Theorema binomium,) ad $x^m + m \times x^{m-1} \times \dot{x}$ + &c, et quantitas x^{m+n} creſcet ad $\overline{x + \dot{x}}^{m+n}$, hoc eſt, (per idem Theorema,) ad $x^{m+n} + \overline{m+n} \times x^{m+n-1} \times \dot{x}$ + &c, et pro-inde quantitas Px^m creſcet à Px^m ad $P \times \overline{x^m + m \times x^{m-1} \times \dot{x} + \&c}$, ſeu ad $Px^m + Pm \times x^{m-1}\dot{x}$ + &c. Ergò, dum x acquirit incrementum valdè parvum $\dot{x}$, quantitas Px^m acquiret incrementum $Pm \times x^{m-1}\dot{x}$ + &c, ſeu (negligendo omnes terminos hujus incrementi ſub notâ &c comprehenſos, ob extremam eorum parvitatem præ primo ejuſdem termino $Pm \times x^{m-1}\dot{x}$,) incrementum $Pm \times x^{m-1}\dot{x}$, et x^{m+n} acquiret incrementum $\overline{m+n} \times x^{m+n-1} + \dot{x}$. Eſt verò incrementum $Pm \times x^{m-1}\dot{x}$ ad incrementum $\overline{m+n} \times x^{m+n-1}\dot{x}$ ut $Pm \times x^{m-1}$ ad $\overline{m+n} \times x^{m+n-1}$, ſeu ut $Pm \times x^{m-1}$ ad $\overline{m+n} \times x^n \times x^{m-1}$, ſeu ut Pm ad $\overline{m+n} \times x^n$, ſeu ut $\frac{Pm}{m+n}$ ad x^n. Ergò, tam diù quàm quantitas $\frac{Pm}{m+n}$ erit major quantitate x^n, ſeu quantitas $\overline{\frac{Pm}{m+n}}^{\frac{1}{n}}$ erit major ipſâ x, incrementum parvum quantitatis Px^m erit majus incremento contemporaneo quantitatis x^{m+n}; ſed poſtquam quantitas $\frac{Pm}{m+n}$ fit major quantitate x^n, ſeu quantitas $\overline{\frac{Pm}{m+n}}^{\frac{1}{n}}$ fit major ipſâ x, incrementum quantitatis Px^m erit minus incremento contemporaneo quantitatis x^{m+n}. Atque ideò, dum x creſcit à nihilo uſque ad $\overline{\frac{Pm}{m+n}}^{\frac{1}{n}}$, et pro-inde x^m creſcit à nihilo uſque ad $\overline{\overline{\frac{Pm}{m+n}}^{\frac{1}{n}}}^m$, ſeu ad $\overline{\frac{Pm}{m+n}}^{\frac{m}{n}}$, quantitas binomia

nomia $Px^m - x^{m+n}$ creſcet continuò à nihilo uſque ad $P \times \overline{\frac{Pm}{m+n}}^{\frac{m}{n}} - \overline{\frac{Pm}{m+n}}^{\frac{m+n}{n}}$ ſeu ad $P \times \overline{\frac{Pm}{m+n}}^{\frac{m}{n}} - \overline{\frac{Pm}{m+n}}^{\frac{m+n}{n}}$; et, dum x creſcit ulteriùs à $\overline{\frac{Pm}{m+n}}^{\frac{1}{n}}$ ad $P_{\overline{n}}$ quantitas binomia $Px^m - x^{m+n}$ decreſcet continuò, et tunc fiet æqualis $P \times \overline{P\frac{1}{n}}^{m} - \overline{P\frac{1}{n}}^{m+n}$, (ſeu $P \times P\frac{m}{n} - P\frac{m+n}{n}$, ſeu $P \times P\frac{m}{n} - P\frac{m}{n} \times P\frac{n}{n}$,) ſeu $P \times P\frac{m}{n} - P\frac{m}{n} \times P$, hoc eſt, æqualis nihilo. Ergò maximus valor quantitatis binomiæ $Px^m - x^{m+n}$ erit ille quem habet quando x eſt æqualis quantitati $\overline{\frac{Pm}{m+n}}^{\frac{1}{n}}$. Q. E. I.

Coroll. 1. Quando x eſt æqualis quantitati $\overline{\frac{Pm}{m+n}}^{\frac{1}{n}}$, quantitas binomia $Px^m - x^{m+n}$ erit æqualis quantitati $P \times \overline{\frac{Pm}{m+n}}^{\frac{m}{n}} - \overline{\frac{Pm}{m+n}}^{\frac{m+n}{n}}$, ut oſtenſum eſt. Ergò hæc quantitas $P \times \overline{\frac{Pm}{m+n}}^{\frac{m}{n}} - \overline{\frac{Pm}{m+n}}^{\frac{m+n}{n}}$ erit maximus valor quantitatis binomiæ $Px^m - x^{m+n}$ qui eſſe poſſit. Et pro-inde, ſi in æquatione aliquâ binomiali hujus formæ, $Px^m - x^{m+n} = Q$, homogeneum comparationis, ſeu quantitas cognita Q, fuerit æqualis quantitati $P \times \overline{\frac{Pm}{m+n}}^{\frac{m}{n}} - \overline{\frac{Pm}{m+n}}^{\frac{m+n}{n}}$, æquatio habebit tantùm unam radicem, ſcilicet, $\overline{\frac{Pm}{m+n}}^{\frac{1}{n}}$; ſed, ſi quantitas cognita Q fuerit minor quantitate $P \times \overline{\frac{Pm}{m+n}}^{\frac{m}{n}} - \overline{\frac{Pm}{m+n}}^{\frac{m+n}{n}}$, æquatio habebit duas radices, quarum una erit minor ipſâ $\overline{\frac{Pm}{m+n}}^{\frac{1}{n}}$, et altera erit major ipſâ $\overline{\frac{Pm}{m+n}}^{\frac{1}{n}}$, ſed minor ipſâ $P\frac{1}{n}$, ſeu $\sqrt[n]{P}$.

Solutio

Solutio omnium exemplorum præcedentium per expreſſionem generalem $\left(\frac{Pm}{m+n}\right)^{\frac{1}{n}}$, *hic inventam pro Solutione hujus exempli ſexti.*

ART. XI. Hoc exemplum ſextum comprehendit omnia exempla præcedentia de valoribus quantitatis x in quantitatibus binomiis $px - xx$, $qx - x^3$, $px^2 - x^3$, $px^3 - x^4$, et $rx - x^4$ quando iſtæ quantitates binomiæ ſunt maximæ quæ eſſe poſſint; adeò ut ſolutiones diverſæ iſtorum exemplorum derivari poſſint ex ſolutione hujus exempli ſexti, tanquam caſus ejuſdem particulares, ſubſtituendo in quantitate generali binomiâ $Px^m - x^{m+n}$ pro P (co-efficiente quantitatis x^m in primo hujus quantitatis binomiæ termino,) co-efficientes p, q, p, p, et r, et pro m, ſeu indice poteſtatis ipſius x in hoc primo termino Px^m, indices 1, 1, 2, 3, et 1, et pro $m + n$, ſeu indice poteſtatis ipſius x in ſecundo hujus quantitatis binomiæ termino x^{m+n}, indices $1 + 1$, $1 + 2$, $2 + 1$, $3 + 1$, et $1 + 3$, ſeu 2, 3, 3, 4, et 4; quod quidem fieri poteſt hoc modo.

Primò, ſit $P = p$, et $m = 1$, et $n = 1$. Et quantitas binomia generalis $Px^m - x^{m+n}$ fiet quantitas binomia particularis ($px^1 - x^{1+1}$, ſeu) $px - x^2$: et $\left(\frac{Pm}{m+n}\right)^{\frac{1}{n}}$, erit $\left(= \left(\frac{p \times 1}{1+1}\right)^{\frac{1}{1}} = \left(\frac{p}{2}\right)^1\right) = \frac{p}{2}$; hoc eſt, quantitas x in quantitate binomiâ $px - xx$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit $\frac{p}{2}$, ſeu dimidium datæ quantitatis p; ut in exemplo primo determinatum eſt.

Secundò, ſit $P = q$, et $m = 1$, et $n = 2$. Et quantitas binomia $Px^m - x^{m+n}$ fiet $(= qx^1 - x^{1+2}) = qx - x^3$; et $\left(\frac{Pm}{m+n}\right)^{\frac{1}{n}}$ erit $\left(= \left(\frac{q \times 1}{1+2}\right)^{\frac{1}{2}} = \left(\frac{q}{3}\right)^{\frac{1}{2}}\right) = \sqrt{\frac{q}{3}} = \frac{\sqrt{q}}{\sqrt{3}}$; hoc eſt, quantitas x in quantitate binomiâ $qx - x^3$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit $\frac{\sqrt{q}}{\sqrt{3}}$, ut in exemplo ſecundo determinatum eſt.

Tertiò,

Tertiò, ſit $P = p$, et $m = 2$, et $n = 1$. Et quantitas binomia $Px^m - x^{m+n}$ fiet $= (px^2 - x^{2+1}) = px^2 - x^3$; et $\sqrt[n]{\frac{Pm}{m+n}}$ erit $(= \sqrt[1]{\frac{p \times 2}{2+1}} = \sqrt[1]{\frac{p \times 2}{3}}) = \frac{2p}{3}$; hoc eſt, quantitas x in quantitate binomiâ $px^2 - x^3$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit $= \frac{2p}{3}$; ut in exemplo tertio determinatum eſt.

Quartò, ſit $P = p$, et $m = 3$, et $n = 1$. Et quantitas $Px^m - x^{m+n}$ fiet $(= px^3 - x^{3+1}) = px^3 - x^4$; et $\sqrt[n]{\frac{mP}{m+n}}$ erit $(= \sqrt[1]{\frac{3p}{3+1}} = \sqrt[1]{\frac{3p}{4}})$ $= \frac{3p}{4}$; hoc eſt, quantitas x in quantitate binomiâ $px^3 - x^4$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit $\frac{3p}{4}$; ut in exemplo quarto determinatum eſt.

Quintò, ſit $P = r$, et $m = 1$, et $n = 3$. Et quantitas binomia $Px^m - x^{m+n}$ fiet $(= rx^1 - x^{1+3},) = rx - x^4$; et $\sqrt[n]{\frac{Pm}{m+n}}$ erit $(= \sqrt[3]{\frac{r \times 1}{1+3}}) = \sqrt[3]{\frac{r}{4}}$, ſeu $\sqrt[3]{\frac{r}{4}}$; hoc eſt, quantitas x in quantitate binomiâ $rx - x^4$, quando iſta quantitas eſt maxima quæ eſſe poſſit, erit $\sqrt[3]{\frac{r}{4}}$, ſeu $\sqrt[3]{\frac{r}{4}}$; ut in exemplo quinto determinatum eſt.

Hæc ſecunda inveſtigatio valorum quantitatis x in quantitatibus binomiis $px - xx$, $qx - x^3$, $px^2 - x^3$, $px^3 - x^4$, et $rx - x^4$, quando iſtæ quantitates ſunt maximæ quæ eſſe poſſint, per applicationem ſucceſſivam expreſſionis $\sqrt[n]{\frac{Pm}{m+n}}$, (quæ deſignat valorem ipſius x in quantitate binomiâ generali $Px^m - x^{m+n}$, quando iſta quantitas ſit maxima quæ eſſe poſſit,) ad illorum inventionem, cùm, ut jàm oſtenſum eſt, eoſdem omninò valores quantitatis x in iſtis caſibus nobis indicet quos anteà inveneramus in exemplis quinque præcedentibus,

tibus, magnoperè confirmat inveſtigationes iſtorum valorum exhibitas in iſtis exemplis, et novo argumento probat iſtos valores fuiſſe ibidem rectè determinatos. Et hæc exempla ſufficere poſſe videntur ad illuſtrandam prædictam Methodum inveniendi maximos valores quantitatum compoſitarum pro quantitatibus binomiis quibuſcunque ſub hâc expreſſione generali $Px^m - x^{m+n}$ comprehenſis. Itaque nunc pergimus ad quantitates trinomias.

Exemplum Septimum, in quantitate trinomiâ $qx + px^2 - x^3$.

ART. XII. Deſignent p lineam aliquam datam, et q aliquam ſuperficiem datam; et ſit x quantitas quædam variabilis, quæ creſcit continuò à nihilo donec x^3, ſeu ejus cubus, fiat æqualis quantitati binomiæ $qx + px^2$, ſeu ſummæ ſolidorum qx et px^2; quod quidem fiet cum xx eſt æqualis quantitati binomiæ $q + px$, ſeu cum quantitas binomia $xx - px$ eſt æqualis quantitati datæ q, et pro-inde cum quantitas trinomia $xx - px + \frac{pp}{4}$ eſt æqualis quantitati $\frac{pp}{4} + q$, ſeu $\frac{pp+4q}{4}$, et cum $x - \frac{p}{2}$ eſt æqualis $\frac{\sqrt{pp+4q}}{2}$, et x eſt æqualis $\frac{p+\sqrt{pp+4q}}{2}$. Et inquiratur, quænam erit magnitudo lineæ x quando quantitas trinomia $qx + px^2 - x^3$ eſt maxima quæ eſſe poſſit.

Solutio.

Dividatur quantitas $\frac{p+\sqrt{pp+4q}}{2}$ (cui x fit ultimò æqualis,) in partes valdè parvas et æquales, quarum numerus ſit 10,000,000,000, ſeu decèm millia millionum; et unaquæque harum partium vocetur $\dot{x}$.

Ergò, dum x creſcit ab x ad $x + \dot{x}$, quantitas x creſcet ab x^2 ad $\overline{x + \dot{x}}^2$, ſeu ad $x^2 + 2x\dot{x}$ + &c, et quantitas x^3 creſcet ab x^3 ad $\overline{x + \dot{x}}^3$, ſeu ad $x^3 + 3x^2\dot{x}$

$3x^2\dot{x}$ + &c, et quantitas px^2 crescet à px^2 ad $p \times \overline{x + \dot{x}}^2$, (seu ad $p \times \overline{x^2 + 2x\dot{x} + \&c}$,) seu ad $px^2 + 2px\dot{x} + \&c$, et quantitas qx crescet à qx ad $q \times \overline{x + \dot{x}}$, seu ad $qx + q\dot{x}$, et pro-inde quantitas binomia $qx + px^2$ crescet ad quantitatem quadrinomiam $qx + q\dot{x} + px^2 + 2px\dot{x}$; adeò ut, dum quantitas x acquirit incrementum $\dot{x}$, quantitas binomia $qx + px^2$ acquiret incrementum $q\dot{x} + 2px\dot{x}$, et quantitas x^3 acquiret incrementum $3x^2\dot{x}$. Est verò incremen-

tum $q\dot{x} + 2px\dot{x}$ ad incrementum $3x^2\dot{x}$ ut $q + 2px$ ad $3x^2$, seu ut $\frac{q}{3} + \frac{2px}{3}$ ad x^2. Ergò, tam diù quàm quantitas binomia $\frac{q}{3} + \frac{2px}{3}$ erit major quantitate x^2, incrementum quantitatis binomiæ $qx + px^2$ erit majus incremento contemporaneo quantitatis x^3; et postquam quantitas binomia $\frac{q}{3} + \frac{2px}{3}$ fit minor quantitate x^2, incrementum quantitatis binomiæ $qx + px^2$ erit minus incremento contemporaneo quantitatis x^3. Atque ideò, dum x crescit à nihilo usque ad istam magnitudinem (quam vocemus M,) quam habet quando quantitas binomia $\frac{q}{3} + \frac{2px}{3}$ est æqualis ipsi x^2, quantitas trinomia $qx + px^2 - x^3$ crescet continuò; et, dum x crescit ulteriùs ab istâ magnitudine M ad $\frac{p + \sqrt{pp + 4q}}{2}$ quantitas ista trinomia decrescet continuò, et tunc fiet æqualis nihilo. Ergò maximus valor quantitatis trinomiæ $qx + px^2 - x^3$ erit ille quem habet quandò $\frac{q}{3} + \frac{2px}{3}$ est æqualis x^2. Sed, cum $\frac{q}{3} + \frac{2px}{3}$ est æqualis ipsi x^2, erit $xx - \frac{2px}{3} = \frac{q}{3}$, et $xx - \frac{2px}{3} + \frac{pp}{9} \left(= \frac{pp}{9} + \frac{q}{3}\right) = \frac{pp + 3q}{9}$, et pro-inde $x - \frac{p}{3} = \frac{\sqrt{pp + 3q}}{3}$, et $x = \frac{\sqrt{pp + 3q} + p}{3}$, seu $\frac{p + \sqrt{pp + 3q}}{3}$. Ergò M, seu valor quantitatis x, in eo temporis puncto in quo quantitas trinomia $qx + px^2 - x^3$ crescere definit et incipit decrescere, et pro-inde est maxima quæ esse possit, erit æqualis $\frac{p + \sqrt{pp + 3q}}{3}$.

Q. E. I.

Coroll. Ergò, si in æquatione cubicâ hujus formæ, $qx + px^2 - x = r$, ponatur M pro quantitate $\frac{p + \sqrt{pp + 3q}}{3}$, et substituatur M pro x in

quantitate trinomiâ $qx + px^2 - x^3$, et quantitas $q\,\mathrm{M} + p\,\mathrm{M}^2 - \mathrm{M}^3$, ſeu valor hujus quantitatis trinomiæ ex hâc ſubſtitutione ortus, ſit æqualis quantitati r, ſeu homogeneo comparationis, ſeu quantitati cognitæ, in æquatione propoſitâ $qx + px^2 - x^3 = r$, æquatio habebit tantùm unam radicem, ſcilicet, M, ſeu $\frac{p + \sqrt{pp + 3q}}{3}$; ſed, ſi quantitas cognita r ſit minor quantitate trinomiâ $q\,\mathrm{M} + p\,\mathrm{M}^2 - \mathrm{M}^3$, æquatio habebit duas radices, quarum una erit minor ipſâ M, ſeu $\frac{p + \sqrt{pp + 3q}}{3}$, et altera erit major ipſâ M, ſeu $\frac{p + \sqrt{pp + 3q}}{3}$, ſed minor ipſâ $\frac{p + \sqrt{pp + 4q}}{2}$.

Exemplum Octavum, in quantitate trinomiâ $qx - px^2 - x^3$.

ART. XIII. Deſignet p lineam aliquam datam, et q aliquam ſuperficiem datam; et ſit x quantitas quædam variabilis, quæ creſcit continuò à nihilo donec quantitas binomia $px^2 + x^3$ fiat æqualis quantitati qx; quod quidem fiet cum quantitas binomia $px + xx$ eſt æqualis datæ ſuperficiei q, et pro-inde cum quantitas trinomia $xx + px + \frac{pp}{4}$ eſt æqualis quantitati $\frac{pp}{4} + q$, ſeu $\frac{pp+4q}{4}$, et $x + \frac{p}{2}$ eſt æqualis $\frac{\sqrt{pp+4q}}{2}$, et x eſt æqualis $\frac{\sqrt{pp+4q}}{2} - \frac{p}{2}$. Et inquiratur, quænam erit magnitudo lineæ x quando quantitas trinomia $qx - px^2 - x^3$ eſt maxima quæ eſſe poſſit.

Solutio.

Dividatur quantitas $\frac{\sqrt{pp+4q} - p}{2}$ (cui x fit ultimò æqualis,) in partes valdè parvas et æquales, quarum numerus ſit 10,000,000,000, ſeu decèm millia millionum; et unaquæque harum partium vocetur $\dot{x}$.

Ergò, dum x creſcit ab x ad $x + \dot{x}$, quantitas x^2 creſcet ab x^2 ad $\overline{x + \dot{x}}^2$, ſeu ad $xx + 2x\dot{x} +$ &c, et quantitas x^3 creſcet ab x^3 ad $\overline{x + \dot{x}}^3$, ſeu ad $x^3 + 3x^2\dot{x}$

$3x^2\dot{x}$ + &c, et quantitas qx crescit à qx ad $q \times \overline{x + \dot{x}}$, seu ad $qx + q\dot{x}$, et quantitas px^2 crescet à px^2 (ad $p \times \overline{x + \dot{x}}^2$, seu ad $p \times \overline{xx + 2x\dot{x} +}$ &c, seu) ad $px^2 + 2px\dot{x}$ + &c, et pro-inde quantitas binomia $px^2 + x^3$ crescet ad quantitatem quadrinomiam $px^2 + 2px\dot{x} + x^3 + 3x^2\dot{x}$; adeò ut, dum x acquirit incrementum valdè parvum $\dot{x}$, quantitas qx acquiret incrementum $q\dot{x}$, et quantitas binomia $px^2 + x^3$ acquiret incrementum $2px\dot{x} + 3x^2\dot{x}$. Est verò incrementum $q\dot{x}$ ad incrementum $2px\dot{x} + 3x^2\dot{x}$ ut q ad $2px + 3x^2$, seu ut $\frac{q}{3}$ ad $\frac{2px}{3} + xx$. Ergò, tam diù quàm quantitas $\frac{q}{3}$ erit major quantitate binomia $\frac{2px}{3} + xx$, incrementum quantitatis qx erit majus incremento contemporaneo quantitatis binomiæ $px^2 + x^3$; et postquam quantitas $\frac{q}{3}$ fit minor quantitate binomiâ $\frac{2px}{3} + xx$, incrementum quantitatis qx erit minus incremento contemporaneo quantitatis binomiæ $px^2 + x^3$. Atque ideò, dum x crescit à nihilo ad istam magnitudinem (quam vocemus M,) quam habet quandò quantitas $\frac{q}{3}$ est æqualis quantitati binomiæ $\frac{2px}{3} + xx$, seu ad magnitudinem quæ est radix æquationis quadraticæ $xx + \frac{2px}{3} = \frac{q}{3}$, quantitas trinomia $qx - px^2 - x^3$ crescet continuò; et, dum x crescit ulteriùs ab istâ magnitudine M ad $\frac{\sqrt{pp+4q} - p}{2}$, quantitas trinomia $qx - px^2 - x^3$ decrescet continuò, et tunc fiet æqualis nihilo. Ergò maximus valor quantitatis trinomiæ $qx - px^2 - x^3$ erit ille quem habet quando quantitas binomia $\frac{2px}{3} + xx$ est æqualis quantitati $\frac{q}{3}$, seu quando quantitas x est radix æquationis quadraticæ $xx + \frac{2px}{3} = \frac{q}{3}$. Sed, cum $xx + \frac{2px}{3}$ est æqualis $\frac{q}{3}$, erit $xx + \frac{2px}{3} + \frac{pp}{9}$ $(= \frac{q}{3} + \frac{pp}{9}) = \frac{pp + 3q}{9}$, et pro-inde $x + \frac{p}{3}$ erit $= \frac{\sqrt{pp + 3q}}{3}$, et x erit $= \frac{\sqrt{pp+3q} - p}{3}$. Ergò, M, seu valor quantitatis x, in eo temporis puncto in quo quantitas trinomia $qx - px^2 - x^3$ crescere definit, et incipit decrescere, et pro-inde est maxima quæ esse possit, erit $= \frac{\sqrt{pp + 3q} - p}{3}$.

Q. E. I.

Coroll. Ergò, si in æquatione cubicâ hujus formæ, $qx - px^2 - x^3 = r$, ponatur M pro quantitate $\frac{\sqrt{pp+3q}-p}{3}$, et substituatur M pro x in quantitate trinomiâ $qx - px^2 - x^3$, et valor quantitatis trinomiæ $q\,M - p\,M^2 - M^3$ ex hâc substitutione ortæ sit æqualis quantitati r, seu homogeneo comparationis, seu quantitati cognitæ, in dictâ æquatione $qx - px^2 - x^3 = r$, æquatio habebit tantùm unam radicem, scilicet, M, seu $\frac{\sqrt{pp+3q}-p}{3}$; sed, si quantitas cognita r sit minor quantitate trinomiâ $q\,M - p\,M^2 - M^3$, æquatio habebit duas radices, quarum una erit minor ipsâ M, seu $\frac{\sqrt{pp+3q}-p}{3}$, et altera erit major ipsâ M, seu $\frac{\sqrt{pp+3q}-p}{3}$, sed minor ipsâ $\frac{\sqrt{pp+4q}-p}{2}$

Exemplum Nonum, in quantitate trinomiâ $rx - qx^2 - x^4$.

Art. XIV. Designent Litteræ q et r quantitates aliquas datas; et x designet aliam quantitatem variabilem, quæ crescit continuò à nihilo donec quantitas binomia $qx + x^3$ fiat æqualis quantitati datæ r, et pro-inde quantitas binomia $qx^2 + x^4$ fiat æqualis quantitati rx, et quantitas trinomia $rx - qx^2 - x^4$ fiat æqualis nihilo. Et inquiratur, quænam erit magnitudo quantitatis x, quandò quantitas trinomia $rx - qx^2 - x^4$ est maxima quæ esse possit.

Solutio.

Quandò quantitas trinomia $rx - qx^2 - x^4$ est æqualis nihilo, rx erit æqualis quantitati binomiæ $qx^2 + x^4$, et r erit æqualis quantitati binomiæ $qx + x^3$, et x erit radix æquationis cubicæ $x^3 + qx = r$, quæ unam solummodò radicem habet. Resolvatur jam hæc æquatio sive per Cardani regulam primam, (à Scipione Ferreo olìm inventam,) sive per Raphsoni Methodum approximationis, sive alio quocunque modo; et designetur ejus radix, sic inventa, per Litteram a. Et quantitas a erit magnitudo quantitatis variabilis x, quando quantitas trinomia $rx - qx^2 - x^4$ est æqualis nihilo.

Dividatur

Dividatur jam quantitas a (cui quantitas x fit ultimò æqualis, cum quantitas trinomia $rx - qx^2 - x^4$ fit æqualis nihilo,) in partes valdè parvas et æquales, quarum numerus fit 10,000,000,000, ſeu decèm millia millionum ; et unaquæque harum partium vocetur $\dot{x}$.

Ergò, dum x creſcit ab x ad $x + \dot{x}$, quantitas x^2 creſcet ab x^2 ad $\overline{x + \dot{x}}^2$, ſeu ad $x^2 + 2x\dot{x} +$ &c, et quantitas x^4 creſcet ab x^4 ad $\overline{x + \dot{x}}^4$, ſeu ad $x^4 + 4x^3\dot{x} +$ &c, et quantitas rx creſcet ab rx ad $r \times \overline{x + \dot{x}}$, ſeu ad $rx + r\dot{x}$, et quantitas qx^2 creſcet à qx^2 ad $q \times \overline{x + \dot{x}}^2$ (ſeu ad $q \times x^2 + 2x\dot{x} +$ &c,) ſeu ad $qx^2 + 2qx\dot{x} +$ &c, et pro-inde quantitas binomia $qx^2 + x^4$ creſcet à $qx^2 + x^4$ ad quantitatem quadrinomiam $qx + 2qx\dot{x} + x^4 + 4x^3\dot{x}$; adeò ut, dum x acquirit incrementum valdè parvum $\dot{x}$, quantitas rx acquiret incrementum $r\dot{x}$, et quantitas binomia $qx^2 + x^4$ acquiret incrementum $2qx\dot{x} + 4x^3\dot{x}$. Eſt verò incrementum $r\dot{x}$ ad incrementum $2qx\dot{x} + 4x^3\dot{x}$ ut r ad $2qx + 4x^3$, ſeu ut $\frac{r}{4}$ ad $\frac{qx}{2} + x^3$. Ergò, tam diù quàm quantitas $\frac{r}{4}$ erit major quantitate binomiâ $\frac{qx}{2} + x^3$, incrementum quantitatis rx erit majus incremento contemporaneo quantitatis binomiæ $qx^2 + x^4$; et, poſtquam quantitas $\frac{r}{4}$ fit minor quantitate binomiâ $\frac{qx}{2} + x^3$, incrementum quantitatis rx erit minus incremento quantitatis binomiæ $qx^2 + x^4$. Atque ideò, dum x creſcit à nihilo ad iſtam magnitudinem (quam vocemus M,) quam habet quando quantitas $\frac{r}{4}$ eſt æqualis quantitati binomiæ $\frac{qx}{2} + x^3$, ſeu ad magnitudinem quæ eſt radix æquationis cubicæ $x^3 + \frac{qx}{2} = \frac{r}{4}$, quantitas trinomia $rx - qx^2 - x^4$ continuò creſcet à nihilo ad valorem $r\,\mathrm{M} - q\,\mathrm{M}^2 - \mathrm{M}^4$; et, dum x creſcit ulteriùs ab iſtâ magnitudine M ad magnitudinem a, ſeu ad radicem æquationis cubicæ $x^3 + qx = r$, quantitas trinomia $rx - qx^2 - x^4$ decreſcet continuò à valore $r\,\mathrm{M} - q\,\mathrm{M}^2 - \mathrm{M}^4$ ad nihilum. Ergò magnitudo quantitatis x in eo temporis puncto in quo quantitas trinomia $rx - qx^2 - x^4$ creſcere deſinit et incipit decreſcere, et pro-inde eſt maxima quæ eſſe poſſit, erit M, ſeu radix æquationis cubicæ $x^3 + \frac{qx}{2} = \frac{r}{4}$. Q. E. I.

Corolл. Ergò, ſi in æquatione biquadraticâ trinomiâ hujus formæ, $rx - qx^2 - x^4 = s$, Littera M ponatur pro radice æquationis cubicæ $x^3 + \frac{qx}{2} = \frac{r}{4}$,

$\frac{r}{4}$, et ſi homogeneum comparationis, ſeu quantitas cognita, s, in hâc æquatione ſit æqualis quantitati trinomiæ $r\,M - q\,M^2 - M^4$, (quæ oritur ex ſubſtitutione dictæ radicis M pro x in quantitate trinomiâ $rx - qx^2 - x^4$, ſeu primâ parte dictæ æquationis biquadraticæ $rx - qx^2 - x^4 = s$,) hæc æquatio habebit tantùm unam radicem, ſcilicet, ipſam M, ſeu radicem æquationis cubicæ $x^3 + \frac{qx}{2} = \frac{r}{4}$; ſed, ſi quantitas cognita s ſit minor quantitate trinomiâ $r\,M - q\,M^2 - M^4$, æquatio habebit duas radices, quarum una erit minor ipſâ M, ſeu radice æquationis cubicæ $x^3 + \frac{qx}{2} = \frac{r}{4}$, et altera erit major ipſâ M, ſeu radice æquationis cubicæ $x^3 + \frac{qx}{2} = \frac{r}{4}$, ſed minor quantitate a, ſeu radice æquationis cubicæ $x^3 + qx = r$.

Exemplum Decimum, in quantitate trinomiâ $rx + qx^2 - x^4$.

Art. XV. Deſignent Litteræ q et r quantitates aliquas datas, et x deſignet quantitatem variabilem, quæ creſcit continuò à nihilo donec quantitas x^3 fiat æqualis quantitati binomiæ $r + qx$, et pro-inde donec quantitas x^4 fiat æqualis quantitati binomiæ $rx + qx^2$, et quantitas trinomia $rx + qx^2 - x^4$ fiat æqualis nihilo. Et inquiratur, quænam erit magnitudo quantitatis x quando quantitas trinomia $rx + qx^2 - x^4$ eſt maxima quæ eſſe poſſit.

Solutio.

Quandò quantitas trinomia $rx + qx^2 - x^4$ eſt æqualis nihilo, quantitas binomia $rx + qx^2$ erit æqualis quantitati x^4, et pro-inde quantitas binomia $r + qx$ erit æqualis quantitati x^3, et quantitas r erit æqualis quantitati binomiæ $x^3 - qx$, et x erit radix æquationis cubicæ $r = x^3 - qx$, ſeu $x^3 - qx = r$, quæ unam ſolummodò radicem habet. Reſolvatur jam hæc æquatio $x^3 - qx = r$, ſive per ſecundam Cardani regulam (à D. Nicolao Tartaleâ olìm inventam,) ſi $\frac{rr}{4}$ ſit major ipſâ $\frac{q^3}{27}$, ſeu alio quocunque modo; et, ſi $\frac{rr}{4}$ ſit minor ipſâ $\frac{q^3}{27}$,

reſolvatur

reſolvatur per Raphſoni Methodum approximationis, ſeu alio quocunque modo; et deſignetur ejus radix ſic inventa, per Litteram a. Et quantitas a erit magnitudo quantitatis variabilis x, quandò quantitas trinomia $rx + qx^2 - x^4$ eſt æqualis nihilo.

Dividatur jam quantitas a (cui quantitas x fit ultimò æqualis, cum quantitas trinomia $rx + qx^2 - x^4$ fit æqualis nihilo,) in partes valdè parvas et æquales, quarum numerus ſit 10,000,000,000, ſeu decèm millia millionum; et unaquæque harum partium vocetur $\dot{x}$.

Ergò, dum x creſcit ab x ad $x + \dot{x}$, quantitas x^2 creſcet ab x^2 ad $x + \dot{x}^2$, ſeu ad $x^2 + 2x\dot{x} +$ &c, et quantitas x^4 creſcet ab x^4 ad $\overline{x + \dot{x}}^4$, ſeu ad $x^4 + 4x^3\dot{x} +$ &c, et quantitas rx creſcet ab rx ad $r \times \overline{x + \dot{x}}$, ſeu ad $rx + r\dot{x}$, et quantitas qx^2 creſcet à qx^2 ad $q \times \overline{x + \dot{x}}^2$, ſeu ad $q \times \overline{x^2 + 2x\dot{x}} +$ &c, ſeu ad $qx^2 + 2qx\dot{x} +$ &c, et pro-inde quantitas binomia $rx + qx^2$ creſcet ab $rx + qx^2$ ad quantitatem quadrinomiam $rx + r\dot{x} + qx^2 + 2qx\dot{x}$; adeò ut, dum x acquirit incrementum valdè parvum $\dot{x}$, quantitas binomia $rx + qx^2$ acquiret incrementum $r\dot{x} + 2qx\dot{x}$, et quantitas x^4 acquiret incrementum $4x^3\dot{x}$. Eſt verò incrementum $r\dot{x} + 2qx\dot{x}$ ad incrementum $4x^3\dot{x}$ ut quantitas binomia $r + 2qx$ ad quantitatem $4x^3$, ſeu ut quantitas binomia $\frac{r}{4} + \frac{qx}{2}$ ad quantitatem x^3.

Ergò, tamdiù quàm quantitas binomia $\frac{r}{4} + \frac{qx}{2}$ eſt major quantitate x^3, incrementum quantitatis binomiæ $rx + qx^2$ erit majus incremento contemporaneo quantitatis x^4; et, poſtquam quantitas binomia $\frac{r}{4} + \frac{qx}{2}$ fit minor quantitate x^3, incrementum quantitatis binomiæ $rx + qx^2$ erit minus incremento contemporaneo quantitatis x^4. Atque ideò, dum x creſcit à nihilo ad iſtam magnitudinem quam habet quando quantitas binomia $\frac{r}{4} + \frac{qx}{2}$ eſt æqualis quantitati x^3, et pro-inde quantitas binomia $x^3 - \frac{qx}{2}$ eſt æqualis quantitati $\frac{r}{4}$, ſeu ad magnitudinem (quam vocemus M,) quæ eſt æqualis radici æquationis cubicæ $x^3 - \frac{qx}{2} = \frac{r}{4}$ (quæ æquatio unam ſolummodò radicem habet,) quantitas trinomia $rx + qx^2 - x^4$ continuò creſcet à nihilo ad quantitatem $r\,\mathrm{M} + q\,\mathrm{M}^2 - \mathrm{M}^4$; et, dum x creſcit ulteriùs ab M, ſeu radice æquationis cubicæ $x^3 - \frac{qx}{2} = \frac{r}{4}$, ad a, ſeu radicem æquationis cubicæ $x^3 - qx = r$, quantitas trinomia $rx + qx^2 - x^4$ decreſcet continuò à quantitate $r\,\mathrm{M} + q\,\mathrm{M}^2 - \mathrm{M}^4$ ad nihilum. Igitùr valor quantitatis trinomiæ $rx + qx^2 - x^4$

erit

erit maximus quando x eſt æqualis ipſi M, ſeu radici æquationis cubicæ $x^3 - \frac{qx}{2} = \frac{r}{4}$; et M, ſeu radix hujus æquationis cubicæ, erit magnitudo quantitatis x, quando quantitas trinomia $rx + qx^2 - x^4$ eſt maxima quæ eſſe poſſit. Q. E. I.

Coroll. Ergò, ſi in æquatione biquadraticâ hujus formæ, $rx + qx^2 - x^4 = s$, ponatur Littera M prò radice æquationis cubicæ $x^3 - \frac{qx}{2} = \frac{r}{4}$, et ſi homogeneum comparationis, ſeu quantitas cognita, s, in hâc æquatione biquadraticâ $rx + qx^2 - x^4 = s$, ſit æqualis quantitati trinomiæ $r\,M + q\,M^2 - M^4$, (quæ oritur ex ſubſtitutione dictæ radicis M prò x in quantitate trinomiâ $rx + qx^2 - x^4$, ſeu in primâ parte dictæ æquationis biquadraticæ $rx + qx^2 - x^4 = s$,) hæc æquatio habebit tantùm unam radicem, ſcilicet, ipſam M, ſeu radicem æquationis cubicæ $x^3 - \frac{qx}{2} = \frac{r}{4}$; ſed, ſi quantitas cognita s ſit minor quantitate trinomiâ $r\,M + q\,M^2 - M^4$, hæc æquatio habebit duas radices, quarum una erit minor ipſâ M, ſeu radice æquationis cubicæ $x^3 - \frac{qx}{2} = \frac{r}{4}$, et altera erit major ipſâ M, ſed minor quantitate a, ſeu radice æquationis cubicæ $x^3 - qx = r$.

Scholium.

Art. XVI. Ex his exemplis ſatìs, ut ſpero, manifeſtum erit hanc Methodum inveniendi maximos valores quantitatum compoſitarum, quarum omnia membra involvunt eandem quantitatem variabilem Litterâ quâvis, ut x, deſignatam, quandò quædam earum membra ſigno — præfixo notantur, ſeu à reliquis ſubtrahuntur, eſſe huic negotio apprimè idoneam, immò et breviorem et faciliorem priore illâ à Fermatio et Hugenio traditâ. Et, ut collatio harum Methodorum inter ſe quoàd inventionem maximorum valorum hujuſmodi quantitatum compoſitarum (quæ ad determinandos Limites radicum æquationum plurimarum algebraïcarum ſunt neceſſariò conſiderandæ,) meliùs et clariùs inſtitui poſſit, iiſdem in hâc Methodo illuſtrandâ uſi ſumus exemplis decèm quæ anteà adduxeramus in ſuperiore tractatu tanquam exempla prioris Methodi de inventione Maximorum à Fermatio et Hugenio traditæ. Nunc verò eandem applicabimus ad duas alias quantitates trinomias, nempe, quantitates $qx - px^2 + x^3$ et $rx - qx^2 + x^4$, et ad quantitates trinomias his duabus, quoàd ſigna + et — præfixa terminis, præciſè contrarias, nempe, quantitates $px^2 - qx - x^3$ et $qx^2 - rx - x^4$, (quæ quidem ita ſunt duabus prioribus affines et correſpondentes, ut ferè pro earundem caſibus haberi poſſint,) quarum examinatio aliquanto

quanto ſubtilior et difficilior invenietur quàm ſolutio priorum decèm exemplorum, ſed tamèn non minùs certè ex principiis hujus Methodi pendet, et eorum ope perfici poteſt.

Exemplum Undecimum, in quantitatibus trinomiis

$qx - px^2 + x^3$ *et* $px^2 - qx - x^3$.

Art. XVII. Deſignent Litteræ p et q quantitates aliquas datas, ſcilicet, Littera p aliquam datam rectam lineam, et Littera q aliquam datam ſuperficiem; et x deſignet quantitatem, ſcilicet, lineam rectam, variabilem, quæ creſcit continuò à nihilo ad infinitum. Et inquiratur, primò, an quantitas trinomia $qx - px^2 + x^3$ continuò creſcet (ſicut ipſa x,) à nihilo ad infinitum, an, primùm, creſcet à nihilo uſque ad certam quandam magnitudinem, et deinde decreſcet: et, ſi inveniatur quòd in quibuſdam caſibus non creſcet continuò à nihilo ad infinitum, ſed, primùm, creſcet à nihilo ad certam quandam magnitudinem, et deinde decreſcet, tunc inquiratur, in ſecundo loco, quænam erit magnitudo quantitatis x quandò quantitas trinomia $qx - px^2 + x^3$ creſcere definit, et incipit decreſcere, et proinde quando valor ejus erit maximus qui eſſe poſſit per totum id tempus creſcendi et decreſcendi: et inquiratur, tertio loco, quænam erit ratio creſcendi et decreſcendi quantitatis trinomiæ $px^2 - qx - x^3$, quæ in ſignis + et − (quæ ipſius terminis præfiguntur,) eſt priori quantitati trinomiæ $qx - px^2 + x^3$ præciſè contraria.

Solutio.

De quantitate trinomiâ $qx - px^2 + x^3$.

Si data quantitas q eſt major $\frac{pp}{4}$, quantitas binomia $qx + x^3$ erit ſempèr major quantitate px^2, quæcunque fuerit magnitudo quantitatis x. Hoc ita demonſtrari poteſt.

Quandò quantitas x eſt major quantitate p, quantitas x^3, ſola, erit major quantitate px^2. Ergò, à fortiori, quantitas binomia $qx + x^3$, erit major quantitate px^2.

Porrò, ſi x ſit æqualis p, quantitas x^3 erit æqualis quantitati px^2, et pro-inde quantitas binomia $qx + x^3$, erit major quantitate px^2.

Ergò, cum x eſt aut major ipſâ p, aut ipſi æqualis, quantitas binomia $qx + x^3$ erit major quantitate px^2.

Sed, cum quantitas x eſt minor ipſâ p, quantitas x^3 erit minor ipſâ px^2.

Fieri tamèn poteſt ut quantitas binomia $qx + x^3$ ſit etiàm in hoc caſu perpetuò major ipſâ px^2; nempe, ſi qx ſit major quàm quantitas binomia $px^2 - x^3$, ſeu ſi quantitas data q ſit major quantitate binomiâ $px - xx$. Eſt verò maximus valor quantitatis binomiæ $px - xx$, ſeu $\overline{p - x} \times x$, æqualis $\frac{pp}{4}$. Ergò, ſi data quantitas q ſit major quantitate $\frac{pp}{4}$, erit etiàm major quantitate binomiâ $px - xx$, et pro-inde quantitas binomia $q + xx$ erit major quantitate px, et quantitas $qx + x^3$ erit major quantitate px^2, etiàm dum x eſt minor ipſâ p. Igitùr, ſi quantitas data q ſit major quantitate $\frac{pp}{4}$, quantitas binomia $qx + x^3$ erit ſempèr major quantitate px^2, quæcunque fuerit magnitudo quantitatis x; et pro-inde quantitas px^2 poterit ſempèr ſubtrahi à quantitate binomiâ $qx + x^3$, et quantitas trinomia $qx + x^3 - px^2$, ſeu $qx - px^2 + x^3$, erit ſempèr poſſibilis, quæcunque fuerit magnitudo ipſius x, et quantitas contraria $px^2 - qx - x^3$ erit ſempèr impoſſibilis.

Jàm verò inquirendum eſt, an, cum q eſt major ipſâ $\frac{pp}{4}$, et pro-inde quantitas trinomia $qx - px^2 + x^3$ ſempèr eſt poſſibilis quæcunque ſit magnitudo ipſius x, hæc quantitas continuò creſcet à nihilo ad infinitum, an in quibuſdam caſibus ita continuò creſcet, ſed in aliis caſibus, proùt diverſæ fuerint inter ſe magnitudines datarum quantitatum p et q, creſcet continuò à nihilo uſque ad certam quandam magnitudinem, et deinde decreſcet. Et ut hoc appareat, neceſſe erit conferre inter ſe incrementa valdè parva et contemporanea quantitatis ſimplicis px^2 et quantitatis binomiæ $qx + x^3$, quarum differentia conſtituit quantitatem trinomiam $qx - p\,x + x^3$.

Dividatur ergò quantitas, ſeu linea, data p, in partes valdè parvas et æquales, quarum numerus ſit 10,000,000,000, ſeu decèm millia millionum; et unaquæque harum partium vocetur $\dot{x}$.

Ergò, dum x acquirit incrementum valdè parvum $\dot{x}$, qx acquiret incrementum $q\dot{x}$, et px^2 acquiret incrementum $2px\dot{x}$, et x^3 acquiret incrementum $3x^2\dot{x}$, et quantitas binomia $qx + x^3$ acquiret incrementum $q\dot{x} + 3x^2\dot{x}$. Eſt verò incrementum $q\dot{x} + 3x^2\dot{x}$ ad incrementum $2px\dot{x}$ ut $q + 3xx$ ad $2px$, ſeu ut $\frac{q}{3} + xx$ ad $\frac{2px}{3}$. Ergò, tam diù quàm quantitas binomia $\frac{q}{3} + xx$ erit major quantitate ſimplice $\frac{2px}{3}$, ſeu quantitas ſimplex $\frac{q}{3}$ erit major quantitate binomiâ $\frac{2px}{3} - xx$, incrementum quantitatis binomiæ $qx + x^3$ erit majus incremento contemporaneo quantitatis ſimplicis px^2, et pro-inde quantitas trinomia $qx + x^3 - px^2$, ſeu $qx - px^2 + x^3$, continuò creſcet; ſed poſtquàm quantitas binomia $\frac{q}{3} + xx$ fit minor quantitate $\frac{2px}{3}$, (ſi id fieri poteſt,) ſeu

quantitas

quantitas $\frac{q}{3}$ fit minor quantitate binomiâ $\frac{2px}{3} - xx$, incrementum quantitatis binomiæ $qx + x^3$ erit minus incremento contemporaneo quantitatis simplicis px^2, et pro-inde quantitas trinomia $qx - px^2 + x^3$ continuò decreſcet. Inquirendum eſt igitùr an quantitas binomia $\frac{q}{3} + xx$ ſit ſempèr major quantitate ſimplice $\frac{2px}{3}$, an ſit aliquandò major iſtâ quantitate ſimplice, aliquandò ipſi æqualis, et quandóque ipſâ minor. Hæc verò inquiſitio ita inſtitui poteſt.

Si quantitas $\frac{q}{3} + xx$ eſt æqualis quantitati $\frac{2px}{3}$, quantitas $\frac{q}{3}$ erit æqualis quantitati binomiæ $\frac{2px}{3} - xx$, hoc eſt, quantitati $\overline{\frac{2p}{3} - x} \times x$, cujus maximus valor eſt $\overline{\frac{2p}{2 \times 3}}^2$, (ſeu $\overline{\frac{p}{3}}^2$, ſeu $\frac{p^2}{3^2}$,) ſeu $\frac{pp}{9}$. Ergò, ſi quantitas $\frac{q}{3}$ eſt major quantitate $\frac{pp}{9}$, (ſeu ſi $\frac{3q}{9}$ eſt major $\frac{pp}{9}$, ſeu ſi $3q$ eſt major pp,) ſeu ſi q eſt major $\frac{pp}{3}$, quantitas $\frac{q}{3}$ erit major quantitate binomiâ $\frac{2px}{3} - xx$ (quæ eſt minor ipſâ $\frac{pp}{9}$,) et quantitas binomia $\frac{q}{3} + xx$ erit major quantitate ſimplice $\frac{2px}{3}$, et pro-inde incrementum valdè parvum quantitatis binomiæ $qx + x^3$ erit majus incremento contemporaneo quantitatis ſimplicis px^2, atque ideò quantitas trinomia $qx - px^2 + x^3$ continuò creſcet à nihilo ad infinitum, dum quantitas primitiva, ſeu radicalis, x creſcit à nihilo ad infinitum. In hoc igitùr caſu, ſeu cum quantitas data q eſt major quantitate $\frac{pp}{3}$, nullus erit maximus valor quantitatis trinomiæ $qx - px^2 + x^3$. Quod erat in hâc inquſitione imprimis determinandum.

Si quantitas q eſt major quantitate $\frac{pp}{3}$, quantitas trinomia $qx - px^2 + x^3$ nullum habebit maximum valorem, ſed continuò creſcet à nihilo ad infinitum.

De caſu in quo quantitas q eſt minor quantitate $\frac{pp}{3}$, *ſed major quantitata* $\frac{pp}{4}$.

Sed, ſi quantitas data q ſit minor quantitate $\frac{pp}{3}$, licèt major quantitate $\frac{pp}{4}$, quantitas binomia $\frac{q}{3} + xx$ non erit ſempèr major quantitate ſimplice $\frac{2px}{3}$, nec pro-inde quantitas $\frac{q}{3}$ erit ſempèr major quantitate binomiâ $\frac{2px}{3} - xx$. Sed quantitas $\frac{q}{3}$ erit in hoc caſu bìs æqualis quantitati binomiæ $\frac{2px}{3} - xx$,

ſeu, potiùs, quantitas $\frac{2px}{3} - xx$, (quæ variabilis eſt, et, primùm, creſcit, à nihilo ad $\frac{pp}{9}$, dum x creſcit à nihilo ad $\frac{p}{3}$, et deinde decreſcit à $\frac{pp}{9}$ ad nihilum dum x creſcit ulteriùs à $\frac{p}{3}$ ad $\frac{2p}{3}$,) fiet bìs æqualis quantitati eidem $\frac{q}{3}$, nempe, primùm, cum x fit æqualis quantitati $\frac{p-\sqrt{pp-3q}}{3}$, et iterùm cum x fit æqualis quantitati $\frac{p+\sqrt{pp-3q}}{3}$. Quoniam enim, dum x creſcit à nihilo uſque ad $\frac{2p}{3}$, quantitas binomia $\frac{2px}{3} - xx$ creſcit primùm à nihilo ad $\frac{pp}{9}$, (cui æqualis eſt cum x eſt æqualis ipſi $\frac{p}{3}$,) et deinde decreſcit à $\frac{pp}{9}$ ad nihilum (cui fit æqualis quandò x eſt æqualis ipſi $\frac{2p}{3}$,) manifeſtum eſt quòd dicta quantitas binomia $\frac{2px}{3} - xx$ fiet bìs æqualis cuilibet quantitati quæ ſit minor quantitate $\frac{pp}{9}$. Sed quantitas $\frac{q}{3}$ eſt in hoc caſu minor quantitate $\frac{pp}{9}$. Ergò, dum x creſcit à nihilo ad $\frac{2p}{3}$, quantitas binomia $\frac{2px}{3} - xx$ fiet bìs æqualis quantitati $\frac{q}{3}$, ſcilicèt, primùm, cum x eſt minor ipſâ $\frac{p}{3}$, et quantitas binomia $\frac{2px}{3} - xx$ eſt minor quantitate $\frac{2p}{3} \times \frac{p}{3} - \frac{p}{3} \times \frac{p}{3}$, (ſeu $\frac{2pp}{9} - \frac{pp}{9}$, ſeu $\frac{pp}{9}$, et iterùm cum x eſt major ipſâ $\frac{p}{3}$, ſed minor ipſâ $\frac{2p}{3}$, et quantitas binomia $\frac{2px}{3} - xx$ eſt minor ipſâ $\frac{pp}{9}$, ſed major nihilo, cui fit æqualis, ſecundâ vice, cum x eſt æqualis ipſi $\frac{2p}{3}$. Valores autèm quantitatis x quandò quantitas $\frac{2px}{3} - xx$ eſt æqualis $\frac{q}{3}$ invenientur reſolvendo æquationem quadraticam $\frac{2px}{3} - xx = \frac{q}{3}$; quod ita fieri poterit. Subtrahantur utræque partes hujus æquationis à quantitate $\frac{pp}{9}$, quæ eſt iis major. Et erit reſiduum $\frac{pp}{9} - \frac{2px}{3} + xx$ æquale reſiduo $\frac{pp}{9} - \frac{q}{3}$, (ſeu $\frac{pp}{9} - \frac{3q}{9}$,) ſeu $\frac{pp-3q}{9}$. Ergò, (extrahendo radices quadraticas harum quantitatum æqualium,) erit $\frac{p}{3}$

$\frac{p}{3} - x = \frac{\sqrt{pp-3q}}{3}$, et etiàm $x - \frac{p}{3} = \frac{\sqrt{pp-3q}}{3}$. Et pro-inde minor valor ipſius x erit $\frac{p-\sqrt{pp-3q}}{3}$, et major ejus valor erit $\frac{p+\sqrt{pp-3q}}{3}$.

Q. E. I.

Igitùr, dum x creſcit à nihilo ad $\frac{p-\sqrt{pp-3q}}{3}$, quantitas binomia $\frac{2px}{3} - xx$ erit minor ipſâ $\frac{q}{3}$; et, cum x eſt æqualis ipſi $\frac{p-\sqrt{pp-3q}}{3}$, quantitas binomia $\frac{2px}{3} - xx$ erit æqualis ipſi $\frac{q}{3}$; et, dum x creſcit à $\frac{p-\sqrt{pp-3q}}{3}$ ad $\frac{p}{3}$, quantitas $\frac{2px}{3} - xx$ erit major ipſâ $\frac{q}{3}$, ſed minor ipſâ $\frac{pp}{9}$; et, dum x creſcit ulteriùs à $\frac{p}{3}$ ad $\frac{p+\sqrt{pp-3q}}{3}$, quantitas $\frac{2px}{3} - xx$ erit minor quidem ipſâ $\frac{pp}{9}$, ſed tamèn major ipſâ $\frac{q}{3}$; et, cum x eſt æqualis ipſi $\frac{p+\sqrt{pp-3q}}{3}$, quantitas $\frac{2px}{3} - xx$ erit iterùm æqualis ipſi $\frac{q}{3}$; et, dum x creſcit à $\frac{p+\sqrt{pp-3q}}{3}$ ad $\frac{2p}{3}$, quantitas $\frac{2px}{3} - xx$ decreſcet à $\frac{q}{3}$ ad nihilum. Et pro-inde, dum x creſcit à nihilo ad $\frac{p-\sqrt{pp-3q}}{3}$, quantitas binomia $\frac{q}{3} + xx$ erit major quantitate $\frac{2px}{3}$; et, dum x creſcit ulteriùs à $\frac{p-\sqrt{pp-3q}}{3}$ ad $\frac{p+\sqrt{pp-3q}}{3}$, quantitas binomia $\frac{q}{3} + xx$ erit minor quantitate $\frac{2px}{3}$; et, dum x creſcit à $\frac{p+\sqrt{pp-3q}}{3}$ ad $\frac{2p}{3}$, quantitas binomia $\frac{q}{3} + xx$ erit iterùm major quantitate $\frac{2px}{3}$. Ergò, dum x creſcit à nihilo ad $\frac{p-\sqrt{pp-3q}}{3}$, quantitas trinomia $qx - px^2 + x^3$ creſcet continuò à nihilo ad magnitudinem quæ oritur ex ſubſtitutione quantitatis $\frac{p-\sqrt{pp-3q}}{3}$ prò x in ejus terminis, id eſt, ad quantitatem $\frac{pq}{3} - \frac{2p^3}{27} + \frac{2pp-6q \times \sqrt{pp-3q}}{27}$; et, dum x creſcit ulteriùs à $\frac{p-\sqrt{pp-3q}}{3}$ ad $\frac{p}{3}$, quantitas trinomia $qx - px^2 +$

x^3 decreſcet

x^3 decreſcet continuò à quantitate $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\overline{2pp-6q} \times \sqrt{pp-3q}}{27}$ ad quantitatem $q \times \frac{p}{3} - p \times \frac{pp}{9} + \overline{\frac{p}{3}}^3$, (ſeu ad quantitatem $\frac{pq}{3} - \frac{p^3}{9} + \frac{p^3}{27}$, ſeu ad quantitatem $\frac{pq}{3} - \frac{3p^3}{27} + \frac{p^3}{27}$,) ſeu ad quantitatem $\frac{pq}{3} - \frac{2p^3}{27}$; et, dum x creſcit ulteriùs à $\frac{p}{3}$ ad $\frac{p+\sqrt{pp-3q}}{3}$, quantitas trinomia $qx - px^2 + x^3$ decreſcet ulteriùs à quantitate $\frac{pq}{3} - \frac{2p^3}{27}$ ad quantitatem quæ oritur ex ſubſtitutione quantitatis $\frac{p+\sqrt{pp-3q}}{3}$ prò x in ejus terminis, id eſt, ad quantitatem $\frac{pq}{3} - \frac{2p^3}{27} - \frac{\sqrt{pp-3q} \times \overline{2pp-6q}}{27}$; et, dum x creſcit ulteriùs à quantitate $\frac{p+\sqrt{pp-3q}}{3}$ ad quantitatem $\frac{2p}{3}$, quantitas trinomia $qx - px^2 + x^3$ iterùm creſcet à quantitate $\frac{pq}{3} - \frac{2p^3}{27} - \frac{\sqrt{pp-3q} \times \overline{2pp-6q}}{27}$ ad quantitatem $q \times \frac{2p}{3} - p \times \frac{4pp}{9} + \frac{8p^3}{27}$, (ſeu ad quantitatem $\frac{2pq}{3} - \frac{4p^3}{9} + \frac{8p^3}{27}$, ſeu ad quantitatem $\frac{2pq}{3} - \frac{12p^3}{27} + \frac{8p^3}{27}$,) ſeu ad quantitatem $\frac{2pq}{3} - \frac{4p^3}{27}$; et, dum x creſcit ulteriùs à $\frac{2p}{3}$ ad infinitum, quantitas xx, ſola, erit major quantitate $\frac{2px}{3}$, et, à fortiori, quantitas binomia $\frac{q}{3} + xx$ erit major quantitate $\frac{2px}{3}$, et pro-inde incrementum valdè parvum quantitatis binomiæ $qx + x^3$ erit majus incremento contemporaneo quantitatis px^2, et idcircò quantitas trinomia $qx - px^2 - x^3$ creſcet continuò à quantitate $\frac{2pq}{3} - \frac{4p^3}{27}$ *ad infinitum*. Itaque quantitas $\frac{p-\sqrt{pp-3q}}{3}$ erit valor quantitatis variabilis x, quandò quantitas trinomia $qx - px^2 + x^3$ fit *maxima* quæ eſſe poſſit antequàm incipit decreſcere; et pro-inde iſte maximus valor quantitatis trinomiæ $qx - px^2 + x^3$ erit quantitas $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\overline{2pp-6q} \times \sqrt{pp-3q}}{27}$: Et quantitas $\frac{p+\sqrt{pp-3q}}{3}$ erit valor quantitatis x, quandò quantitas trinomia $qx - px^2 + x^3$ fit *minima* quæ eſſe poſſit poſtquàm à priore valore maximo $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\overline{2p^2-6q} + \sqrt{pp-3q}}{27}$ decreſcere incipit;

Maximus valor quantitatis trinomiæ $qx - px^2 + x^3$.

incipit; et pro-inde minimus iſte valor quantitatis trinomiæ $qx - px^2 + x^3$ erit $\frac{pq}{3} - \frac{2p^3}{27} - \frac{\sqrt{pp-3q} \times 2p^2-6q}{27}$: Et ab hoc valore minimo quantitas trinomia $qx - px^2 + x^3$ iterùm creſcet continuò ad quantitatem $\frac{2pq}{3} - \frac{4p^3}{27}$, dum x creſcit à quantitate $\frac{p+\sqrt{pp-3q}}{3}$ ad quantitatem $\frac{2p}{3}$, et poſteà perget creſcere continuò à quantitate $\frac{2pq}{3} - \frac{4p^3}{27}$ ad infinitum, dum x creſcit à $\frac{2p}{3}$ ad infinitum. Q. E. I.

Minimus valor ejuſdem quantitatis trinomiæ.

De quantitate trinomiâ $qx - px^2 + x^3$ quandò quantitas q eſt minor quantitate $\frac{pp}{4}$.

Art. XVIII. Hactenùs poſuimus datam quantitatem q eſſe aut majorem quantitate $\frac{pp}{3}$, aut iſtâ quantitate minorem, ſed tamèn majorem quantitate $\frac{pp}{4}$. Nunc verò ponamus quantitatem q eſſe etiàm minorem quantitate $\frac{pp}{4}$; et inquiramus quo modo quantitas trinomia $qx - px^2 + x^3$ creſcet et decreſcet, et quæ erunt magnitudines quantitatis x quandò iſta quantitas trinomia fit maxima quæ eſſe poſſit, in hâc hypotheſi; et inquiramus etiàm quonam modo quantitas trinomia $px^2 - qx - x^3$, (quæ, quoàd ſigna + et − præfixa ejus terminis, priori quantitati $qx - px^2 + x^3$ eſt præciſè contraria,) creſcet et decreſcet, et quænam erit magnitudo quantitatis x quandò hæc quantitas trinomia $px^2 - qx - x^3$ eſt maxima quæ eſſe poſſit. Hæc autèm omnia ita indagari et determinari poterunt.

De quantitate trinomiâ $qx - px^2 + x^3$ et quantitate trinomiâ $px^2 - qx - x^3$ eidem (quoàd ſigna + et − ſingulis terminis præfixa,) præciſè contrariâ.

Maximus valor quantitatis binomiæ $px - xx$ eſt $\frac{pp}{4}$. Ergò, quandò quantitas data q eſt minor ipſâ $\frac{pp}{4}$, (ut nunc ponitur,) poterit etiàm eſſe minor quantitate binomiâ $px - xx$, et pro-inde quantitas binomia $q + xx$ poterit eſſe minor quantitate ſimplice px, et quantitas binomia $qx + x^3$ poterit eſſe

minor

minor quantitate ſimplice px^2. Et in hoc caſu differentia quantitatum $qx + x^3$ et px^2 non erit quantitas trinomia $qx + x^3 - px^2$, ſeu $qx - px^2 + x^3$, ut anteà, ſed quantitas trinomia $px^2 - qx - x^3$. Hoc verò non accidere poteſt etiàm in hâc hypotheſi quamdiù x eſt parvæ cujuſvis magnitudinis, ſed tantùm poſtquam x crevit à nihilo ad certam quandam magnitudinem, et eſt major iſtâ magnitudine, ſed minor aliâ quâdam magnitudine quæ iſtam priorem magnitudinem non multùm excedit. Nam, cum x creſcere incipit à nihilo, quantitas q eſt major quantitate px, et pro-inde quantitas qx eſt major quantitate px^2, et, à fortiori, quantitas binomia $qx + x^3$ eſt major quantitate px^2. Sed poſteà, creſcente x, quantitas px^2 fit primùm æqualis quantitati qx, et mòx æqualis quantitati binomiæ $qx + x^3$, in quo caſu quantitas trinomia $qx - px^2 + x^3$ fit æqualis nihilo, et quantitas $px^2 - qx - x^3$ incipit eſſe poſſibilis. Hoc autèm temporis puncto px eſt $= q + xx$, et $px - xx$ eſt $= q$, et pro-inde x eſt æqualis minori ex duabus radicibus æquationis quadraticæ $px - xx = q$, id eſt, quantitati $\frac{p - \sqrt{pp - 4q}}{2}$. Poſteà verò, creſcente x à quantitate $\frac{p - \sqrt{pp-4q}}{2}$ ad majorem ex duabus radicibus ejuſdem æquationis quadraticæ, quæ eſt $\frac{p + \sqrt{pp-4q}}{2}$, quantitas binomia $px - xx$ erit major quantitate q, et pro-inde quantitas px erit major quantitate binomiâ $q + xx$, et quantitas px^2 erit major quantitate binomiâ $qx + x^3$, et quantitas $px^2 - qx - x^3$ erit poſſibilis; et, cum x ſit æqualis quantitati $\frac{p + \sqrt{pp-4q}}{2}$, quantitas binomia $px - xx$ ſit, ſecundâ vice, æqualis ipſi q, et pro-inde quantitas px ſit æqualis quantitati binomiæ $q + xx$, et quantitas px^2 ſit æqualis quantitati binomiæ $qx + x^3$, et quantitas $px^2 - qx - x^3$ ſit æqualis nihilo. Unde manifeſtum eſt quantitatem $px^2 - qx - x^3$ non eſſe poſſibilem niſi cum quantitas x ſit major quantitate $\frac{p - \sqrt{pp-4q}}{2}$ et minor quantitate $\frac{p + \sqrt{pp-4q}}{2}$. Q. E. D.

Art. XIX. Ex iis quæ jam oſtenſa ſunt manifeſtum eſt, quòd, dum x creſcit à nihilo ad quantitatem $\frac{p - \sqrt{pp-4q}}{2}$, quantitas trinomia $qx - px^2 + x^3$, primùm, creſcet à nihilo ad certam quandam magnitudinem, et deinde ab iſtâ magnitudine decreſcet ad nihilum, et quòd tunc quantitas trinomia $px^2 - qx - x^3$ (priori, quoàd ſigna terminis præfixa, præciſè contraria,) fit poſſibilis; et, dum x creſcit

cit ulteriùs à $\frac{p-\sqrt{pp-4q}}{2}$ ad $\frac{p+\sqrt{pp-4q}}{2}$, hæc quantitas trinomia $px^2 - qx - x^3$ primùm creſcet à nihilo (cui eſt æqualis quandò x eſt æqualis quantitati $\frac{p-\sqrt{pp-4q}}{2}$,) ad certam quandam magnitudinem, et deinde decreſcet ab iſtâ magnitudine ad nihilum, cui fit iterùm æqualis quandò x eſt æqualis quantitati $\frac{p+\sqrt{pp-4q}}{2}$. Nunc igitùr inquiremus, primò, quænam erit magnitudo quantitatis x quandò quantitas trinomia $qx - px^2 + x^3$ fit maxima quæ eſſe poſſit dum x non ſit major quantitate $\frac{p-\sqrt{pp-4q}}{2}$; et, ſecundò, quænam erit magnitudo ejuſdem quantitatis variabilis x, quandò quantitas trinomia $px^2 - qx - x^3$ fit maxima quæ eſſe poſſit, dum x eſt major quantitate $\frac{p-\sqrt{pp-4q}}{2}$ et minor quantitate $\frac{p+\sqrt{pp-4q}}{2}$. Hæc inquiſitio fieri poteſt hoc modo.

Si $\dot{x}$ deſignet, ut ſuprà, partem 10,000,000,000, mam quantitatis datæ p, ſeu unam ex permultis iſtius quantitatis partibus æqualibus et valdè parvis, quarum numerus ſit decèm millia millionum, et x acquirat, in quovis valdè parvo tempore incrementum $\dot{x}$, incrementum contemporaneum quantitatis qx erit $q \times \dot{x}$, ſeu $q\dot{x}$, et incrementum contemporaneum quantitatis px^2 erit $2px\dot{x}$, et incrementum contemporaneum quantitatis x^3 erit $3x^2\dot{x}$, et incrementum contemporaneum quantitatis binomiæ $qx + x^3$ erit quantitas binomia $q\dot{x} + 3x^2\dot{x}$. Eſt verò incrementum $q\dot{x} + 3x^2\dot{x}$ ad incrementum $2px\dot{x}$ ut quantitas binomia $q + 3x^2$ ad quantitatem $2px$, ſeu ut quantitas binomia $\frac{q}{3} + xx$ ad quantitatem $\frac{2px}{3}$. Ergò, tam diù quàm quantitas binomia $\frac{q}{3} + xx$ erit major quantitate $\frac{2px}{3}$, ſeu quantitas $\frac{q}{3}$ erit major quantitate $\frac{2px}{3} - xx$, incrementum quantitatis binomiæ $qx + x^3$ erit majus incremento contemporaneo quantitatis px^2, et pro-inde quantitas trinomia $qx - px^2 + x^3$ continuò creſcet; ſed poſtquam quantitas binomia $\frac{q}{3} + xx$ fit minor quantitate $\frac{2px}{3}$, ſeu quantitas $\frac{q}{3}$ fit minor quantitate $\frac{2px}{3} - xx$, incrementum quantitatis $qx + x^3$ erit minus incremento contemporaneo quantitatis px^2, et pro-inde quantitas trinomia $qx - px^2 + x^3$

continuò decreſcet, donec fiat æqualis nihilo, quandò x eſt æqualis quantitati $\frac{p-\sqrt{pp-4q}}{2}$. Ergò maximus valor hujus quantitatis trinomiæ $qx - px^2 + x^3$ erit ille quem habet quandò quantitas $\frac{q}{3}$ eſt æqualis quantitati binomiæ $\frac{2px}{3} - xx$, ſeu cum x eſt æqualis minori ex duabus radicibus æquationis quadraticæ $\frac{2px}{3} - xx = \frac{q}{3}$, hoc eſt, quantitati $\frac{p-\sqrt{pp-3q}}{3}$. Ergò hæc quantitas $\frac{p-\sqrt{pp-3q}}{3}$ eſt magnitudo quantitatis x quando quantitas trinomia $qx - px^2 + x^3$ fit maxima quæ eſſe poſſit dum x non eſt major quantitate $\frac{p-\sqrt{pp-4q}}{2}$.

Q. E. I.

Secundo loco invenienda eſt magnitudo quantitatis x quandò quantitas trinomia $px^2 - qx - x^3$ fit maxima quæ eſſe poſſit, dum x eſt major quantitate $\frac{p-\sqrt{pp-4q}}{2}$ et minor quantitate $\frac{p+\sqrt{pp-4q}}{2}$. Hæc verò ita determinari poteſt.

Incrementum valdè parvum quantitatis px^2 eſt $2px\dot{x}$, et incrementum contemporaneum quantitatis binomiæ $qx + x^3$ eſt $q\dot{x} + 3x^2\dot{x}$. Eſt verò incrementum $2px\dot{x}$ ad incrementum $q\dot{x} + 3x^2\dot{x}$ ut $2px$ ad quantitatem binomiam $q + 3x^2$, ſeu ut $\frac{2px}{3}$ ad $\frac{q}{3} + xx$. Ergò, tam diù quàm quantitas $\frac{2px}{3}$ erit major quantitate $\frac{q}{3} + xx$, ſeu $\frac{2px}{3} - xx$ erit major quantitate $\frac{q}{3}$, incrementum quantitatis ſimplicis px^2 erit majus incremento contemporaneo quantitatis binomiæ $qx + x^3$, et pro-inde quantitas trinomia $px^2 - qx - x^3$ continuò creſcet; ſed, poſtquam $\frac{2px}{3}$ fit minor quantitate $\frac{q}{3} + xx$, ſeu quantitas binomia $\frac{2px}{3} - xx$ fit minor quantitate $\frac{q}{3}$, incrementum quantitatis ſimplicis px^2 erit minus incremento contemporaneo quantitatis binomiæ $qx + x^3$, et pro-inde quantitas trinomia $px^2 - qx - x^3$ continuò decreſcet, et fiet æqualis nihilo quandò x eſt æqualis quantitati $\frac{p+\sqrt{pp-4q}}{2}$. Ergò maximus valor hujus quantitatis trinomiæ $px^2 - qx - x^3$ erit ille quem habet quandò quantitas binomia

nomia $\frac{2px}{3} - xx$ eſt æqualis quantitati $\frac{q}{3}$, ſeu cum x eſt æqualis majori ex duabus radicibus æquationis quadraticæ $\frac{2px}{3} - xx = \frac{q}{3}$, hoc eſt, quantitati $\frac{p+\sqrt{pp-3q}}{3}$. Ergò hæc quantitas $\frac{p+\sqrt{pp-3q}}{3}$ eſt magnitudo quantitatis x quandò quantitas trinomia $px^2 - qx - x^3$ eſt maxima quæ eſſe poſſit, dum x eſt major quantitate $\frac{p-\sqrt{pp-4q}}{2}$ et minor quantitate $\frac{p+\sqrt{pp-4q}}{2}$.

Q. E. I.

Et, quoniam oſtenſum eſt quòd quantitas $\frac{p-\sqrt{pp-3q}}{3}$ eſt magnitudo ipſius x quando quantitas trinomia $qx - px^2 + x^3$ eſt maxima quæ eſſe poſſit dum x non eſt major quantitate $\frac{p-\sqrt{pp-4q}}{2}$, valor iſte maximus dictæ quantitatis trinomiæ erit ille qui oritur ex ſubſtitutione quantitatis $\frac{p-\sqrt{pp-3q}}{3}$ prò x in dictâ quantitate trinomiâ. Ille verò erit quantitas $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\sqrt{2p^2-6q} \times \sqrt{pp-3q}}{27}$. Ergò hæc quantitas $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\sqrt{2p^2-6q} \times \sqrt{pp-3q}}{27}$ eſt maximus valor quantitatis trinomiæ $qx - px^2 + x^3$, dum x non eſt major quantitate $\frac{p-\sqrt{pp-4q}}{2}$.

Et, quoniam oſtenſum eſt quod quantitas $\frac{p+\sqrt{pp-3q}}{3}$ eſt magnitudo ipſius x quandò quantitas trinomia $px^2 - qx - x^3$ eſt maxima quæ eſſe poſſit, valor iſte maximus dictæ quantitatis trinomiæ erit ille qui oritur ex ſubſtitutione quantitatis $\frac{p+\sqrt{pp-3q}}{3}$ prò x in dictâ quantitate trinomiâ. Ille verò erit quantitas $\frac{2p^3}{27} - \frac{pq}{3} + \frac{\sqrt{2p^2-6q} \times \sqrt{pp-3q}}{27}$. Ergò hæc quantitas $\frac{2p^3}{27} - \frac{pq}{3} + \frac{\sqrt{2p^2-6q} \times \sqrt{pp-3q}}{27}$ erit maximus valor quantitatis trinomiæ $px^2 - qx - x^3$, in quâ x eſt major quantitate $\frac{p-\sqrt{pp-4q}}{2}$, et minor quantitate $\frac{p+\sqrt{pp-4q}}{2}$.

Poſtquam quantitas x facta eſt æqualis quantiati $\frac{p+\sqrt{pp-4q}}{2}$ et quantitas bi-

nomia

nomia $qx + x^3$ eſt, ſecundâ vice, æqualis quantitati ſimplici px^2, et x creſcit à quantitate $\frac{p+\sqrt{pp-4q}}{2}$ ad infinitum, quantitas binomia $qx + x^3$ erit perpetuò major quantitate ſimplice px^2, et incrementum valdè parvum quantitatis binomiæ $qx + x^3$, nempe, $q\dot{x} + 3x^2\dot{x}$, erit perpetuò majus incremento contemporaneo quantitatis ſimplicis px^2, ſcilicet, $.2px\dot{x}$, ut demonſtrari poteſt hoc modo.

Dum quantitas x creſcit à quantitate $\frac{p+\sqrt{pp-3q}}{3}$ ad $\frac{2p}{3}$, quantitas $\frac{2px}{3} - xx$ decreſcit à $\frac{q}{3}$ ad nihilum, et pro-inde erit ſempèr minor ipſâ $\frac{q}{3}$. Ergò $\frac{2px}{3}$ erit minor quantitate binomiâ $\frac{q}{3} + xx$, et $2px$ erit minor quantitate binomiâ $q + 3xx$, et pro-inde incrementum $q\dot{x} + 3x^2\dot{x}$ erit majus incremento $2px\dot{x}$. Porrò, cum x eſt major $\frac{2p}{3}$, xx erit major $\frac{2px}{3}$, et $3x^2\dot{x}$ erit major $2px\dot{x}$, et ideò, à fortiori, incrementum $q\dot{x} + 3x^2\dot{x}$ erit majus incremento $2px\dot{x}$. Ergò, dum x creſcit à quantitate $\frac{p+\sqrt{pp-3q}}{3}$ ad $\frac{2p}{3}$, et deinde à $\frac{2p}{3}$ ad aliam quamvis majorem magnitudinem, ſeu, ut plerumque loquimur, ad infinitum, incrementum $q\dot{x} + 3x^2\dot{x}$ erit majus incremento $2px\dot{x}$.

Sed quantitas $\frac{p+\sqrt{pp-4q}}{2}$ eſt major quantitate $\frac{p+\sqrt{pp-3q}}{3}$.

Nam quantitas $\frac{p+\sqrt{pp-4q}}{2}$ eſt $= \frac{3p+3\sqrt{pp-4q}}{6} = \frac{3p+\sqrt{9pp-36q}}{6}$; et quantitas $\frac{p+\sqrt{pp-3q}}{3}$ eſt $= \frac{2p+2\sqrt{pp-3q}}{6} = \frac{2p+\sqrt{4pp-12q}}{6}$. Ergò quantitas $\frac{p+\sqrt{pp-4q}}{2}$ erit major quantitate $\frac{p+\sqrt{pp-3q}}{3}$, ſi quantitas $3p + \sqrt{9pp-36q}$ eſt major quantitate $2p + \sqrt{4pp-12q}$, ſeu ſi quantitas $p + \sqrt{9pp-36q}$ eſt major quantitate $\sqrt{4pp-12q}$, ſeu ſi quantitas $pp + 9pp - 36q + 2p \times \sqrt{9pp-36q}$, ſeu quantitas $10pp - 36q + 2p \times \sqrt{9pp-36q}$, eſt major quantitate $4pp - 12q$, ſeu (addendo utrinque $36q$,) ſi quantitas $10pp + 2p \times \sqrt{9pp-36q}$ eſt major quantitate $4pp + 24q$, ſeu (auferendo utrinque $4pp$,) ſi quantitas $6pp + 2p \times \sqrt{9pp-36q}$ eſt major quantitate $24q$, ſeu ſi quantitas

quantitas $3pp + p \times \sqrt{9pp - 36q}$ eſt major quantitate $12q$, ſeu ſi quantitas $3pp + \sqrt{9p^4 - 36p^2q}$ eſt major quantitate $12q$, ſeu ſi quantitas $pp + \sqrt{p^4 - 4p^2q}$ eſt major quantitate $4q$. Sed, quoniam in hoc caſu q eſt minor quantitate $\frac{pp}{4}$, pp erit major quàm $4q$, et pro-inde, à fortiori, quantitas binomia $pp + \sqrt{p^4 - 4p^2q}$ erit major quantitate $4q$. Ergò quantitas $\frac{p + \sqrt{pp - 4q}}{2}$ erit major quantitate $\frac{p + \sqrt{pp - 3q}}{3}$. Q. E. D.

Sed oſtenſum eſt, quòd, dum x creſcit à quantitate $\frac{p + \sqrt{pp - 3q}}{3}$ ad aliam quamvis majorem magnitùdinem, incrementum valdè parvum $q\dot{x} + 3x^2\dot{x}$ quantitatis binomiæ $qx + x^3$ erit ſempèr majus incremento contemporaneo $2px\dot{x}$ quantitatis px^2.

Ergò, dum x creſcit à quantitate $\frac{p + \sqrt{pp - 4q}}{2}$ (quæ eſt major quantitate $\frac{p + \sqrt{pp - 3q}}{3}$,) ad aliam quamvis majorem magnitudinem, incrementum $q\dot{x} + 3x\dot{x}$ valdè parvum quantitatis binomiæ $qx + x^3$ erit ſempèr majus incremento contemporaneo $2px\dot{x}$ quantitatis ſimplicis px^2, et pro-inde earum differentia, ſeu quantitas trinomia $qx - px^2 + x^3$, continuò creſcet ad infinitum.

De æquatione cubicâ $qx - px^2 + x^3 = r$, *ſeu* $x^3 - px^2 + qx = r$.

Coroll. 1. Ergò, ſi in æquatione cubicâ hujus formæ $qx - px^2 + x^3 = r$, ſeu $x^3 - px^2 + qx = r$, quantitas q, (ſeu co-efficiens quantitatis ignotæ x,) ſit major quantitate $\frac{pp}{3}$, (ſeu tertiâ parte quadrati quantitatis p, quæ eſt co-efficiens quadrati ex quantitate ignotâ x,) æquatio habebit tantùm unam radicem, quæcunque fuerit magnitudo quantitatis cognitæ r. Et hæc radix unica erit eò major, quo major eſt quantitas cognita r.

Coroll. 2. Sed, ſi in dictâ æquatione cubicâ $qx - px^2 + x^3 = r$ quantitas q ſit minor quantitate $\frac{pp}{3}$, ſed major quantitate $\frac{pp}{4}$, et quantitas cognita r ſit æqualis quantitati $\frac{pq}{3} - \frac{2p^3}{27} + \frac{2pp - 6q}{27}\sqrt{pp - 3q}$, (quæ oritur ex ſubſtitutione

ſubſtitutione quntitatis $\frac{p-\sqrt{pp-3q}}{3}$ prò x in quantitate binomiâ $qx - px^2 + x^3$,) æquatio habebit duas radices, quarum prima, ſeu minor, erit quantitas $\frac{p-\sqrt{pp-3q}}{3}$, et ſecunda, ſeu major, erit major quantitate $\frac{p+\sqrt{pp-3q}}{3}$, ex cujus ſubſtitutione prò x in quantitate trinomiâ $qx - px^2 + x^3$ oritur quantitas $\frac{pq}{3} - \frac{2p^3}{27} - \frac{\sqrt{pp-3q} \times 2pp-6q}{27}$.

Coroll. 3. Et, ſi in dictâ æquatione cubicâ $qx - px^2 + x^3 = r$ quantitas q ſit minor quantitate $\frac{pp}{3}$, ſed major quantitate $\frac{pp}{4}$, et quantitas cognita r ſit minor quantitate $\frac{pq}{3} - \frac{2p^3}{27} + \frac{2pp-6q \times \sqrt{pp-3q}}{27}$, (quæ oritur ex ſubſtitutione quantitatis $\frac{p-\sqrt{pp-3q}}{3}$ prò x in quantitate trinomiâ $qx - px^2 + x^3$,) ſed major quantitate $\frac{pq}{3} - \frac{2p^3}{27} - \frac{\sqrt{pp-3q} + 2pp-6q}{27}$, (quæ oritur ex ſubſtitute quantitatis $\frac{p+\sqrt{pp-3q}}{3}$ pro x in quantitate trinomiâ $qx - px^2 + x^3$,) æquatio habebit tres radices; quarum prima, ſeu minima, erit minor quantitate $\frac{p-\sqrt{pp-3q}}{3}$; et ſecunda ſeu media, erit major quantitate $\frac{p-\sqrt{pp-3q}}{3}$, ſed minor quantitate $\frac{p+\sqrt{pp-3q}}{3}$; et tertia, ſeu maxima, erit major quantitate $\frac{p+\sqrt{pp-3q}}{3}$.

Coroll. 4. Et, ſi in dictâ æquatione cubicâ $qx - px^2 + x^3 = r$ quantitas q ſit minor quantitate $\frac{pp}{3}$, ſed major quantitate $\frac{pp}{4}$, et quantitas cognita r ſit æqualis quantitati $\frac{pq}{3} - \frac{2p^3}{27} - \frac{\sqrt{pp-3q} \times 2pp-3q}{27}$, quæ oritur ex ſubſtitutione quantitatis $\frac{p+\sqrt{pp-3q}}{3}$ prò x in quantitate trinomiâ $qx - px^2 + x^3$,) æquatio habebit duas radices, quarum prima, ſeu minor, erit minor quantitate $\frac{p-\sqrt{pp-3q}}{3}$, et ſecunda, ſeu major, erit quantitas $\frac{p+\sqrt{pp-3q}}{3}$.

Coroll.

Coroll. 5. Et, ſi in dictâ æquatione cubicâ $qx - px^2 + x^3 = r$ quantitas q ſit minor quantitate $\frac{pp}{3}$, ſed major quantitate $\frac{pp}{4}$, et quantitas cognita r ſit minor quantitate $\frac{pq}{3} - \frac{2p^3}{27} - \frac{\sqrt{pp-3q} \times 2pp-6q}{27}$ (quæ oritur ex ſubſtitutione quantitatis $\frac{p+\sqrt{pp-3q}}{3}$ prò x in quantitate trinomiâ $qx - px^2 + x^3$,) æquatio habebit tantùm unam radicem, quæ erit minor quantitate $\frac{p-\sqrt{pp-3q}}{3}$.

Coroll. 6. Et, ſi in datâ æquatione cubicâ $qx - px^2 + x^3 = r$ quantitas q ſit minor quantitate $\frac{pp}{3}$, ſed major quantitate $\frac{pp}{4}$, et quantitas cognita r ſit major quantitate $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\sqrt{2pp-6q} \times \sqrt{pp-3q}}{27}$, (quæ oritur ex ſubſtitutione quantitatis $\frac{p-\sqrt{pp-3q}}{3}$ pro x in quantitate trinomiâ $qx - px^2 + x^3$,) æquatio habebit tantùm unam radicem, quæ erit major quantitate $\frac{p+\sqrt{pp-3q}}{3}$.

Coroll. 7. Et, ſi in dictâ æquatione cubicâ $qx - px^2 + x^3 = r$ quantitas q ſit minor quantitate $\frac{pp}{4}$, et quantitas cognita r ſit minor quantitate $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\sqrt{2pp-6q} \times \sqrt{pp-3q}}{27}$, (quæ oritur ex ſubſtitutione quantitatis $\frac{p-\sqrt{pp-3q}}{3}$ prò x in quantitate trinomiâ $qx - px^2 + x^3$,) æquatio habebit tres radices; quarum prima, ſeu minima, erit minor quantitate $\frac{p-\sqrt{pp-3q}}{3}$; et ſecunda, ſeu media, erit major quantitate $\frac{p-\sqrt{pp-3q}}{3}$, ſed minor quantitate $\frac{p-\sqrt{pp-4q}}{2}$; et tertia, ſeu maxima, erit major quantitate $\frac{p+\sqrt{pp-4q}}{2}$.

Coroll. 8. Et, ſi in dictâ æquatione cubicâ $qx - px^2 + x^3 = r$, quantitas q ſit minor quantitate $\frac{pp}{4}$, et quantitas cognita r ſit æqualis quantitati

titati $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\overline{2pp-6q} \times \sqrt{pp-3q}}{27}$, (quæ oritur ex ſubſtitutione quantitatis $\frac{p-\sqrt{pp-3q}}{3}$ prò x in quantitate trinomiâ $qx - px^2 + x^3$,) æquatio habebit duas radices; quarum prima, ſeu minor, erit quantitas $\frac{p-\sqrt{pp-3q}}{3}$, et ſecunda, ſeu major, erit major quantitate $\frac{p+\sqrt{pp-3q}}{3}$.

Coroll. 9. Et, ſi in dictâ æquatione cubicâ $qx - px^2 + x^3 = r$ quantitas q ſit minor quantitate $\frac{pp}{4}$, et quantitas cognita r ſit major quantitate $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\overline{2pp-6q} \times \sqrt{pp-3q}}{27}$, (quæ oritur ex ſubſtitutione quantitatis $\frac{p-\sqrt{pp-3q}}{3}$ prò x in quantitate trinomiâ $qx - px^2 + x^3$,) æquatio habebit tantùm unam radicem; quæ erit major quantitate $\frac{p+\sqrt{pp+4q}}{2}$.

Coroll. 10. Et, ſi in dictâ æquatione cubicâ $qx - px^2 + x^3 = r$, quantitas q ſit minor quantitate $\frac{pp}{4}$, et quantitas cognita r ſit aut æqualis quantitati $\frac{2p^3}{27} - \frac{pq}{3} + \frac{\overline{2p^2-6q} \times \sqrt{pp-3q}}{27}$, (quæ oritur ex ſubſtitutione quantitatis $\frac{p+\sqrt{pp-3q}}{3}$ prò x in quantitate trinomiâ $px^2 - qx - x^3$, quæ eſt priori quantitati trinomiæ $qx - px^2 + x^3$, quoàd ſigna + et − ejus terminis præfixa, præciſè contraria,) aut iſtâ quantitate minor, æquatio $px^2 - qx - x^3 = r$ (quæ eſt dictæ æquationi $qx - px^2 + x^3 = r$, quoàd ſigna + et − terminis qx, px^2, et x^3 præfixa, præciſè contraria,) erit etiàm poſſibilis, non minùs quàm prior æquatio $qx - px^2 + x^3 = r$. Sed, ſi quantitas q non ſit minor quantitate $\frac{pp}{4}$; aut, ſi quantitas q ſit minor quantitate $\frac{pp}{4}$, ſed quantitas cognita r ſit major quantitate $\frac{2p^3}{27} - \frac{pq}{3} + \frac{\overline{2p^2-6q} \times \sqrt{pp-3q}}{27}$, æquatio $px^2 - qx - x^3 = r$ non erit poſſibilis.

Et,

Et, cum q eſt minor quantitate $\frac{pp}{4}$, et quantitas cognita r eſt aut æqualis quantitati $\frac{2p^3}{27} - \frac{pq}{3} + \frac{\overline{2p^2-6q} \times \sqrt{pp-3q}}{27}$, aut ipſâ minor, et pro-inde æquatio $px^2 - qx - x^3$ eſt poſſibilis; in primo horum caſuum, ſcilicet, cum quantitas cognita r eſt æqualis quantitati $\frac{2p^3}{27} - \frac{pq}{3} + \frac{\overline{2pp-6q} \times \sqrt{pp-3q}}{27}$, dicta æquatio $px^2 - qx - x^3 = r$ habebit tantùm unam radicem, nempe, quantitatem $\frac{p+\sqrt{pp-3q}}{3}$; et in ſecundo caſu, ſeu cum quantitas cognita r eſt minor quantitate $\frac{2p^3}{27} - \frac{pq}{3} + \frac{\overline{2pp-6q} \times \sqrt{pp-3q}}{27}$, dicta æquatio $px^2 - qx - x^3 = r$ habebit duas radices; quarum minor erit major quantitate $\frac{p-\sqrt{pp-4q}}{2}$, ſed minor quantitate $\frac{p+\sqrt{pp-3q}}{3}$; et major erit major quantitate $\frac{p+\sqrt{pp-3q}}{3}$, ſed minor quantitate $\frac{p+\sqrt{pp-4q}}{2}$.

Exemplum Duodecimum, in quantitatibus trinomiis $rx - qx^2 + x^4$ *et* $qx^2 - rx - x^4$.

Art. XX. Deſignent Litteræ q et r quantitates aliquas datas; et x deſignet quantitatem variabilem, quæ creſcit continuò à nihilo ad infinitum. Et inquiratur, primò, an quantitas trinomia $rx - qx^2 + x^4$ continuò creſcet (ſicut ipſa x) à nihilo ad infinitum, an, primùm, creſcet à nihilo uſque ad certam quandam magnitudinem, et deinde decreſcet: et, ſi inveniatur quòd, in quibuſdam caſibus, non creſcet continuò à nihilo ad infinitum, ſed, primùm, creſcet à nihilo ad certam quandam magnitudinem, et deinde decreſcet, tunc inquiratur, in ſecundo loco, quænam erit magnitudo quantitatis x, quandò quantitas trinomia $rx - qx^2 + x^4$ creſcere deſinit, et incipit decreſcere, et pro-inde quandò valor ejus erit maximus qui eſſe poſſit per totum id tempus creſcendi et decreſcendi: et inquiratur, tertio loco, quænam erit ratio creſcendi

et decreſcendi quantitatis trinomiæ $qx^2 - rx - x^4$, quæ in ſignis + et — (quæ ipſius terminis præfiguntur,) eſt priori quantitati trinomiæ $rx - qx^2 + x^4$ præciſè contraria.

Solutio.

Si $r + x^3$ eſt $= qx$, et pro-inde r eſt $= qx - x^3$, quantitas binomia $rx + x^4$ erit $= qx^2$, et quantitas trinomia $rx - qx^2 + x^4$ erit $= o$. Sed, dum x creſcit continuò à nihilo ad $\sqrt{q}$, quantitas binomia $qx - x^3$, primùm, creſcet à nihilo uſque ad certam quandam magnitudinem, et deinde decreſcet ab iſtâ magnitudine ad nihilum, cui fit æqualis quando xx eſt $= q$, et x eſt $= \sqrt{q}$. Hæc verò maxima magnitudo quantitatis binomiæ $qx - x^3$ eſt $\frac{2q\sqrt{q}}{3\sqrt{3}}$, proùt ſuperiùs, in Exemplo ſecundo, eſt oſtenſum. Ergò, ſi r ſit major quantitate $\frac{2q\sqrt{q}}{3\sqrt{3}}$, erit, à fortiori, major quantitate binomiâ $qx - x^3$, et pro-inde rx erit major quantitate binomiâ $qx^2 - x^4$, et quantitas binomia $rx + x^4$ erit major quantitate ſimplice qx^2. In hoc igitùr caſu quantitas trinomia $qx^2 - rx - x^4$ erit impoſſibilis, et quantitas $rx - qx^2 + x^4$ erit ſempèr poſſibilis, quæcunque fuerit magnitudo quantitatis x.

Jàm verò inquirendum eſt, an, cum r eſt major quantitate $\frac{2q\sqrt{q}}{3\sqrt{3}}$, et pro-inde quantitas trinomia $rx - qx^2 + x^4$ ſempèr eſt poſſibilis, quæcunque ſit magnitudo quantitatis x, hæc quantitas trinomia continuò creſcet à nihilo ad infinitum, an in quibuſdam caſibus ita continuò creſcet, ſed in aliis caſibus, proùt diverſæ fuerint inter ſe magnitudines datarum quantitatum q et r, creſcet continuò à nihilo uſque ad certam quandam magnitudinem, et deinde decreſcet. Et ut hoc appareat, neceſſe erit conferre intèr ſe incrementa valdè parva et contemporanea quantitatis ſimplicis qx^2 et quantitatis binomiæ $rx + x^4$, quarum differentia conſtituit quantitatem trinomiam $rx - qx^2 + x^4$.

Dividatur ergò quantitas data $q^{\frac{1}{2}}$, ſeu $\sqrt{q}$, in partes valdè parvas et æquales, quarum numerus ſit 10,000,000,000, ſeu decèm millia millionum; et unaquæque harum partium vocetur $\dot{x}$.

Ergò, dum x acquirit incrementum valdè parvum $\dot{x}$, rx acquiret incrementum $r\dot{x}$, et qx^2 acquiret incrementum $2qx\dot{x}$, et x^4 acquiret incrementum $4x^3\dot{x}$, et quantitas binomia $rx + x^4$ acquiret incrementum $r\dot{x} + 4x^3\dot{x}$. Eſt verò incrementum $r\dot{x} + 4x^3\dot{x}$ ad incrementum $2qx\dot{x}$ ut $r + 4x^3$ ad $2qx$, ſeu

ut

ut $\frac{r}{4} + x^3$ ad $\frac{qx}{2}$. Ergò, tam diù quàm quantitas binomia $\frac{r}{4} + x^3$ erit major quantitate ſimplice $\frac{qx}{2}$, ſeu quantitas ſimplex $\frac{r}{4}$ erit major quantitate binomiâ $\frac{qx}{2} - x^3$, incrementum quantitatis binomiæ $rx + x^4$ erit majus incremento contemporaneo quantitatis ſimplicis qx^2, et pro-inde quantitas trinomia $rx + x^4 - qx^2$, ſeu $rx - qx^2 + x^4$, continuò creſcet; ſed, poſtquàm quantitas binomia $\frac{r}{4} + x^3$ fit minor quantitate $\frac{qx}{2}$, (ſi id fieri poteſt,) ſeu quantitas $\frac{r}{4}$ fit minor quantitate binomiâ $\frac{qx}{2} - x^3$, incrementum quantitatis binomiæ $rx + x^4$ erit minus incremento contemporaneo quantitatis ſimplicis qx^2, et pro inde quantitas trinomia $rx - qx^2 + x^4$ continuò decreſcet. Inquirendum eſt igitùr an quantitas binomia $\frac{r}{4} + x^3$ ſit ſempèr major quantitate ſimplice $\frac{qx}{2}$, an ſit aliquandò major iſtâ quantitate ſimplice, aliquandò autem ipſi æqualis, et quandóque ipſâ minor, ſeu, (quod eodem recidit,) an quantitas ſimplex $\frac{r}{4}$ ſit ſempèr major quantitate binomiâ $\frac{qx}{2} - x^3$, an ſit aliquandò major iſtâ quantitate binomiâ, aliquandò autem ipſi æqualis, et quandóque ipſâ minor. Hæc verò inquiſitio ita inſtitui poteſt.

Ponatur $k = \frac{q}{2}$; et quantitas binomia $\frac{qx}{2} - x^3$ fiet = quantitati $kx - x^3$. Sed oſtenſum eſt ſuprà, in exemplo ſecundo, quod, dum x creſcit à nihilo uſque ad $\sqrt{k}$, quantitas binomia $kx - x^3$, primùm, creſcet à nihilo uſque ad $\frac{2k\sqrt{k}}{3\sqrt{3}}$, et deinde decreſcet à $\frac{2k\sqrt{k}}{3\sqrt{3}}$ ad nihilum, cui fit æqualis quandò x eſt æqualis $\sqrt{k}$, ſeu xx eſt æqualis ipſi k. Sed, quoniam k eſt $= \frac{q}{2}$, quantitas $k \times \sqrt{k}$ erit $= \frac{q}{2} \times \frac{\sqrt{q}}{\sqrt{2}}$, et $2k\sqrt{k}$ erit $= \frac{q \times \sqrt{q}}{\sqrt{2}}$, et $\frac{2k\sqrt{k}}{3\sqrt{3}}$ erit $\left(= \frac{q\sqrt{q}}{3\sqrt{3} \times \sqrt{2}}\right) = \frac{q\sqrt{q}}{3\sqrt{6}}$. Ergò, dum x creſcit à nihilo ad $\frac{\sqrt{q}}{\sqrt{2}}$, ſeu xx creſcit à nihilo ad $\frac{q}{2}$, quantitas binomia $\frac{q}{2} \times x - x^3$, primùm, creſcet à nihilo ad $\frac{q\sqrt{q}}{3\sqrt{6}}$, et deinde decreſcet à $\frac{q\sqrt{q}}{3\sqrt{6}}$ ad nihilum; et pro-inde $\frac{q\sqrt{q}}{3\sqrt{6}}$ erit maximus valor quantitatis

binomiæ $\frac{q}{2} \times x - x^3$. Ergò, ſi quantitas $\frac{r}{4}$ ſit major quantitate $\frac{q\sqrt{q}}{3\sqrt{6}}$, ſeu ſi quantitas data r ſit major quantitate $\frac{4q\sqrt{q}}{3\sqrt{6}}$, ſeu ſi quantitas rr ſit major quantitate $\frac{16q^3}{54}$, ſeu $\frac{8q^3}{27}$, ſeu $\overline{\frac{2q}{3}}^3$, quantitas $\frac{r}{4}$ erit major quantitate binomiâ $\frac{q}{2} \times x - x^3$ (quæ eſt minor ipſâ $\frac{q\sqrt{q}}{3\sqrt{6}}$,) et quantitas binomia $\frac{r}{4} + x^3$ erit major quantitate ſimplice $\frac{q}{2} \times x$, et pro-inde incrementum valdè parvum quantitatis binomiæ $rx + x^4$ erit majus incremento contemporaneo quantitatis ſimplicis qx^2; atque ideò quantitas trinomia $rx - qx^2 + x^4$ continuò creſcet à nihilo ad infinitum, dum quantitas primitiva x creſcit à nihilo ad infinitum. In hoc igitùr caſu, ſeu cum quantitas data r eſt major quantitate $\frac{4q\sqrt{q}}{3\sqrt{6}}$, ſeu cum quantitas rr eſt major quantitate $\frac{8q^3}{27}$, ſeu $\overline{\frac{2q}{3}}^3$, nullus erit maximus valor quantitatis trinomiæ $rx - qx^2 + x^4$. Quod erat in hâc inquiſitione imprimis determinandum.

Sed, ſi quantitas data r ſit minor quantitate $\frac{4q\sqrt{q}}{3\sqrt{6}}$, licèt major quantitate $\frac{2q\sqrt{q}}{3\sqrt{3}}$, ſeu ſi quantitas rr ſit minor quantitate $\frac{8q^3}{27}$, licèt major quantitate $\frac{4q^3}{27}$, quantitas binomia $\frac{r}{4} + x^3$ non erit ſempèr major quantitate ſimplice $\frac{qx}{2}$, nec pro-inde quantitas $\frac{r}{4}$ erit ſempèr major quantitate binomiâ $\frac{q}{2} \times x - x^3$. Sed quantitas $\frac{r}{4}$ erit in hoc caſu bìs æqualis quantitati binomiæ $\frac{q}{2} \times x - x^3$; ſeu, potiùs, quantitas binomia $\frac{q}{2} \times x - x^3$ (quæ variabilis eſt, et, primùm, creſcit, à nihilo ad $\frac{q\sqrt{q}}{3\sqrt{6}}$, dum x creſcit à nihilo ad $\frac{\sqrt{q}}{\sqrt{2} \times \sqrt{3}}$, ſeu ad $\frac{\sqrt{q}}{\sqrt{6}}$, et deinde decreſcit à $\frac{q\sqrt{q}}{3\sqrt{6}}$, ad nihilum, dum x creſcit ulteriùs à $\frac{\sqrt{q}}{\sqrt{6}}$ ad $\frac{\sqrt{q}}{\sqrt{2}}$,) fiet bìs æqualis quantitati eidem $\frac{r}{4}$, nempe, primùm, cum x fit æqualis minori ex duabus radicibus æquationis cubicæ $\frac{q}{2} \times x - x^3 = \frac{r}{4}$,

$\frac{r}{4}$, et iterùm, cum x fit æqualis majori ex iſtis radicibus. Quoniam enim, dum x creſcit à nihilo uſque ad $\frac{\sqrt{q}}{\sqrt{2}}$, quantitas binomia $\frac{q}{2} \times x - x^3$, primùm, creſcit, à nihilo uſque ad $\frac{q\sqrt{q}}{3\sqrt{6}}$, (cui æqualis eſt cum x eſt æqualis $\frac{\sqrt{q}}{\sqrt{6}}$,) et deinde decreſcit à $\frac{q\sqrt{q}}{3\sqrt{6}}$ ad nihilum, (cui fit æqualis quandò x eſt æqualis $\frac{\sqrt{q}}{\sqrt{2}}$,) manifeſtum eſt quòd dicta quantitas binomia $\frac{qx}{2} - x^3$ fiet bìs æqualis cuilibet quantitati quæ ſit minor quantitate $\frac{q\sqrt{q}}{3\sqrt{6}}$. Ergò, dum x creſcit à nihilo ad $\frac{\sqrt{q}}{\sqrt{2}}$, quantitas binomia $\frac{qx}{2} - x^3$ fiet bìs æqualis quantitati $\frac{r}{4}$, (quæ eſt minor quantitate $\frac{q\sqrt{q}}{3\sqrt{6}}$,) ſcilicet, primùm, cum x eſt minor ipſâ $\frac{\sqrt{q}}{\sqrt{6}}$, et quantitas binomia $\frac{q}{2} \times x - x^3$ eſt minor quantitate $\frac{q}{2} \times \frac{\sqrt{q}}{\sqrt{6}} - \frac{q\sqrt{q}}{6\sqrt{6}}$, (ſeu $\frac{3q\sqrt{q}}{6\sqrt{6}} - \frac{q\sqrt{q}}{6\sqrt{6}}$, ſeu $\frac{2q\sqrt{q}}{6\sqrt{6}}$,) ſeu $\frac{q\sqrt{q}}{3\sqrt{6}}$, et iterùm cum x eſt major ipſâ $\frac{\sqrt{q}}{\sqrt{6}}$, ſed minor ipſâ $\frac{\sqrt{q}}{\sqrt{2}}$, et quantitas binomia $\frac{q}{2} \times x - x^3$ eſt iterùm minor ipſâ $\frac{q\sqrt{q}}{3\sqrt{6}}$, ſed major nihilo, cui fit æqualis, ſecundâ vice, cum x eſt æqualis ipſi $\frac{\sqrt{q}}{\sqrt{2}}$. Valores autèm quantitatis x quandò quantitas $\frac{q}{2} \times x - x^3$ eſt æqualis $\frac{r}{4}$ inveniri poterunt reſolvendo æquationem cubicam $\frac{q}{2} \times x - x^3 = \frac{r}{4}$, quod commodiſſimè fieri poteſt per Raphſoni Methodum Approximationum. Reſolvatur igitùr hæc æquatio cubica, et prima, ſeu minor, ex ejus radicibus vocetur a, et ſecunda, ſeu major, vocetur b.

Igitùr, dum x creſcit à nihilo ad a, (ſeu minorem ex radicibus æquationis cubicæ $\frac{q}{2} \times x - x^3 = \frac{r}{4}$,) quantitas binomia $\frac{q}{2} \times x - x^3$ erit minor ipſâ $\frac{r}{4}$; et, cum x eſt æqualis ipſi a, quantitas binomia $\frac{q}{2} \times x - x^3$ erit æqualis ipſi $\frac{r}{4}$; et, dum x creſcit à magnitudine a ad $\frac{\sqrt{q}}{\sqrt{6}}$, quantitas binomia $\frac{q}{2} \times x - x^3$ erit major ipſâ $\frac{r}{4}$, ſed minor ipſâ $\frac{q\sqrt{q}}{3\sqrt{6}}$; et, dum x creſcit

à $\frac{\sqrt{q}}{\sqrt{6}}$ ad magnitudinem b, (ſeu majorem ex radicibus dictæ æquationis cubicæ,) quantitas binomia $\frac{q}{2} \times x - x^3$ erit etiamnùm major ipſâ $\frac{r}{4}$; et, cum quantitas x eſt æqualis ipſi b, quantitas binomia $\frac{q}{2} \times x - x^3$ erit iterùm æqualis ipſi $\frac{r}{4}$; et, dum x creſcit à b ad $\frac{\sqrt{q}}{\sqrt{2}}$, quantitas binomia $\frac{q}{2} \times x - x^3$ decreſcet à quantitate $\frac{r}{4}$ ad nihilum. Et pro-inde, dum x creſcit à nihilo ad magnitudinem a, quantitas binomia $\frac{r}{4} + x^3$ erit major quantitate ſimplice $\frac{q}{2} \times x$; et, dum x creſcit ulteriùs à magnitudine a ad magnitudine b, quantitas binomia $\frac{r}{4} + x^3$ erit minor quantitate ſimplice $\frac{q}{2} \times x$; et, dum x creſcit à magnitudine b ad $\frac{\sqrt{q}}{\sqrt{2}}$, quantitas binomia $\frac{r}{4} + x^3$ erit iterùm major quantitate ſimplice $\frac{q}{2} \times x$. Ergò, dum x creſcit à nihilo ad magnitudinem a, quantitas trinomia $rx - qx^2 + x^4$ creſcet continuò à nihilo ad magnitudinem $ra - qa^2 + a^4$; et, dum x creſcit ulteriùs à magnitudine a ad $\frac{\sqrt{q}}{\sqrt{6}}$, quantitas trinomia $rx - qx^2 + x^4$ decreſcet continuò ab $ra - qa^2 + a^4$ ad $\frac{r \times \sqrt{q}}{\sqrt{6}} - \frac{qq}{6} + \frac{qq}{36}$, (ſeu ad $\frac{r \times \sqrt{q}}{\sqrt{6}} - \frac{6qq}{36} + \frac{qq}{36}$,) ſeu ad $\frac{r \times \sqrt{q}}{\sqrt{6}} - \frac{5qq}{36}$; et, dum x creſcit ulteriùs à $\frac{\sqrt{q}}{\sqrt{6}}$ ad magnitudinem b, quantitas trinomia $rx - qx^2 + x^4$ etiamnùm decreſcet à magnitudine $\frac{r \times \sqrt{q}}{\sqrt{6}} - \frac{5qq}{36}$ ad magnitudinem $rb - qb^2 + b^4$; et, dum x creſcit ulteriùs à magnitudine b ad magnitudinem $\frac{\sqrt{q}}{\sqrt{2}}$, quantitas trinomia $rx - qx^2 + x^4$ iterùm creſcet à magnitudine $rb - qb^2 + b^4$ ad magnitudinem $\frac{r \times \sqrt{q}}{\sqrt{2}} - \frac{qq}{2} + \frac{qq}{4}$, ſeu ad magnitudinem $\frac{r \times \sqrt{q}}{\sqrt{2}} - \frac{qq}{4}$; et, dum x creſcit ulterius à $\frac{\sqrt{q}}{\sqrt{2}}$ ad infinitum, quantitas a^3, ſola, erit major quantitate $\frac{q}{2} \times x$, et, à fortiori, quantitas binomia $\frac{r}{4} + x^3$ erit major iſtâ quantitate, et pro-inde incrementum valdè parvum quantitatis binomiæ $rx + x^4$ erit majus incremento contemporaneo quantitatis ſimplicis qx^2, et idcircò

idcircò quantitas trinomia $rx - qx^2 + x^4$ creſcet continuò à magnitudine $\frac{r \times \sqrt{q}}{\sqrt{2}} - \frac{qq}{4}$ ad infinitum. Itaque quantitas a, ſeu minor ex duabus radicibus æquationis cubicæ $\frac{q}{2} \times x - x^3 = \frac{r}{4}$, erit valor quantitatis variabilis x, quandò quantitas trinomia $rx - qx^2 + x^4$ fit maxima quæ eſſe poſſit antequàm incipit decreſcere; et pro-inde iſte maximus valor quantitatis trinomiæ $rx - qx^2 + x^4$ erit quantitas $ra - qa^2 + a^4$: et quantitas b, ſeu major ex duabus radicibus æquationis cubicæ $\frac{q}{2} \times x - x^3 = \frac{r}{4}$ erit valor quantitatis x, quandò quantitas trinomia $rx - qx^2 + x^4$ fit *minima* quæ eſſe poſſit poſtquàm à priore valore maximo $ra - qa^2 + a^4$ decreſcere cœpit; et pro inde minimus iſte valor quantitatis trinomiæ $rx - qx^2 + x^4$ erit $rb - qb^2 + b^4$: Et ab hoc valore minimo quantitas trinomia $rx - qx^2 + x^4$ iterùm creſcet continuò ad quantitatem $\frac{r \times \sqrt{q}}{\sqrt{2}} - \frac{q \times q}{2} + \frac{qq}{4}$, ſeu ad quantitatem $\frac{r \times \sqrt{q}}{\sqrt{2}} - \frac{qq}{4}$, dum x creſcit à magnitudine b ad $\frac{\sqrt{q}}{\sqrt{2}}$; et poſteà hæc quantitas trinomia perget creſcere continuò a magnitudine $\frac{r \times \sqrt{q}}{2} - \frac{qq}{4}$ ad infinitum dum x creſcet continuò à $\frac{\sqrt{q}}{\sqrt{2}}$ ad infinitum.

Maximus valor quantitatis trinomiæ $rx - qx^2 + x^4$, dum quantitas x non eſt major quantitate $\frac{\sqrt{q}}{\sqrt{2}}$.

Minimus valor ejuſdem quantitatis trinomiæ.

Q. E. I.

De quantitate trinomiâ $qx^2 - rx - x^4$.

Art. XXI. Hactenùs poſuimus quantitatem r in quantitate trinomiâ $rx - qx^2 + x^4$ eſſe majorem quantitate $\frac{2q\sqrt{q}}{3\sqrt{3}}$, ſeu quantitatem rr eſſe majorem quantitate $\frac{4q^3}{27}$, ſeu quantitatem $\frac{rr}{4}$ eſſe majorem quantitate $\frac{q^3}{27}$; in quo caſu quantitas binomia $rx + x^4$ erit ſempèr major quantitate ſimplice qx^2, et pro-inde quantitas trinomia $qx^2 - rx - x^4$ non eſt poſſibilis. Nunc verò ponemus quantitatem r eſſe minorem quantitate $\frac{2q\sqrt{q}}{3\sqrt{3}}$, ſeu quantitatem rr eſſe minorem quantitate $\frac{4q^3}{27}$, ſeu quantitatem $\frac{rr}{4}$ eſſe minorem quantitate $\frac{q^3}{27}$; in quo caſu quantitas trinomia $qx^2 - rx - x^4$ eſt poſſibilis. Et in hâc hypotheſi inquiremus, primò, quænam erit ratio creſcendi et decreſcendi quantitatis trinomiæ $rx - qx^2 + x^4$, et, ſecundò, quænam erit ratio creſcendi et decre-

ſcendi

ſcendi quantitatis trinomiæ $qx^2 - rx - x^4$, quæ eſt priori quantitati trinomiæ $rx - qx^2 + x^4$, quoàd ſigna + et — terminis rx, qx^2, et x^4, præfixa, præciſè contraria. Hæ verò inquiſitiones inſtitui poſſunt hoc modo.

Quantitas trinomia $rx - qx^2 + x^4$ fiet æqualis nihilo cum quantitas ſimplex qx^2 fit æqualis quantitati binomiæ $rx + x^4$, ſeu cum quantitas ſimplex qx fit æqualis quantitati binomiæ $r + x^3$, ſeu cum quantitas binomia $qx - x^3$ fit æqualis quantitati ſimplici r. Hoc autèm fit bìs dum x creſcit à nihilo ad $\surd q$, ſeu xx creſcit à nihilo ad q; ſcilicet, primùm, cum x fit æqualis minori ex duabus radicibus æquationis cubicæ $qx - x^3 = r$, atque iterùm, cum x fit æqualis majori ex iſtis duabus radicibus. Deſignetur minor harum radicum per litteram e, et major per litteram f. Ergò, dum x creſcit à nihilo ad magnitudinem e, quantitas binomia $qx - x^3$ erit minor quantitate r, et pro-inde quantitas binomia $qx^2 - x^4$ erit minor quantitate rx, et quantitas qx^2 erit minor quantitate binomiâ $rx + x^4$, et quantitas trinomia $rx - qx^2 + x^4$ erit poſſibilis, et quantitas trinomia $qx^2 - rx - x^4$ erit impoſſibilis; et, dum x creſcit ulteriùs à magnitudine e, (ſeu minore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$,) ad magnitudinem f, (ſeu majorem ex iſtis radicibus,) quantitas binomia $qx - x^3$ erit major quantitate r, et pro-inde quantitas binomia $qx^2 - x^4$ erit major quantitate rx, et quantitas qx^2 erit major quantitate binomiâ $rx + x^4$, et quantitas trinomia $qx^2 - rx - x^4$ fiet poſſibilis, et quantitas trinomia $rx - qx^2 + x^4$ erit impoſſibilis; et cum x creſcit ulteriùs à magnitudine f, (ſeu majore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$,) ad magnitudinem $\surd q$, quantitas binomia $qx - x^3$ fiet iterùm minor quantitate r; et pro-inde quantitas binomia $qx^2 - x^4$ fiet iterùm minor quantitate rx, et quantitas qx^2 erit iterùm minor quantitate binomiâ $rx + x^4$, atque ideò quantitas trinomia $rx + x^4 - qx^2$, ſeu $rx - qx^2 + x^4$, fiet iterùm poſſibilis, et ita perpetuò manebit, dum x creſcit ulteriùs à magnitudine $\surd q$ ad infinitum.

Quoniam ergò, dum x creſcit à nihilo ad magnitudinem e, ſeu minorem ex radicibus æquationis cubicæ $qx - x^3 = r$, quantitas trinomia $rx - qx + x^4$ eſt ſemper poſſibilis et creſcit primùm à nihilo ad certam quandam magnitudinem, et deinde decreſcit ab iſtâ magnitudine ad nihilum, (cui fit æqualis cum x eſt æqualis magnitudini e,) inquirendum nunc venit quænam erit magnitudo quantitatis x in eo temporis puncto in quo quantitas trinomia $rx - qx^2 + x^4$ creſcere deſinit et incipit decreſcere, et pro-inde eſt maxima

quæ

quæ eſſe poſſit dum x non ſit major quàm e, ſeu minor duarum radicum æquationis cubicæ $qx - x^3 = r$. Hoc verò ita inveniri poteſt.

Quantitas trinomia $rx - qx^2 + x^4$ creſcet continuò tam diu quàm incrementum valdè parvum quantitatis binomiæ $rx + x^4$ eſt majus incremento contemporaneo quantitatis qx^2, quæ ab ipſâ ſubtrahitur; hoc eſt, tam diù quam quantitas binomia $r\dot{x} + 4x^3\dot{x}$ eſt major quantitate ſimplice $2qx\dot{x}$, ſeu quantitas binomia $r + 4x^3$ eſt major quantitate $2qx$, ſeu quantitas binomia $\frac{r}{4} + x^3$ eſt major quantitate $\frac{qx}{2}$, ſeu quantitas ſimplex $\frac{r}{4}$ eſt major quantitate binomiâ $\frac{qx}{2} - x^3$; et cum incrementum quantitatis binomiæ $rx + x^4$ fit minus incremento contemporaneo quantitatis qx^2, ſeu cum quantitas $\frac{r}{4}$ fit minor quantitate binomiâ $\frac{qx}{2} - x^3$, quantitas trinomia $rx - qx^2 + x^4$ incipiet decreſcere. Ergò hæc quantitas trinomia creſcere deſinet et incipiet decreſcere cum quantitas binomia $\frac{qx}{2} - x^3$ fit æqualis quantitati $\frac{r}{4}$, ſeu cum x fit æqualis minori ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$. Deſignetur jam hæc minor radix hujus æquationis cubicæ per litteram a, et deſignetur altera, ſeu major, radix ejuſdem æquationis per litteram b, ut ſuprà eſt factum. Et a, ſeu minor ex duabus radicibus hujus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$, erit magnitudo quantitatis variabilis x in eo temporis puncto in quo quantitas trinomia $rx - qx^2 + x^4$ fit maxima quæ eſſe poſſit dum x creſcit à nihilo ad magnitudinem e, ſeu minorem ex duabus radicibus æquationis cubicæ $qx - x^3 = r$. Q. E. I.

Porrò, quoniam, dum x creſcit à magnitudine e, ſeu minore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$, ad magnitudinem f, ſeu majorem ex duabus radicibus ejuſdem æquationis, quantitas $qx^2 - rx - x^4$ fit poſſibilis, et primùm creſcit a nihilo ad certam quandam magnitudinem, et deinde decreſcit ab iſtâ magnitudine ad nihilum, (cui fit iterùm æqualis cum x eſt æqualis magnitudini f, ſeu majori ex duabus radicibus æquationis cubicæ $qx - x^3 = r$,) inquirendum nobis reſtat, quænam erit magnitudo quantitatis x in eo temporis puncto in quo quantitas trinomia $qx^2 - rx - x^4$ creſcere deſinit

et incipit decreſcere, et pro-inde eſt maxima quæ eſſe poſſit. Hoc autèm ita inveniri poteſt.

Quantitas trinomia $qx^2 - rx - x^4$ creſcet continuò tàm diù quàm incrementum valdè parvum quantitatis ſimplicis qx^2 erit majus incremento contemporaneo quantitatis binomiæ $rx + x^4$, quæ ab ipſâ ſubtrahitur; hoc eſt, tàm diù quàm quantitas ſimplex $2qx\dot{x}$ erit major quantitate binomiâ $r\dot{x} + 4x^3\dot{x}$, ſeu quantitas $2qx$ erit major quantitate $r + 4x^3$, ſeu quantitas $\frac{qx}{2}$ erit major quantitate $\frac{r}{4} + x^3$, ſeu quantitas binomia $\frac{qx}{2} - x^3$ erit major quantitate ſimplice $\frac{r}{4}$; et cum incrementum quantitatis qx^2 fit minus incremento quantitatis binomiæ $rx + x^4$, ſeu quantitas binomia $\frac{qx}{2} - x^3$ fit minor quantitate ſimplice $\frac{r}{4}$, quantitas trinomia $qx^2 - rx - x^4$ creſcere definet, et incipiet decreſcere. Ergò hæc quantitas trinomia $qx^2 - rx - x^4$ creſcere definet, et incipiet decreſcere cum quantitas binomia $\frac{qx}{2} - x^3$ fit æqualis quantitati ſimplici $\frac{r}{4}$, ſeu cum x fit æqualis majori ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$, quam per Litteram b deſignavimus. Hæc igitùr quantitas b, ſeu major ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$, erit magnitudo quantitatis variabilis x in eo temporis puncto in quo quantitas trinomia $qx^2 - rx - x^4$ creſcere definet et incipiet decreſcere, et pro-inde erit maxima quæ eſſe poſſit. Q. E. I.

Maximus valor quantitatis trinomiæ $qx^2 - rx - x^4$.

Et hinc manifeſtum eſt quòd maximus valor quantitatis trinomiæ $qx^2 - rx - x^4$ erit $qb^2 - rb - b^4$, in quâ Littera b deſignat majorem ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$.

Denique, dum x creſcit à magnitudine f, ſeu majore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$, ad magnitudinem $\surd q$, quantitas binomia $qx - x^3$ fiet, ſecundâ vice, minor quantitate r, et pro-inde quantitas qx fiet minor quantitate binomiâ $r + x^3$, atque ideò quantitas ſimplex qx^2 erit minor quantitate binomiâ $rx + x^4$, et differentia harum quantitatum erit iterùm quantitas trinomia $rx - qx^2 + x^4$; et, dum x creſcit ulteriùs à magnitudine

$\surd q$

$\sqrt{q}$ ad infinitum, quantitas x^4 ſola erit major quantitate qx^2, et, à fortiori, quantitas binomia $rx + x^4$ erit major eâdem quantitate. Ergò, dum x creſcit à magnitudine f, ſeu majore ex duabus radicibus æquationis $qx - x^3 = r$, ad infinitum, quantitas binomia $rx + x^4$ erit perpetuò major quantitate qx^2, et pro-inde differentia harum quantitatum erit quantitas trinomia $rx - qx^2 + x^4$.

Porrò, dum x creſcit à magnitudine f, ſeu majore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$, ad infinitum, incrementum valdè parvum, $r\dot{x} + 4x^3\dot{x}$, quantitatis binomiæ $rx + x^4$ erit ſempèr majus incremento contemporaneo, $2qx\dot{x}$, quantitatis qx^2, et pro-inde quantitas $rx - qx^2 + x^4$ continuò creſcet à nihilo (cui eſt æqualis quandò x eſt æqualis quantitati f, ſeu majori ex duabus radicibus æquationis cubicæ $qx - x^3 = r$,) ad infinitum. Hoc verò ita demonſtrari poteſt.

Incrementum $r\dot{x} + 4x^3\dot{x}$ erit majus incremento $2qx\dot{x}$ quandò quantitas binomia $r + 4x^3$ eſt major quantitate ſimplice $2qx$, ſeu quantitas binomia $\frac{r}{4} + x^3$ eſt major quantitate ſimplice $\frac{qx}{2}$, ſeu quandò quantitas ſimplex $\frac{r}{4}$ eſt major quantitate binomiâ $\frac{qx}{2} - x^3$. Sed, dum x creſcit à nihilo ad magnitudinem $\frac{\sqrt{q}}{\sqrt{2} \times \sqrt{3}}$, ſeu $\frac{\sqrt{q}}{\sqrt{6}}$, quantitas binomia $\frac{qx}{2} - x^3$ creſcet continuò à nihilo ad magnitudinem $\frac{q}{2} \times \frac{\sqrt{q}}{\sqrt{6}} - \frac{q\sqrt{q}}{6\sqrt{6}}$, (ſeu $\frac{3q\sqrt{q}}{6\sqrt{6}} - \frac{q\sqrt{q}}{6\sqrt{6}}$, ſeu $\frac{2q\sqrt{q}}{6\sqrt{6}}$,) ſeu $\frac{q\sqrt{q}}{3\sqrt{6}}$; et, dum x creſcit ulteriùs à magnitudine $\frac{\sqrt{q}}{\sqrt{6}}$ ad magnitudinem $\frac{\sqrt{q}}{\sqrt{2}}$, quantitas binomia $\frac{qx}{2} - x^3$ decreſcet continuò à magnitudine $\frac{q\sqrt{q}}{3\sqrt{6}}$ ad nihilum, cui fit æqualis quandò x eſt æqualis magnitudini $\frac{\sqrt{q}}{\sqrt{2}}$. Ergò, ſi quantitas $\frac{r}{4}$ eſt minor quantitate $\frac{q\sqrt{q}}{3\sqrt{6}}$, quantitas binomia $\frac{qx}{2} - x^3$ erit bìs æqualis quantitati $\frac{r}{4}$, dum x creſcit à nihilo ad magnitudinem $\frac{\sqrt{q}}{\sqrt{2}}$; nempe, primùm, cum x eſt minor quantitate $\frac{\sqrt{q}}{\sqrt{6}}$, et iterùm cum x eſt major quantitate $\frac{\sqrt{q}}{\sqrt{6}}$, ſed minor quantitate $\frac{\sqrt{q}}{\sqrt{2}}$; et hæc quantitas binomia $\frac{qx}{2} - x^3$ erit primùm minor quantitate $\frac{r}{4}$, dum x creſcit à nihilo ad magnitudinem a, ſeu

minorem ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$; et tunc erit major eâdem quantitate $\frac{r}{4}$, dum x creſcit à magnitudine a, ſeu minore ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$, ad magnitudinem b, ſeu majorem ex iſtis radicibus; et poſtremò fiet iterùm minor dictâ quantitate $\frac{r}{4}$, dum x creſcit à magnitudine b, ſeu majore ex iſtis radicibus, ad magnitudinem $\frac{\sqrt{q}}{\sqrt{2}}$. Ergò, dum x creſcit à magnitudine b, ſive majore ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$, ad magnitudinem $\frac{\sqrt{q}}{\sqrt{2}}$, quantitas $\frac{r}{4}$ erit major quantitate binomiâ $\frac{qx}{2} - x^3$, et pro-inde quantitas binomia $\frac{r}{4} + x^3$ erit major quantitate ſimplice $\frac{qx}{2}$, et incrementum $r\dot{x} + 4x^3\dot{x}$ erit majus incremento contemporaneo $2qx\dot{x}$, atque ideò quantitas trinomia $rx - qx^2 + x^4$ continuò creſcet.

Sed quantitas b, ſeu major ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$, eſt minor quantitate f, ſeu majore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$, ut mox demonſtrabimus. Ergò, dum x creſcit à magnitudine f, ſive majore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$, (cui æqualis eſt quando quantitas binomia $rx + x^4$ fit ſecundâ vice æqualis quantitati ſimplice qx^2, et pro-inde quantitas trinomia $rx - qx^2 + x^4$ fit iterùm æqualis nihilo,) ad magnitudinem $\frac{\sqrt{q}}{\sqrt{2}}$, dicta quantitas trinomia $rx - qx^2 + x^4$ creſcet continuò à nihilo ad magnitudinem $r \times \frac{\sqrt{q}}{\sqrt{2}} - q \times \frac{q}{2} + \frac{qq}{4}$, ſeu $\frac{r\sqrt{q}}{\sqrt{2}} - \frac{qq}{4}$.

Porrò, quando x eſt æqualis quantitati $\frac{\sqrt{q}}{\sqrt{2}}$, quantitas $\frac{qx}{2}$ erit æqualis $\frac{q\sqrt{q}}{2\sqrt{2}}$, et quantitas x^3 erit etiàm æqualis eidem quantitati $\frac{q\sqrt{q}}{2\sqrt{2}}$, et pro-inde quantitas $\frac{qx}{2} - x^3$ erit $= 0$. Et cum quantitas x eſt major quantitate $\frac{\sqrt{q}}{\sqrt{2}}$,

quantitas

quantitas xx erit major quantitate $\frac{q}{2}$, et pro-inde quantitas x^3, seu $xx \times x$, erit major quantitate $\frac{qx}{2}$, et, à fortiori, quantitas binomia $\frac{r}{4} + x^3$ erit major quantitate $\frac{qx}{2}$, et incrementum $r\dot{x} + 4x^3\dot{x}$ erit majus incremento contemporaneo $2qx\dot{x}$. Ergò, dum x crescit à magnitudine $\frac{\sqrt{q}}{\sqrt{2}}$ ad infinitum, quantitas trinomia $rx - qx^2 + x^4$ crescet continuò ad infinitum.

Sed jàm anteà demonstratum est, quòd, si quantitas b, seu major ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$ sit minor quantitate f, seu majore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$, dicta quantitas trinomia $rx - qx^2 + x^4$ crescet continuò à nihilo ad magnitudinem $r \times \frac{\sqrt{q}}{\sqrt{2}} - \frac{q^4}{4}$, dum quantitas x crescit à magnitudine f ad magnitudinem $\frac{\sqrt{q}}{\sqrt{2}}$. Ergò, dum quantitas x crescit continuò à magnitudine f ad infinitum, quantitas trinomia $rx - qx^2 + x^4$ crescet etiàm continuò ad infinitum.

Q. E. D.

Supereſt jam ut demonſtremus id quod in hâc demonſtratione hactenùs aſſumptum eſt, nempe, quòd quantitas b, ſeu major ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$, eſt minor quantitate f, ſeu majore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$. Hoc autem ita demonſtrari poteſt ex abſurdo.

Si b non eſt minor ipſâ f, erit vel ipſi æqualis, vel ipſâ major.

Primò ponamus, illam eſſe æqualem ipſi f. Et inde ſequetur quantitatem binomiam $qb - b^3$ eſſe æqualem quantitati binomiæ $qf - f^3$, et pro-inde eſſe æqualem quantitati r. Sed, quoniam quantitas binomia $\frac{qb}{2} - b^3$ eſt $= \frac{r}{4}$, erit quantitas binomia $2qb - 4b^3 = r$. Ergò $2qb - 4b^3$ erit $= qb - b^3$, et (addendo utrinque $4b^3$,) quantitas $2qb$ erit $= qb + 3b^3$, et (auferendo utrinque qb,) quantitas qb erit $= 3b^2$, et pro-inde q erit $= 3bb$, et bb erit $= \frac{q}{3}$ et b erit $= \frac{\sqrt{q}}{\sqrt{3}}$. Sed quantitas f eſt $= b$. Ergò quantitas f erit etiàm

æqualis

æqualis quantitati $\frac{\sqrt{q}}{\sqrt{3}}$. Sed in æquatione cubicâ $qx - x^3 = r$, major ex duabus ejus radicibus ſcilicet f, erit ſempèr neceſſariò major quantitate $\frac{\sqrt{q}}{\sqrt{3}}$. Ergò f erit ſimùl æqualis ipſi $\frac{\sqrt{q}}{\sqrt{3}}$, et ipſâ major; quod eſt impoſſibile. Ergò b non poteſt eſſe æqualis ipſi f.

Jàm verò ponamus hanc quantitatem b eſſe majorem quantitate f. Et ſequetur quantitatem binomiam $qb - b^3$ eſſe minorem quantitate binomiâ $qf - f^3$. Valor enim quantitatis cujuſvis binomiæ $qx - x^3$ decreſcit continuò à maximo ejus valore ad nihilum, dum x creſcit à valore medio quem habet ubi quantitas binomia $qx - x^3$ eſt maxima quæ eſſe poſſit, ad maximum ſui ipſius valorem, ſeu quantitatem $\sqrt{q}$. Sed $qf - f^3$ eſt $= r$. Ergò $qb - b^3$ erit minor ipſâ r. Sed $\frac{qb}{2} - b^3$ eſt $= \frac{r}{4}$, et pro-inde $2qb - 4b^3$ erit $= r$. Ergò $qb - b^3$ erit minor quantitate $2qb - 4b^3$; et pro-inde (addendo utrinque $4b^3$,) $qb + 3b^3$ erit minor ipſâ $2qb$, et (auferendo utrinque qb,) $3b^3$ erit minor quantitate qb, et $3bb$ erit minor ipſâ q, et bb erit minor ipſâ $\frac{q}{3}$, et b erit minor ipſâ $\frac{\sqrt{q}}{\sqrt{3}}$. Sed f ponitur eſſe minor ipſâ b. Ergò, à fortiori, erit minor etiàm ipſâ $\frac{\sqrt{q}}{\sqrt{3}}$. Sed in æquatione cubicâ $qx - x^3 = r$ major ex duabus ejus radicibus, ſcilicet f, erit ſempèr neceſſariò major quantitate $\frac{\sqrt{q}}{\sqrt{3}}$. Ergò quantitas f erit ſimùl minor ipſâ $\frac{\sqrt{q}}{\sqrt{3}}$, et eâdem major; quod fieri non poteſt. Ergò quantitas b non poteſt eſſe major ipſâ f.

Sed ante eſt oſtenſum eam non poſſe æqualem eſſe ipſi f.

Ergò neceſſe eſt ut quantitas b ſit minor ipſâ f. Q. E. D.

Ergò nunc certò concludi poteſt, quòd, dum x creſcit à magnitudine f, ſeu majore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$, ad infinitum, quantitas trinomia $rx - qx^2 + x^4$ creſcet etiàm continuò à nihilo (cui eſt æqualis quandò x eſt æqualis quantitati f,) ad infinitum.

Q. E. D.

Art.

Art. XXII. Ex præmiſſis deduci poſſunt Corollaria ſequentia, quæ in reſolutione æquationis biquadraticæ trinomialis hujus formæ, ſcilicet, $rx - qx^2 + x^4 = s$, ſeu $x^4 - qx^2 + rx = s$, et in reſolutione æquationis biquadraticæ trinomialis hujus formæ, ſcilicet, $qx^2 - rx - x^4 = s$, (quæ priori eſt, quoàd ſigna terminis ignotis rx, qx^2, et x^4, præfixa, præciſè contraria,) per Raphſoni Methodum Approximationis erunt plerumque magnæ utilitatis.

Coroll. 1. Quoniam oſtenſum eſt in præmiſſis, quòd, ſi in quantitate trinomiâ $rx - qx^2 + x^4$, ſeu $x^4 - qx^2 + rx$, quantitas data r (quæ eſt coefficiens quantitatis ſimplicis x,) ſit major quantitate $\frac{4q\sqrt{q}}{3\sqrt{6}}$, ſeu ſi quantitas rr ſit major quantitate $\frac{16q^3}{54}$, ſeu $\frac{8q^3}{27}$, ſeu $\left|\frac{2q}{3}\right|^3$, quantitas binomia $x^4 + rx$ erit ſempèr major quantitate ſimplice qx^2, et pro-inde quantitas trinomia $rx - qx^2 + x^4$, ſeu $x^4 - qx^2 + rx$, erit ſempèr poſſibilis, et quantitas contraria $qx^2 - rx - x^4$ erit ſemper impoſſibilis, quæcunque fuerit magnitudo quantitatis x; manifeſtum eſt quòd in hoc caſu æquatio biquadratica trinomialis $rx - qx^2 + x^4 = s$, ſeu $x^4 - qx^2 + rx = s$, erit ſempèr poſſibilis, et æquatio contraria $qx^2 - rx - x^4 = s$ erit ſemper impoſſibilis, quæcunque ſit magnitudo quantitatis cognitæ, ſeu homogenei comparationis, s.

Et, quoniam oſtenſum eſt etiàm in præmiſſis, quòd in hoc caſu, ſeu ſi quantitas data r ſit major quantitate $\frac{4q\sqrt{q}}{3\sqrt{6}}$, ſeu quantitas rr ſit major quantitate $\frac{8q^3}{27}$, quantitas trinomia $rx - qx^2 + x^4$ creſcet continuò à nihilo ad infinitum dum quantitas ſimplex x creſcit à nihilo ad infinitum, ſequitur quòd in hoc caſu dicta æquatio biquadratica trinomialis $rx - qx^2 + x^4 = s$, ſeu $x^4 - qx^2 + rx = s$, habebit tantùm unam radicem, et hæc radix unica erit eo major quo major eſt quantitas cognita, ſeu homogeneum comparationis, s.

Coroll. 2. Et, ſi quantitas data r ſit æqualis quantitati $\frac{4q\sqrt{q}}{3\sqrt{6}}$, ſeu quantitas rr ſit æqualis quantitati $\frac{8q^3}{27}$, dicta æquatio biquadratica trinomialis $rx - qx^2 + x^4 = s$, ſeu $x^4 - qx^2 + rx = s$, habebit paritèr tantùm unam radicem; et hæc radix unica erit eo major quo major eſt quantitas cognita, ſeu homogeneum comparationis, s.

Coroll. 3. Et, ſi quantitas data r ſit minor quantitate $\frac{4q\sqrt{q}}{3\sqrt{6}}$, ſed major quantitate $\frac{2q\sqrt{q}}{3\sqrt{3}}$, ſeu ſi quantitas rr ſit minor quantitate $\frac{8q^3}{27}$, ſed major quantitate $\frac{4q^3}{27}$, et ſi minor ex duabus radicibus æqationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$ vocetur a, et major ex iiſdem radicibus vocetur b, et ſi quantitas cognita s æquationis biquadraticæ $rx - qx^2 + x^4 = s$ ſit major quantitate trinomiâ $ra - qa^2 + a^4$, æquatio iſta biquadratica habebit tantùm unam radicem; et hæc radix unica erit major quantitate b, ſeu majore ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$.

Si verò quantitas cognita s æquationis biquadraticæ $rx - qx^2 + x^4 = s$ ſit æqualis quantitati trinomiæ $ra - qa^2 + a^4$, æquatio iſta biquadratica habebit duas radices; quarum prima, ſeu minor, erit quantitas a, ſeu minor ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$; et ſecunda, ſeu major, erit major quantitate b, ſeu majore ex duabus radicibus ejuſdem æquationis cubicæ.

Et, ſi quantitas cognita s æquationis biquadraticæ $rx - qx^2 + x^4 = s$ ſit minor quantitate trinomiâ $ra - qa^2 + a^4$, ſed major quantitate trinomiâ $rb - qb^2 + b^4$, æquatio iſta biquadratica habebit tres radices; quarum prima, ſeu minima, erit minor quantitate a, ſeu minore ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$; et ſecunda, ſeu media, erit major quantitate a, ſed minor quantitate b, ſeu majore ex duabus radicibus iſtius æquationis cubicæ; et tertia, ſeu maxima, erit major eâdem quantitate b.

Et,

Et, ſi quantitas cognita s in dictâ æquatione biquadraticâ $rx - qx^2 + x^4 = s$ ſit æqualis quantitati trinomiæ $rb - qb^2 + b^4$, æquatio iſta biquadratica habebit duas radices; quarum prima, ſeu minor, erit minor quantitate a, ſeu minore ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$; et ſecunda, ſeu major, erit quantitas b, ſeu major ex duabus radicibus ejuſdem æquationis cubicæ.

Denique, ſi quantitas cognita s in æquatione biquadraticâ $rx - qx^2 + x^4 = s$ ſit minor quantitate trinomiâ $rb - qb^2 + b^4$, æquatio iſta biquadratica habebit tantùm unam radicem; et hæc radix unica erit minor quantitate a, ſeu minore ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$.

Coroll. 4. Et, ſi quantitas r ſit minor quantitate $\frac{2q\sqrt{q}}{3\sqrt{3}}$, ſeu quantitas rr ſit minor quantitate $\frac{4q^3}{27}$, ſeu quantitas $\frac{rr}{4}$ ſit minor quantitate $\frac{q^3}{27}$; et, ſi minor ex duabus radicibus æquationis cubicæ $qx - x^3 = r$ vocetur e, et major ex iis radicibus vocetur f; et, ſi minor ex duabus radicibus æquationis cubicæ $\frac{q}{2} \times x - x^3 = \frac{r}{4}$ vocetur a, et major ex iiſdem radicibus vocetur b; et, ſi quantitas cognita s in æquatione biquadraticâ trinomiali hujus formæ, $rx - qx^2 + x^4 = s$, ſit major quantitate trinomiâ $ra - qa^2 + a^4$; æquatio iſta biquadratica habebit tantùm unam radicem; et hæc radix unica erit major quantitate f, ſeu majore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$.

Et, ſi quantitas cognita s ſit æqualis quantitati trinomiæ $ra - qa^2 + a^4$, æquatio biquadratica $rx - qx^2 + x^4 = s$ habebit duas radices; quarum prima, ſeu minor, erit quantitas a, ſeu minor ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$; et ſecunda, ſeu major, erit major quantitate f, ſeu majore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$.

Et, ſi quantitas cognita s ſit minor quantitate trinomiâ $ra - qa^2 + a^4$, æquatio biquadratica $rx - qx^2 + x^4 = s$ habebit tres radices; quarum prima, ſeu minima, erit minor quantitate a, ſeu minore ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = r$; et ſecunda, ſeu media, erit major ipſâ a, ſed minor quantitate e, ſeu minore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$; et tertia, ſeu maxima, erit major quantitate f, ſeu majore ex duabus radicibus ejuſdem æquationis cubicæ $qx - x^3 = r$.

COROLL. 5. Et, si quantitas r sit minor quantitate $\frac{2q\sqrt{q}}{3\sqrt{3}}$, seu quantitas rr sit minor quantitate $\frac{4q^3}{27}$, seu quantitas $\frac{rr}{4}$ sit minor quantitate $\frac{q^3}{27}$; et, si minor ex duabus radicibus æquationis cubicæ $\frac{qx}{2} - x^3 = \frac{r}{4}$ vocetur a, et major ex iisdem radicibus vocetur b; et, si minor ex duabus radicibus æquationis cubicæ $qx - x^3 = r$ vocetur e, et major ex iisdem radicibus vocetur f; et, si quantitas cognita s in æquatione biquadraticâ trinomiali $qx^2 - rx - x^4 = s$ sit major quantitate trinomiâ $qb^2 - rb - b^4$; æquatio ista biquadratica erit impossibilis.

Et, si quantitas cognita s sit æqualis dictæ quantitati trinomiæ $qb^2 - rb - b^4$, æquatio biquadratica $qx^2 - rx - x^4 = s$ habebit tantùm unam radicem; et hæc radix unica erit ipsa quantitas b, seu major ex duabus radicibus æquationis cubicæ $\frac{q}{2} \times x - x^3 = \frac{r}{4}$.

Et, si quantitas cognita s sit minor dictâ quantitate trinomiâ $qb^2 - rb - b^4$, æquatio biquadratica $qx^2 - rx - x^4 = s$ habebit duas radices; quarum prima, seu minor, erit major quantitate e, seu minore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$, sed minor quantitate b, seu majore ex duabus radicibus æquationis cubicæ $\frac{q}{2} \times x - x^3 = \frac{r}{4}$; et secunda, seu major, erit major quantitate b, sed minor quantitate f, seu majore ex duabus radicibus æquationis cubicæ $qx - x^3 = r$.

Finis hujus Methodi, à Methodo Fermatianâ, *seu* Hugenianâ, *diversæ, inveniendi* Maximos et Minimos valores *quarundam quantitatum binomialium et trinomialium et altiorum multinomialium, in quibus unus vel plures ex terminis ex quorum compositione conficiuntur, signo —, seu subtractionis, ipsis præfixo, ab aliis distinguuntur, seu à summâ aliorum terminorum subtrahuntur.*

INVESTIGATIO

INVESTIGATIO MAXIMI VALORIS

QUEM

Quantitas binomia $qx - x^3$ habere poſſit dum quantitas x creſcit à nihilo uſque ad $\surd q$, ſeu dum xx creſcit à nihilo ad q;

Per Methodum novam, à duabus Methodis præcedentibus, nempe,

FERMATIANÂ, SEU HUGENIANÂ,

Et meâ, diverſam, quam mecum communicavit vir doctiſſimus,

GULIELMUS FREND, M. A.
COLLEGII JESU APUD CANTABRIGIENSES SOCIUS.

ARTICULUS I.

QUANTITAS binomia $qx - x^3$ erit bìs æqualis nihilo; nempe, primò, cùm x eſt æqualis nihilo, ſeu o, et pro-inde $qx - x^3$ erit $= q \times o - o \times o \times o \times o (= o - o) = o$; et ſecundò, cum xx eſt æquale ſuperficiei datæ q, et pro-inde x eſt $= \surd q$, et qx eſt $= q\surd q$, et x^3 eſt $(= xx \times x) = q\surd q$, et $qx - x^3$ eſt $(= q\surd q - q\surd q) = o$. Ergò, dum linea x creſcit à nihilo uſque ad $\surd q$, quantitas binomia $qx - x^3$ creſcet à nihilo uſque ad certam quandam magnitudinem, et deinde decreſcet ab iſtâ magnitudine ad o, ſeu nihilum. Requiritur autem invenire magnitudinem lineæ x, quando quantitas binomia $qx - x^3$ attigit hanc ſuam maximam magnitudinem, ex quâ creſcere deſinit et incipit decreſcere. Hoc verò ita inveniri poteſt.

Solutio.

Deſignetur hic valor, quem quærimus, lineæ x per Litteram M. Et capiatur alius valor ipſius x minor ipſo M, et deficiens ab ipſo differentiâ quæ vocetur y; adeò ut x ſit æqualis quantitati $M - y$, et x^3 æqualis $\overline{M - y}^3$. Erit igitùr quantitas binomia $q \times \overline{M - y} - \overline{M - y}^3$ minor quantitate binomiâ

$qM - M^3$. Ergò, dum linea x creſcit à valore M ad $\surd q$, et quantitas $qx - x^3$ decreſcit à valore $qM - M^3$ ad o, hæc quantitas binomia $qx - x^3$ fiet in uno aliquo hujus temporis puncto æqualis quantitati binomiæ $q \times \overline{M - y} - \sqrt{M - y}^3$, quæ eſt minor ipſâ $qM - M^3$. Ponatur jàm z pro exceſſu valoris lineæ x ſuprà lineam M in iſto temporis puncto; adeò ut hic valor lineæ x ſit æqualis $M + z$. Et erit $q \times \overline{M + z} - \sqrt{M + z}^3 = q \times \overline{M - y} - \sqrt{M - y}^3$.

Sed $q \times \overline{M + z}$ eſt $= qM + qz$, et $\overline{M + z}^3$, eſt $= M^3 + 3M^2z + 3Mz^2 + z^3$; et pro-inde $q \times \overline{M + z} - \overline{M + z}^3$ erit $= qM + qz - M^3 - 3M^2z - 3Mz^2 - z^3$. Et $q \times \overline{M - y}$ eſt $= qM - qy$, et $\overline{M - y}^3$ eſt $= M^3 - 3M^2y + 3My^2 - y^3$; et pro-inde $q \times \overline{M - y} - \overline{M - y}^3$ erit $(= qM - qy - \sqrt{M^3 - 3M^2y + 3My^2 - y^3}) = qM - qy - M^3 + 3M^2y - 3My^2 + y^3$.

Ergò quantitas ſextinomia $qM + qz - M^3 - 3M^2z - 3Mz^2 - z^3$ erit æqualis quantitati ſextinomiæ $qM - qy - M^3 + 3M^2y - 3My^2 + y^3$; et (addendo utrinque $M^3 + 3M^2z + 3Mz^2 + z^3$,) erit $qM + qz = qM - qy + 3M^2z + 3M^2y + 3Mz^2 - 3My^2 + z^3 + y^3$; et (addendo utrinque qy,) erit $qM + qz + qy = qM + 3M^2z + 3M^2y + 3Mz^2 - 3My^2 + z^3 + y^3$; et (auferendo utrinque qM,) erit $qz + qy = 3M^2z + 3M^2y + 3Mz^2 - 3My^2 + z^3 + y^3$, hoc eſt, $q \times \overline{z + y}$ erit $= 3M^2 \times \overline{z + y} + 3Mz^2 - 3My^2 + z^3 + y^3$; et (dividendo omnes terminos per $z + y$) erit $q = 3M^2 + \frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + \frac{z^3+y^3}{z+y} = 3M^2 + \frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$.

Art. II. Sed in hâc quantitate ſextinomiâ $3M^2 + \frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$ primus ſolummodò terminus $3M^2$ eſt certæ cujuſdam magnitudinis, quanquàm adhùc ignotæ quæ pendet à valore M lineæ x in eo temporis puncto in quo quantitas binomia $qx - x^3$ eſt maxima quæ eſſe poſſit, qui valor eſt unicus, licèt adhùc ignotus; et cæteri quinque termini, $\frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$, poſſunt variari, ſeu fieri majores vel minores quàm anteà fuerant, proùt quantitas y (quæ poteſt eſſe cujuſlibet magnitudinis quæ ſit minor quantitate M,) ſit major, vel minor, quàm fuerat in initio, et pro-inde proùt quantitas z (quæ pendet à quantitate y, et ſimùl cum illâ creſcit aut decreſcit,) ſit major,

major, vel minor, quàm fuerat in initio. Ergò, ut hæc quantitas ſextinomia $3M^2 + \frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$ fiat ſempèr æqualis eidem quantitati q, (cui ipſam eſſe ſemper æqualem jam eſt oſtenſum,) dum valores quantitatum y et z variantur et diminuuntur ad libitum, neceſſe eſt ut omnes termini variabiles hujus quantitatis ſextinomiæ, nempe, termini $+ \frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$, ſimùl ſumptæ, (ratione habitâ ſignorum + et — eiſdem præfixorum,) ſint ſempèr æquales nihilo, ſive ut tres termini $\frac{3M^2z}{z+y} + z^2 + y^2$, quibus ſignum +, ſeu ſignum additionis, præfigitur, ſeu qui ſibi mutuò et primo termino $3M^2$ quantitatis ſextinomiæ $3M^2 + \frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$ adduntur, ſint, ſimùl ſumpti, æquales reliquis duobus terminis ſimùl ſumptis, nempe, $\frac{3My^2}{z+y}$ et zy, quibus ſignum —, ſeu ſignum ſubtractionis, præfigitur, ſeu qui ſubtrahuntur à tribus terminis prioribus $\frac{3Mz^2}{z+y} + z^2 + y^2$. Alitèr enim poſſet fieri (minuendo quantitates y et z, et pro-inde quantitates quinque variabiles $\frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$ quæ pendent ex y et z, et cum iis variantur,) ut tota quantitas quinquinomia $\frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + x^2 - zy + y^2$ minueretur, et pro-inde ut tota quantitas ſextinomia $3M^2 + \frac{3Mz^2}{z+y} - \frac{3M^2y}{z+y} + z^2 - zy + y^2$ (quæ major eſt quantitate quinquinomiâ $\frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$ exceſſu certo et immutato $3M^2$,) ſimùl minueretur, atque adeò fieret minor quantitate datâ q, cui eam eſſe ſempèr æqualem ſuperiùs eſt demonſtratum. Erit igitùr, in omnibus valoribus quantitatum y et z, quantitas quinquinomia $\frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$ æqualis nihilo; et pro-inde quantitas ſextinomia $3M^2 + \frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$ erit æqualis ejuſdem ſoli primo termino $3M^2$. Sed ſuprà demonſtratum eſt totam hanc quantitatem ſextinomiam $3M^2 + \frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$ eſſe æqualem datæ quantitati q. Ergò et ſolus terminus $3M^2$ erit æqualis datæ quantitati q, et pro-inde M^2 erit æqualis quantitati $\frac{q}{3}$, et M erit $= \sqrt{\frac{q}{3}}$, ſeu $\frac{\sqrt{q}}{\sqrt{3}}$; hoc eſt,

magnitudo

magnitudo lineæ x, quando quantitas binomia $qx - x^3$ eſt maxima quæ eſſe poſſit, erit $= \sqrt{\frac{q}{3}}$, ſeu $\frac{\sqrt{q}}{\sqrt{3}}$. Q. E. I.

Co. OLL. 1. In quantitate quinquinomiâ $\frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$ (quæ eſt æqualis nihilo, ſeu in quâ duo termini $\frac{3My^2}{z+y} + zy$ ſunt æquales tribus terminis $\frac{3Mz^2}{z+y} + z^2 + y^2$,) quantitas y eſt ſempèr major quantitate z. Nam, ſi quantitas y eſſet æqualis quantitati z, duo termini $\frac{3My^2}{z+y} + zy$ eſſent æquales duobus terminis $\frac{3Mz^2}{z+y} + z^2$, et pro-inde minores tribus terminis $\frac{3Mz^2}{z+y} + z^2 + y^2$; et, ſi quantitas y eſſet minor quantitate z, duo termini $\frac{3Mz^2}{z+y} + zy$ eſſent minores duobus terminis $\frac{3Mz^2}{z+y} + z^2$, et, à fortiori, minores tribus terminis $\frac{3Mz^2}{z+y} + z^2 + y^2$; quod eſt contrà ſuppoſitionem. Ergò quantitas y non poteſt eſſe aut æqualis quantitati z, aut iſtâ minor; et pro-inde erit iſtâ major. Q. E. D.

Coroll. 2. Cum jam oſtenſum ſit quòd quantitas y eſt major quantitate z, quantitas $\frac{3My^2}{z+y}$ erit major quantitate $\frac{3Mz^2}{z+y}$, et quantitas quinquinomia $\frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + z^2 - zy + y^2$ erit $(= yy - yz + zz - \frac{3M}{z+y} \times \overline{yy - zz} = yy - yz + zz - 3M \times \frac{yy-zz}{y+z} = yy - yz + zz - 3M \times \overline{y - z}) = yy - yz + zz - 3My + 3Mz$. Ergò, quoniam quantitas ſextinomia $3M^2 + \frac{3Mz^2}{z+y} - \frac{3My^2}{z+y} + zz - zy + yy$ eſt ſempèr æqualis datæ quantitati q, erit et quantitas ſextinomia $3M^2 + yy - yz + zz - 3My + 3Mz$ ſempèr æqualis eidem datæ quantitati q, utcunque diminuantur quantitates y et z. Igitùr neceſſe eſt ut quantitas quinquinomia $yy - yz + zz - 3My + 3Mz$ ſit ſempèr æqualis nihilo et pro inde ut duo termini $3My$ et yz, qui à reliquis tribus $3Mz + yy + zz$ ſubtrahuntur, ſint ſempèr æquales iſtis tribus terminis. Nam, ſi duo termini $3My + yz$ eſſent aliquando minores tribus terminis $3Mz + yy + zz$,

quantitas quinquinomia $3Mz - 3My + yy - yz + zz$ foret major quàm fuerat anteà, et pro-inde quantitas ſextinomia $3M^2 + 3Mz - 3My + yy - yz + zz$ (quæ ſuperat dictam quantitatem quinquinomiam datâ differentiâ $3M^2$,) foret major quàm fuerat anteà, et pro-inde major quàm data quantitas q, cui oſtenſum eſt quòd erit ſempèr æqualis. Ergò duo termini $3My + yz$ non poſſunt eſſe minores tribus terminis $3Mz + yy + zz$. Et eodem modo demonſtrari poteſt quòd duo termini prædicti $3My + yz$ nunquam poſſunt eſſe majores tribus terminis $3Mz + yy + zz$. Ergò dicti termini $3My + yz$ erunt perpetuò æquales tribus terminis $3Mz + yy + zz$, et quantitas quinquinomia $3Mz + yy + zz - 3My - yz$, ſeu $3Mz - 3My + yy - yz + zz$, erit ſempèr æqualis nihilo, et pro-inde quantitas ſextinomia $3M^2 + 3Mz - 3My + yy - yz + zz$ erit $(= 3M^2 + o) = 3M^2$. Sed iſta quantitas ſextinomia eſt æqualis datæ quantitati q. Ergò et quantitas $3M^2$ erit æqualis eidem datæ quantitati q; et pro-inde M^2 erit $= \frac{q}{3}$, et M erit $= \sqrt{\frac{q}{3}}$, ſeu $\frac{\sqrt{q}}{\sqrt{3}}$.

Q. E. I.

Art. III. Hæc ſunt ratiocinia quibus innititur hæc inveſtigatio quantitatis M, (ſeu valoris quantitatis variablilis x in eo temporis puncto in quo quantitas binomia $qx - x^3$ fit maxima quæ eſſe poſſit,) mecum communicata, et, ut opinor, inventa, à dicto doctiſſimo viro, GULIELMO FREND, ſi eam rectè intellexerim. Mihi verò hæc inveſtigatio aliquantulùm ſubtilis et conceptu difficilis eſſe videtur, et minùs ſimplex et directa quàm Methodus inveniendi hanc quantitatem M in Diatribé prædictâ tradita et illuſtrata in paginis 156, 157, et 158, quæ fundatur in contemplatione incrementorum, $q\dot{x}$ et $3x^2\dot{x}$, valdè parvorum et æqualibus temporibus acquiſitorum, quibus augentur quantitates qx et x^3 dum x creſcit à nihilo ad magnitudinem $\sqrt{q}$, (quam nunquam ſuperare poteſt,) et in collatione iſtorum incrementorum inter ſe, et determinatione valoris quantitatis x in eo temporis puncto in quo iſta incrementa ſunt inter ſe æqualia.

Londini, in Templo Interiore.
Nov. 9, 1803.

FRANCISCUS MASERES.

TRACTATUS

TRACTATUS

DE

LIMITIBUS ÆQUATIONUM,

Inceptus à FLORIMONDO DE BEAUNE,

Celeberrimo Mathematico Gallo, Regis Galliæ illuſtriſſimi, Ludovici XIV, in Curiâ Bleſenſi Conſiliario; Abſolutus verò, et poſt mortem ejus editus,

AB

ERASMIO BARTHOLINO,

Medicinæ et Mathematum in Regiâ Academiâ Hafnienſi Profeſſore publico;

IN QUO OSTENDITUR

Quo pacto ex formâ Æquationum affectarum definiri poſſint limites, intrà quos radices veræ debent offendi.

Excerptus ex tomo ſecundo editionis celeberrimi Libri cui titulus *Geometria*, à Renato Des Cartes ſcripti, cum commentariis FRANCISCI À SCHOOTEN et aliorum doctorum virorum, impreſſæ Amſtelædami à Ludovico et Daniele Elzeviriis, anno Domini 1659.

DE hoc tractatu de Limitibus radicum æquationum affectarum, seu multinomialium, et alio tractatu ejusdem auctoris *Florimondi de Beaune* de earum Naturâ et Constitutione, ita loquitur vir doctissimus *Franciscus à Schooten* in præfatione editionis Geometriæ Renati des Cartes, cum suis et aliorum commentariis, anno Domini 1659.

" Eâdem ratione ductus, ne Sparta quam Vir ampliſſimus, nunc piæ memoriæ, *Dominus De Beaune* in excolendâ propagandâque hujus Geometriæ methodo susceperat, præcipiti ejus fato interiret; ex officio atque publicâ matheſin amantium utilitate fore existimavi, si clarissimum virum *Dominum Erasmium Bartholinum* nostro rogatu adigerem, ut quæ de Naturâ, Constitutione, ac Limitibus, Æquationum *Dominus de Beaune* vernaculâ suâ linguâ in lucem dare constituerat, cum in manus ipsius incidissent, publico non invideret. Nec frustrà in eo fui: Nactus enim sum, ut quæ ex ejus adversariis, non sine indefesso labore ac difficili fortunâ, ad umbilicum perduxerat, Latinè redderet, nobís que, quò, unà cum his, à me typis mandarentur, concederet."

AD TRACTATUM DE LIMITIBUS ÆQUATIONUM

EPISTOLA PRÆLIMINARIS.

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NISI meminiſſem, quanto majore animo honeſtatis fructus in conſcientiâ, quàm in famâ reponatur; nequaquam opportunum fuiſſet, in edendis hiſce opuſculis Analyticis conſilium. Verùm, quia communibus magìs commodis quàm privatæ jactantiæ ſtudui, eò animus auſus eſt, deliberato conſilio obſequi. Cujus meæ conſcientiæ interpretem, non alium magìs deſidero quàm te, Vir Clariſſime, quem utilitatibus alicrum, plus quàm propriæ laudi, indies deſervire, compertum habeo. Venit in mentem ſtudioſum illud otium, quod Leidæ, mihi ſemper emolumento, utriſque deinde ſolatio erat, cujúſque varietates ſi oratione repetere vellem, procùl animo pleræque obverſantur, non dubito quìn exiſtimationi hominum diligentia et fides noſtra, et in pleriſque etiam pietas ſubjiceretur. Et licèt neſciam, an ullum tempus jucundiùs exegerim; tamen eâ de cauſâ magnifacio, quòd amicitiæ tuæ, uſque ad intimam familiaritatem, capacem me redderet. Neque aliam interpretationem habuit, quòd, Leidâ diſceſſurus, Iſagogen Carteſianam typis excudendam concinnaveram, ut meam famam cum tuâ extenderem. Quâ de cauſâ, cum non modò offenſas, verùm etiam ſimultates varias ſubierim; non ignoro, quæ futura ſit de hiſce jam edendis ſententia. Ne dubites tamen quìn omnia æquo animo toleraverim, præſertìm quia pietas et obſequium cauſam junxêre. Quem enim præterit, fatum litteratorum? Mihi certè non improviſa eſt calumniandi vanitas. Eſt ita naturâ comprobatum, ut benefactis major ex conſcientiâ merces, quàm in ore hominum reponatur: nam plerique, tantum ſuæ detractum iri gloriæ exiſtimant, quantum ceſſerit alienæ: poſtremò, ignaviſſimus quiſque aliorum ſcripta carpere non veretur. Sic contendere, pro moribus temporum, eruditio eſt. Quod recordantem, poſteritatis magna miſeratio ſubit. Quot enim præclara inventa putas obſcurari, propter ſcelus hoc obtrectandi? Plerique ſe intrà perpetuum ſilentium tenere amant, potiùs quam malignitati interpretantium exponi. Ita communem hunc errorem, bonum publicum magnis detrimentis expiabit. Ego, aliorum exemplo, qui-

dem didici nullam ex meis laboribus ſperare laudem; tanta tamen mihi ſemper fuit reverentia poſterûm, ut cenſuram erroris non tam reformidem quam inhumanitatis. Sed, ut de pictore niſi Artifex judicare, ita niſi Mathematicus non ſatis poteſt perſpicere Mathematica; tuæ potiſſimùm ſententiæ hæc exponuntur. Eximium habent uſum ea quæ ſequenti tractatu exponentur, ad numeroſam Æquationum reſolutionem, ut reliquas utilitates pertranſeam, quia Tu eas ignorare non potes. Quarè Lectores rogo, ut judiciis parcant, donec penitùs omnia inſpexerint. Et ſi qui fuerint qui hæc recuſaverint, ſciant ſe nec inventis gratiam adimere, nec mihi laudis conſcientiam. Te verò, Vir Clariſſime, ſi offenderint, omnibus commendationibus deſtituta reputabo. Vale.

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LIMITIBUS ÆQUATIONUM.

CAPUT I.

De Æquationum Quadratarum, ſeu duarum dimenſionum, limitibus.

PROP. 1. $xx - lx + mm = 0$.

PER tranſpoſitionem erit $mm = lx - xx$, &, ſi prima pars fuerit realis, erit etiam altera pars realis, ideóque lx majus quàm xx; &, diviſo utroque termino per x, erit l major quàm x. Quìn &, per tranſpoſitionem propoſitæ æquationis, habebitur $xx = lx - mm$: ideóque altera pars eſt realis, et lx majus quàm mm. Unde, diviſo utroque termino per l, erit x major quàm $\frac{mm}{l}$. Quarè æquationis propoſitæ utraque radix x major erit quàm $\frac{mm}{l}$, ſed minor quàm l.

PROP. 2. $xx - lx - mm = 0$.

Per tranſpoſitionem habebimus $xx = lx + mm$, ideóque xx majus erit quàm mm, et x major quàm m, ac pro-inde mx majus quàm mm. Unde xx minus erit quàm $lx + mx$, adeóque, ſi utraque pars dividatur per x, erit x minor quàm $l + m$. Rurſùs, quoniam xx æquatur $lx + mm$, erit xx majus quàm lx; ac pro-inde, ſi uterque terminus dividatur per x, erit x major quàm l, et lx majus quàm ll. Hinc, cum xx æquetur $lx + mm$, erit xx majus quàm $ll + mm$, hoc eſt, x major quàm $\sqrt{ll + mm}$. Poſtremò, quandoquidem x major eſt quàm m, erit lx majus quàm lm, et xx majus quàm $lm + mm$, hoc eſt, x major quàm $\sqrt{lm + mm}$. Unde radix æquationis propoſitæ erit major quàm maxima harum duarum $\sqrt{ll + mm}$ et $\sqrt{lm + mm}$, ſed minor quàm $l + m$.

PROP.

PROP. 3. $xx + lx - mm = 0$.

Per tranſpoſitionem habebimus $xx + lx = mm$, et per conſequens $\frac{mm}{l}$ major erit quàm x. Rurſùs, exiſtente $xx + lx = mm$, erit mm majus quàm xx, et m major quàm x, ac pro-inde mx majus quàm xx. Atqui habemus $xx + lx = mm$. Ergò $mx + lx$ majus erit quàm mm. Hinc, diviſâ utrâque parte per $m + l$, fiet x major quàm $\frac{mm}{l+m}$. Quarè inventa eſt x, radix æquationis propoſitæ, eſſe major quàm $\frac{mm}{l+m}$, at minor quàm $\frac{mm}{l}$ et m.

CAPUT II.

De limitibus Æquationum Cubicarum, ſeu trium dimenſionum, ſecundo termino carentium.

PROP. 1. $x^3 * - mmx + n^3 = 0$.

Per tranſpoſitionem habebimus $x^3 = + mmx - n^3$, eritque mmx majus quàm n^3. Unde, diviſo utroque termino per mm, erit x major quàm $\frac{n^3}{mm}$. Deinde, per tranſpoſitionem, erit $mmx - x^3 = n^3$, ac per conſequens mm majus quàm xx, et m major quàm x. Quare inventa eſt utraque radix x æquationis propoſitæ major quàm $\frac{n^3}{mm}$, et minor quàm m.

PROP. 2. $x^3 * - mmx - n^3 = 0$.

Per tranſpoſitionem habebimus $x^3 - mmx = n^3$, eritque xx majus quàm mm, et x major quàm m. Erit quoque $x^3 - n^3 = mmx$, ideóque x^3 major quàm n^3, et x major quàm n, ac pro-inde nnx majus quàm n^3. Atqui, per tranſpoſitionem propoſitionis, habemus $mmx + n^3 = x^3$. Quarè $mmx + nnx$ majus erit quàm x^3; et, diviſâ utrâque parte per x, erit $mm + nn$ majus quàm xx; ideó-

que

que x minor quàm $\sqrt{mm + nn}$. Inventa ergò eſt x, radix æquationis propoſitæ, eſſe major quàm m & n, at minor quàm $\sqrt{mm + nn}$. Atque liquet, ad evitandam extractionem radicis cubicæ ipſius n^3, quòd, loco nn ſub vinculo, aſſumi poſſit quantitas aliqua, quæ non ſit minor. Id quod non erit difficile, cognitis, nempe, tribus dimenſionibus ipſius n^2, ſumendóque, loco nn, rectangulum ſub duabus quantitatibus quarum alterutra non ſit ipsâ n minor. Erítque hoc ad ſequentia notatu dignum.

PROP. 3. $x^3 * + mnx - n^3 = 0$.

Per tranſpoſitionem habebimus $x^3 = n^3 - mmx$, erítque $\frac{n^3}{mm}$ major quàm x. Rurſùs erit $mmx = n^3 - x^3$, & conſequentèr n^3 major quàm x^3, et n major quàm x, ac pro-inde nnx majus quàm x^3. Sed, per tranſpoſitionem æquationis propoſitæ, eſt quoque $x^3 + mmx = n^3$. Ergò $mmx + nnx$ majus erit quàm n^3, &, divisâ utrâque parte per $nn + mm$, erit x major quàm $\frac{n^3}{nn+mm}$. Inventa itaque eſt radix x æquationis propoſitæ eſſe major quàm $\frac{n^3}{mm+nn}$, ſed minor quàm $\frac{n^3}{mm}$ et n. Poſſumus etiàm loco nn accipere rectangulum duarum maximarum dimenſionum ipſius n^3, ut radicis cubicæ extractio evitetur.

CAPUT III.

De limitibus Æquationum Cubicarum, penultimo termino carentium.

PROP. 1. $x^3 - lxx^* + n^3 = 0$.

Per tranſpoſitionem erit $x^3 + n^3 = lxx$, ideóque xx majus quàm $\frac{n^3}{l}$. Rurſùs erit $n^3 = lxx - x^3$, et conſequentèr l major quàm x. Quælibet igitùr radicum x æquationis propoſitæ major erit quàm $\sqrt{\frac{n^3}{l}}$, et minor quàm l.

PROP.

Prop. 2. $x^3 - lxx^* - n^3 = 0$.

Per tranſpoſitionem erit $x^3 - lxx = n^3$, ideóque x major quàm l. Rurſùs erit $x^3 - n^3 = lxx$, et conſequentèr x major quàm n, et xx majus quàm nn, et nxx majus quàm n^3. Atqui habemus quoquè, per tranſpoſitionem, $lxx + n^3 = x^3$. Quarè erit $lxx + nxx$ majus quàm x^3. Dividatur utraque pars per xx; erítque $l + n$ major quàm x. Inventa itaque eſt radix x æquationis propoſitæ eſſe major quàm l et n, ſed minor quàm $l + n$. Manifeſtum eſt quoque, ad evitandam extractionem radicis cubicæ ex n^3, quòd loco n ſumi poſſit minor trium dimenſionum ipſius n^3, quando x major eſt; et quando minor perhibetur quàm $l + n$, quòd tunc loco n maxima trium dimenſionum ipſius n^3 accipi queat; et ſic de reliquis, quibus ob nimiam facilitatem non immoramur.

Prop. 3. $x^3 + lxx^* - n^3 = 0$.

Per tranſpoſitionem erit $x^3 = n^3 - lxx$, ac per conſequens $\frac{n^3}{l}$ majus quàm xx. Eſt etiam $lxx = n^3 - x^3$, et conſequentèr n major quàm x, et nxx majus quàm x^3. Sed habetur $x^3 + lxx = n^3$. Ergò $nxx + lxx$ majus erit quàm n^3, hoc eſt, divisâ utrâque parte per $n + l$, erit xx majus quàm $\frac{n^3}{n+l}$. Inventa eſt itaque radix x æquationis propoſitæ eſſe major quàm $\sqrt{\frac{n^3}{l+n}}$, ſed minor quàm $\sqrt{\frac{n^3}{l}}$ et n. Demonſtratur prætereà $nnx + lnx$ majus eſſe quàm n^3, et $nx + lx$ majus quàm nn, et conſequentèr x major quàm $\frac{nn}{l+n}$, quandoquidem n major eſt quàm x.

CAPUT IV.

De Æquationibus Cubicis, in quibus omnes termini extant.

Prop. 1. $x^3 - lxx + mmx - n^3 = 0$.

Per tranſpoſitionem habebimus $x^3 - lxx = n^3 - mmx$. Hinc, ſi x æquetur ipſi l, erit etiàm x ipſi $\frac{n^3}{mm}$ æqualis. Ideóque, ſi viciſsìm l æquetur ipſi $\frac{n^3}{mm}$, hoc

hoc eſt, $lmm = n^3$, erit ſimilitèr x, radix æquationis propoſitæ, æqualis ipſi l et $\frac{n^3}{mm}$. Prætereà, ſi $x^3 - lxx$ eſt realis, hoc eſt, x major quàm l, erit quoque $n^3 - mmx$ realis, et conſequentèr $\frac{n^3}{mm}$ major quàm x. Quòd ſi autem eadem quantitas $x^3 - lxx$ nihilo minor ſit, tranſponatur propoſita æquatio hâc ratione, $lxx - x^3 = mmx - n^3$. Et quandoquidem ſupponitur $lxx - x^3$ eſſe realis, hoc eſt, l major quàm x, erit $mmx - n^3$ etiàm realis, et conſequentèr major erit x quàm $\frac{n^3}{mm}$. Inventa eſt itaque radix æquationis propoſitæ eſſe æqualis ipſi l et ipſi $\frac{n^3}{mm}$, cùm duo hi termini æquantur. Et ſi unam tantùm habeat aut tres, quælibet earum erit intrà hos limites, quando inæquales ſunt; ſi verò æquales hoc eſt, $lmm = n^3$, ſubſtituto lmm loco n^3 in æquatione propoſitâ, [$x^3 - lxx + mmx - n^3 = o$, quæ tunc fiet $x^3 - lxx + mmx - lmm = o$,] et dividendo per $x - l$, * cognoſcemus eam non habere aliam radicem in hoc caſu quàm l.

PROP. 2. $x^3 + lxx - mmx - n^3 = o$.

Per tranſpoſitionem habebimus $x^3 - mmx = n^3 - lxx$. Quòd ſi ergò xx et mm ſunt æqualia, erit etiàm xx ipſi $\frac{n^3}{l}$ æquale; et, ſi xx majus eſt quàm mm,

* Hæc diviſio erit hujuſmodi.

$x - l)\ x^3 - lxx + mmx - lmm\ (xx + mm$
$x^3 - lxx$
$+ mmx - lmm$
$+ mmx - lmm$

Patet ergò quòd quantitas quadrinomia $x^3 - lxx + mmx - lmm$ dividi poteſt accuratè, ſive ſine ullo reſiduo, per quantitatem binomiam $x - l$, et quòd quotiens inde orta erit quantitas binomia $xx + mm$. Sequitur igitùr quòd quantitas iſta quadrinomia poteſt generari ex multiplicatione quantitatis binomiæ $x - l$ in quantitatem binomiam $xx + mm$. Sed quantitas quadrinomia $x^3 - lxx + mmx - lmm$ eſt æqualis nihilo. Ergò neceſſe eſt ut una ſaltèm ex duabus quantitatibus binomiis $x - l$ et $xx + mm$, ex quarum multiplicatione quantitas quadrinomia $x^3 - lxx + mmx - lmm$ orietur, ſit etiàm æqualis nihilo. Sed quantitas binomia $xx + mm$, (quæ eſt ſumma duarum quantitatum xx et mm,) non poteſt æqualis eſſe nihilo; quantitas verò $x - l$ (quæ eſt differentia duarum quantitatum x et l,) poteſt eſſe æqualis nihilo, nempe, ſi quantitas x fuerit æqualis quantitati l. Ergò quantitas binomia $x - l$ erit æqualis nihilo, et pro-inde quantitas x erit æqualis quantitati l, hoc eſt, quantitas l erit radix æquationis $x^3 - lxx + mmx - lmm = o$, et hæc æquatio, ſeu æquatio propoſita $x^3 - lxx + mmx - n^3 = o$ in hoc caſu (in quo l eſt $= \frac{n^3}{mm}$, ſeu lmm eſt $= n^3$,) nullam aliam radicem habebit præter ipſam l. Q. E. D.

F. M.

erit quoque $\frac{n^3}{l}$ majus quàm xx; et, si xx minus est quàm mm, minus quoque erit $\frac{n^3}{l}$ quàm xx. Inventi itaque sunt duo limites m et $\sqrt{\frac{n^3}{l}}$, quorum cuilibet æquatur radix æquationis propositæ, si fuerint æquales, hoc est, si lmm æquatur ipsi n^3; aut necessariò inter duos erit, si inæquales fuerint. Eadem est ratio duorum reliquorum limitum n et $\frac{mm}{l}$ *.

PROP. 3. $x^3 - lxx - mmx - n^3 = 0$.

Per transpositionem erit $x^3 - lxx = mmx + n^3$, ideóque x major quàm l. Rursùs, cum, per transpositionem, sit $x^3 - mmx = lxx + n^3$, erit xx majus quàm mm, et x major quàm m, et mxx majus quàm mmx. Sed, per transpositionem, est quoque $x^3 - n^3 = lxx + mmx$, et per consequens x major quàm n, et nxx majus quàm n^3. Quìn et, per transpositionem propositæ, habetur $lxx + mmx + n^3 = x^3$, atque inventum est mxx majus quàm mmx, et nxx majus quàm n^3. Ergò erit $lxx + mxx + nxx$ majus quàm x^3. Quocircà, si utraque pars divi-

* Nam in æquatione propositâ $x^3 + lxx - mmx - n^3 = o$ (quæ unam tantùm radicem habet,) si radix x sit æqualis quantitati n, quantitas x^3 erit æqualis n^3, et quantitas binomia $x^3 - n^3$ erit æqualis nihilo, et pro-inde quantitas quadrinomia $x^3 + lxx - mmx - n^3$ erit æqualis quantitati binomiæ $lxx - mmx$. Est verò quantitas quadrinomia $x^3 + lxx - mmx - n^3$ æqualis nihilo. Ergò in hoc casu quantitas binomia $lxx - mmx$ erit etiàm æqualis nihilo, et lxx erit $= mmx$, et lx erit $= mm$, et x erit $= \frac{mm}{l}$. Ergò, si x sit æqualis n, erit etiàm æqualis $\frac{mm}{l}$, et pro-inde in hoc casu $\frac{mm}{l}$ erit $= n$, et mm erit $= ln$. Et, è converso, si $\frac{mm}{l}$ sit æqualis n, seu mm sit æqualis ln, radix x æquationis propositæ $x^3 + lxx - mmx - n^3 = o$ erit æqualis quantitati n, seu quantitati $\frac{mm}{l}$.

Secundò, si radix x sit major quàm n, et pro-inde x^3 major quàm n^3, addatur utrinque æquationi propositæ $x^4 + lxx - mmx - n^3 = o$ quantitas mmx; et erit quantitas trinomia $x^3 + lxx - n^3 = mmx$. Sed, quoniam x^3 major est quam n^3, erit quantitas trinomia $x^3 + lxx - n^3$ major quàm lxx, et pro-inde quantitas lxx potest auferri ab ipsâ, et à quantitate mmx, quæ eidem est æqualis. Auferatur itaque: et erit quantitas binomia $x^3 - n^3 =$ quantitati binomiæ $mmx - lxx$. Ergò lx erit minor quàm mm, et x erit minor quàm $\frac{mm}{l}$. In hoc igitùr casu, seu cum x est major quàm n, erit minor quàm $\frac{mm}{l}$, et pro-inde n et $\frac{mm}{l}$ erunt limites magnitudinis radicis x.

Tertiò, si radix x sit minor quàm n, et pro-inde x^3 sit minor quàm n^3, addatur utrinque æquationi propositæ $x^3 + lxx - mmx - n^3 = o$ quantitas binomia $n^3 - x^3$; et erit $lxx - mmx = n^3 - x^3$. Ergò mmx erit minor quàm lxx, et mm quàm lx, et $\frac{mm}{l}$ quàm x. In hoc igitùr casu, seu cum x est minor quàm n, erit major quàm $\frac{mm}{l}$, et pro-inde $\frac{mm}{l}$ et n erunt limites magnitudinis radicis x.

Q. E. D.

F. M.

datur

datur per xx, erit $l + m + n$ major quàm x. Inventa est itaque radix x æquationis propositæ esse major quàm l, m, et n, sed minor quàm $l + m + n$.

PROP. 4. $x^3 + lxx + mmx - n^3 = 0$.

Per transpositionem erit $x^3 + mmx = n^3 - lxx$, ideóque $\frac{n^3}{l}$ majus quàm xx. Sed est quoque $x^3 + lxx = n^3 - mmx$, ideóque $\frac{n^3}{mm}$ major quàm x. At verò est etiàm $lxx + mmx = n^3 - x^3$, et consequentèr n major quàm x; quarè et nnx majus erit quàm x^3, et lnx majus quàm lxx. Atqui est $x^3 + lxx + mmx = n^3$. Ergò $nnx + lnx + mmx$ majus erit quàm n^3, et x major quàm $\frac{n^3}{nn+ln+mm}$. Quarè inventa est radix x æquationis propositæ esse major quàm $\frac{n^3}{mm+ln+mm}$, at minor quàm $\sqrt{\frac{n^3}{l}}$, $\frac{n^3}{mm}$, et n.

PROP. 5. $x^3 - lxx + mmx + n^3 = 0$.

Per transpositionem erit $mmx + n^3 = lxx - x^3$, ideóque l major quàm x. Rursùs erit $x^3 + n^3 = lxx - mmx$, et per consequens x major quàm $\frac{mm}{l}$. Invenimus ergò, quamlibet duarum radicum æquationis propositæ necessariò majorem esse quàm $\frac{mm}{l}$, et minorem quàm l. Sed, per transpositionem, est quoque $x^3 + mmx = lxx - n^3$, et consequentèr xx majus est quàm $\frac{n^3}{l}$. Quarè et x major erit quàm $\sqrt{\frac{n^3}{l}}$.

PROP. 6. $x^3 + lxx - mmx + n^3 = 0$.

Per transpositionem erit $x^3 + lxx = mmx - n^3$, ideóque x major quàm $\frac{n^3}{mm}$. Similitèr erit $x^3 + n^3 = mmx - lxx$, et per consequens $\frac{mm}{l}$ major quàm x. Rursùs erit $lxx + n^3 = mmx - x^3$, et consequentèr m major quàm x. Invenimus ergò, quamlibet duarum radicum æquationis propositæ necessariò majorem esse quàm $\frac{n^3}{mm}$, sed minorem quàm $\frac{mm}{l}$ et m.

PROP. 7. $x^3 - lxx - mmx + n^3 = 0$.

Per transpositionem erit $x^3 - lxx = mmx - n^3$. Unde patet, si x æqualis est ipsi l, quòd tunc quoque x ipsi $\frac{n^3}{mm}$ est æqualis. Ideóque, si l æquatur ipsi

$\frac{n^3}{nm}$, hoc eſt, ſi lmm ſit $= n^3$, una radicum æquationis propoſitæ æquabitur ſingulis terminorum l et $\frac{n^3}{mm}$; et ſi [l et $\frac{n^3}{mm}$, ſeu lmm et n^3,] inæquales fuerint, neutra ex duabus radicibus æquationis propoſitæ poterit eſſe inter hos terminos. Quia videmus, cùm x major eſt quàm l, tum quoque x majorem eſſe quàm $\frac{n^3}{mm}$; et ſi minor eſt quàm l, tum ſimilitèr x minorem eſſe quàm $\frac{n^3}{mm}$.

Sed, per tranſpoſitionem, eſt etiàm $x^3 - mmx = lxx - n^3$. Hinc, ſi xx æquetur ipſi mm, erit quoque $xx = \frac{n^3}{l}$. Ideóque, ſi fuerint hi termini mm et $\frac{n^3}{l}$ æquales, hoc eſt, $lmm = n^3$, una radicum æquationis propoſitæ major erit unoquoque terminorum æqualium m et $\surd\frac{n^3}{l}$; et, ſi inæquales fuerint, neutra duarum radicum æquationis propoſitæ erit inter duos ex his terminis.

Prætereà, per tranſpoſitionem, eſt quoque $x^3 + n^3 = lxx + mmx$, ideóque $lxx + mmx$ majus quàm x^3, et $lx + mm$ majus quàm xx. At x erit realis, et vel æqualis, vel major, vel minor quàm m, ſi æquatio propoſita fuerit realis. Et, ſi æqualis fuerit vel major quàm m, erit et $lx + mx$ majus quàm xx, ac per conſequens $l + m$ major quàm x. Quòd ſi autem x minor fuerit quàm m, multo magìs $l + m$ major erit quàm x. Porrò ex hâc eâdem æquatione conſtat, quòd $lxx + mmx$ etiàm majus eſt quàm n^3. Hinc cum $l + m$ major ſit quàm x, ideóque $llx + lmx$ majus quàm lxx; erit quoque $llx + lmx + mmx$ majus quàm n^3, et x major quàm $\frac{n^3}{ll + lm + mm}$. Invenimus igitùr, quòd quælibet radicum æquationis propoſitæ major eſt quàm $\frac{n^3}{ll + lm + mm}$, et multo major quàm $\frac{n^3}{ll + 2lm + mm}$, at minor quàm $l + m$. Denique, quoniam $l + m$ major eſt quàm x, ſi major fuerit x quàm m, erit inter hóſce terminos $l + m$ et m. Quòd ſi verò m major eſt quàm x, invenimus, quòd $lxx + mmx$ eſt majus quàm n^3; hinc $lmx + mmx$ multo magìs erit majus quàm n^3; adeóque x major quàm $\frac{n^3}{lm + mm}$, et conſequentèr x major quàm minor horum duorum terminorum m et $\frac{n^3}{lm + mm}$ *.

* Hæc ſeptima propoſitio mihi videtur minùs perſpicuè explicari ab auctore quàm propoſitiones quæ illam antecedunt. Conabor itaque eam paulo fuſiùs exponere, ut concluſiones auctoris de limitibus radicum æquationis propoſitæ $x^3 - lxx - mmx + n^3 = 0$ à lectoribus plenè intelligantur, et rectè deduci percipiantur.

Hæc æquatio $x^3 - lxx - mmx + n^3 = 0$ non eſt ſempèr poſſibilis, quæcunque ſit magnitudo termini noti, ſeu abſoluti, n^3, ſed tantùm quandò iſte terminus non excedit certam quandam finitam magnitudinem;—eam, ſcilicet, quæ eſt maxima magnitudo ad quam quantitas trinomia $lxx + mmx$

$mmx - x^3$ creſcere poteſt dum x creſcit à nihilo ad magnitudinem quam habet quando x^3 fit $= lxx + mmx$, et pro-inde xx fit $= lx + mm$, et $xx - lx$ fit $= mm$. Et, ſi terminus n^3 ſit æqualis iſti certæ finitæ magnitudini, æquatio propoſita $x^3 - lxx - mmx + n^3 = 0$, ſeu æquatio inde derivanda $mmx + lxx - x^3 = n^3$, habebit tantùm unam radicem, quæ erit magnitudo quam habet x quandò quantitas trinomia $mmx + lxx - x^3$ eſt maxima quæ eſſe poſſit; quam magnitudinem quantitatis x vocemus M: et, ſi terminus n^3 ſit minor iſtâ certâ finitâ magnitudine (quæ erit $= mm\,\mathrm{M} + l\mathrm{MM} - \mathrm{M}^3$,) æquatio propoſita $x^3 - lxx + mmx + n^3 = 0$, ſeu æquatio $mmx + lxx - x^3 = n^3$, habebit duas radices, quarum minor erit minor quantitate M, major autèm erit iſtâ quantitate major.

His poſitis, (quæ ab auctore ſubintelligi videntur) conferenda eſt radix x primùm, quantitati notæ l, ſeu co-efficienti quantitatis xx in ſecundo termino, lxx, æquationis propoſitæ $x^3 - lxx - mmx + n^3 = 0$; et, deinde, quantitati m, ſeu radici quadratæ quantitatis notæ mm, quæ eſt co-efficiens quantitatis x in tertio termino, mmx, æquationis propoſitæ; et, tertiò, quantitati n, ſeu radici cubicæ quantitatis notæ n^3, quæ eſt quartus terminus æquationis propoſitæ; ut ex his collationibus derivare poſſimus quaſdam notas quantitates quæ ſint limites duarum radicum propoſitæ æquationis.

Imprimis ergò ponamus unam ex duabus radicibus x eſſe æqualem quantitati l. Et erit $x^3 - lxx = 0$, et pro-inde quantitas quadrinomia $x^3 - lxx - mmx + n^3$ erit $= -mmx + n^3$. Sed quantitas quadrinomia $x^3 - lxx - mmx + n^3$ eſt $= 0$. Ergò quantitas binomia $-mmx + n^3$, ſeu $n^3 - mmx$, erit etiàm $= 0$, et mmx erit $= n^3$, et x erit $= \frac{n^3}{m\,m}$; hoc eſt, ſi unus ex duobus valoribus quantitatis x ſit æqualis quantitati l, erit etiàm æqualis quantitati $\frac{n^3}{m\,m}$, et pro-inde l erit $= \frac{n^3}{m\,m}$. Unde ſequitur, è converſo, quòd, ſi l fuerit $= \frac{n^3}{m\,m}$, ſeu lmm fuerit $= n^3$, una ex radicibus æquationis propoſitæ $x^3 - lxx - mmx + n^3 = 0$ erit æqualis quantitati l, ſeu quantitati $\frac{n^3}{m\,m}$. Hæc eſt concluſio prima auctoris, in tribus primis lineis textûs ejus declarata.

Oſtenſum eſt quòd, ſi una ex radicibus x ſit $= l$, quantitas binomia $x^3 - lxx$ erit $= 0$, et quantitas binomia $mmx - n^3$ erit paritèr $= 0$. Ergò, ſi eadem x ſit major quàm l, quantitas x^3 erit major quàm lxx, et pro-inde quantitas mmx erit major quàm n^3; aliter enim quantitas quadrinomia $x^3 - lxx - mmx + n^3$ non eſſet $=$ nihilo, ut in æquatione propoſitâ $x^3 - lxx - mmx + n^3 = 0$ ſupponitur. Ergò in hoc caſu quantitas x, quæ major eſt quàm l, erit etiàm major quàm $\frac{n^3}{m\,m}$. Et, ſi eadem radix x ſit minor quàm l, et pro-inde x^3 eſt minor quàm lxx, addatur utrinque æquationi propoſitæ $x^3 - lxx - mmx + n^3 = 0$ quantitas binomia $lxx - x^3$; et erit $-mmx + n^3$, ſeu $n^3 - mmx = lxx - x^3$; et pro-inde mmx erit minor quàm n^3, et x erit minor quàm $\frac{n^3}{m\,m}$; hoc eſt, quandò una ex duabus radicibus x eſt minor quàm l, erit etiàm minor quàm $\frac{n^3}{m\,m}$. Et hæc eſt concluſio ſecunda auctoris, in lineis $4^{\text{tâ}}$, $5^{\text{tâ}}$, $6^{\text{tâ}}$, et $7^{\text{timâ}}$ textûs ejus declarata.

Secundo loco, conferendæ inter ſe ſunt una ex duabus radicibus x æquationis propoſitæ $x^3 - lxx - mmx + n^3 = 0$ et quantitas m, ſeu radix quadrata co-efficientis mm. Hoc verò fieri poteſt modo ſequente.

Ponamus hanc radicem x eſſe $= m$. Et habebimus $x^3 = mmx$, et $x^3 - mmx = 0$, et $x^3 - lxx - mmx + n^3 = -lxx + n^3$, ſeu $n^3 - lxx$. Sed $x^3 - lxx - mmx + n^3$ eſt $= 0$. Ergò $n^3 - lxx$ erit etiàm $= 0$; et pro-inde lxx erit $= n^3$, et xx erit $= \frac{n^3}{l}$, et x erit $= \sqrt{\frac{n^3}{l}}$. Ergò, ſi x eſt $= m$, erit etiàm æqualis $\sqrt{\frac{n^3}{l}}$, et pro-inde $\sqrt{\frac{n^3}{l}}$ erit $= m$, et $\frac{n^3}{l}$ erit $= mm$. Unde ſequitur, è converſo, quòd, ſi mm ſit æquale $\frac{n^3}{l}$, ſeu lmm ſit æquale n^3, una ex duabus radicibus x erit $= m$.

Ponamus jam x eſſe majorem quàm m. Et erit x^3 major quàm mmx, et $x^3 - mmx + n^3$ erit

major

major quàm n^3. Addatur jam utrinque æquationi propoſitæ $x^3 - lxx - mmx + n^3 = 0$ quantitas lxx; et habebimus $x^3 - mmx + n^3 = lxx$. Auferatur jam utrinque quantitas n^3, quæ oſtenſa eſt eſſe minor quàm $x^3 - mmx + n^3$. Et erit $x^3 - mmx = lxx - n^3$. Ergò lxx eſt major quàm n^3, et pro-inde xx major quàm $\frac{n^3}{l}$, et x major quàm $\sqrt{\frac{n^3}{l}}$. Ergò, quandò x eſt major quàm m, erit etiàm major quàm $\sqrt{\frac{n^3}{l}}$.

Sit jam x minor quàm m. Et erit x^3 minor quàm mmx. Addatur jam utrinque æquationi propoſitæ $x^3 - lxx - mmx + n^3 = 0$ quantitas binomia $mmx - x^3$. Et erit $-lxx + n^3$, ſeu $n^3 - lxx = mmx - x^3$. Ergò lxx eſt minor quàm n^3, et xx minor quàm $\frac{n^3}{l}$, et x minor quàm $\sqrt{\frac{n^3}{l}}$. Ergò, quandò x eſt minor quàm m, erit etiàm minor quàm $\sqrt{\frac{n^3}{l}}$.

Ergò, ſi mm non ſit $= \frac{n^3}{l}$, ſeu ſi lmm non ſit $= n^3$, manifeſtum eſt radicem x non jacere inter quantitates m et $\sqrt{\frac{n^3}{l}}$, ſed, ſi major eſt quàm m, majorem eſſe etiàm quàm $\sqrt{\frac{n^3}{l}}$, et, ſi minor eſt quàm m, minorem eſſe etiàm quàm $\sqrt{\frac{n^3}{l}}$. Et hæ concluſiones veræ ſunt, ſive x ſit major ex duabus radicibus æquationis propoſitæ $x^3 - lxx - mmx + n^3 = 0$, ſive ſit minor ex iiſdem radicibus. Et hoc, ut opinor, eſt id quod intelligi volebat auctor his verbis in lineis $11^{mâ}$ et $12^{mâ}$ hujus $7^{timæ}$ Propoſitionis, ſcilicet, "*Et, ſi inæquales fuerint* [*quantitates m et* $\sqrt{\frac{n^3}{l}}$], *neutra radicum duarum æquationis propoſitæ erit inter duos ex his terminis,*" quæ tamen adhuc mihi aliquantulùm obſcura videntur, niſi forte aliquis præli error in ea ſit illapſus.

Tertio loco, conferendæ inter ſe ſunt una ex duabus radicibus æquationis propoſitæ $x^3 - lxx - mmx + n^3 = 0$ et quantitas n, ſeu radix cubica quantitatis notæ n^3. Hoc autèm fieri poteſt modo ſequente.

Addatur utrinque æquationi propoſitæ $x^3 - lxx - mmx + n^3 = 0$ quantitas binomia $lxx + mmx$. Et erit $x^3 + n^3 = lxx + mmx$. Erit igitùr quantitas binomia $lxx + mmx$ major quàm x^3, et pro-inde quantitas $lx + mm$ major quàm xx. Sed x erit neceſſariò aut æqualis quantitati m, aut illâ major, aut illâ minor. Si fuerit æqualis ipſi m, vel ipſâ major, erit $\overline{l+m} \times x$, ſeu $lx + mx$, major quàm $x \times x$, ſeu xx; et pro-inde $l+m$ erit major quàm x.

Porrò, ex eâdem æquatione $x^3 + n^3 = lxx + mmx$ conſtat quòd $lxx + mmx$ eſt major quàm n^3. Sed $l+m$ eſt major quàm x, ideóque $\overline{l+m} \times lx$, ſeu $llx + lmx$, eſt major quàm $x \times lx$, ſeu lxx. Ergò $llx + lmx + mmx$ erit major quàm $lxx + mmx$, et pro-inde major quàm n^3. Ergò in hoc caſu, ſeu ſi x fuerit æqualis ipſi m, vel ipſâ major, x erit major quàm fractio $\frac{n^3}{ll + lm + mm}$.

Sed ante oſtenſum eſt quòd $l+m$ erat major quàm x. Ergò radix x æquationis propoſitæ $x^3 - lxx - mmx + n^3 = 0$ erit major quàm fractio $\frac{n^3}{ll + lm + mm}$, ſed minor quàm $l+m$. Et hoc erit verum de utrâque radice æquationis propoſitæ $x^3 - lxx - mmx + n^3 = 0$ quæ fuerit aut æqualis ipſi m, aut ipſâ major.

Denique, ſi x ſit major quàm m, quantitas x habebit alios quoque limites, nempe, $l+m$ et m.

Quod ſi verò m ſit major quàm x, quantitas lmx erit major quàm lxx, et pro-inde $lmx + mmx$ erit major quàm $lxx + mmx$. Eſt verò $lxx + mmx$ major quàm n^3. Ergò, quandò m eſt major quàm x, quantitas $lmx + mmx$ erit, à fortiori, major quàm n^3; et pro-inde x erit major quàm fractio $\frac{n^3}{lm + mm}$.

Hi ſunt limites duarum radicum x æquationis propoſitæ $x^3 - lxx - mmx + n^3 = 0$ ab auctore inventi in hâc ſeptimâ propoſitione, ſi ejus mentem rectè ſim aſſecutus: quod tamen me omninò feciſſe vix auſim affirmare. F. M.

CAPUT

CAPUT V.

De Æquationibus quatuor dimensionum, secundo et tertio termino carentibus.

PROP. 1. $x^4 ** - n^3x + p^4 = 0$.

Per transpositionem est $x^4 = n^3x - p^4$, ideóque x major quàm $\frac{p^4}{n^3}$. Sed, per transpositionem, est quoque $p^4 = n^3x - x^4$, et consequentèr n^3 major quàm x^3, ac n major quàm x. Ergò utraque radix x æquationis propositæ major erit quàm $\frac{p^4}{n^3}$, at minor quàm n.

PROP. 2. $x^4 ** - n^3x - p^4 = 0$.

Per transpositionem est $x^4 - n^3x = p^4$, ideóque x^3 major quàm n^3, et x major quàm n, et $nnxx$ major quàm n^3x. Sed est quoque $x^4 - p^4 = n^3x$, ideóque x^4 major quàm p^4, et x major quàm p, et $ppxx$ major quàm p^4. At, per transpositionem, est etiàm $n^3x + p^4 = x^4$. Ergò $nnxx + ppxx$ major erit quàm x^4. Hinc, divisâ utrâque parte per xx, erit xx minor quàm $nn + pp$, et x minor quàm $\sqrt{nn + pp}$. Invenimus igitùr, quòd radix æquationis propositæ est major quàm n et p, sed minor quàm $\sqrt{nn + pp}$, ac pro-inde multo minor quàm $n + p$.

PROP. 3. $x^4 ** + n^3x - p^4 = 0$.

Per transpositionem erit $x^4 = p^4 - n^3x$, ac per consequens $\frac{p^4}{n^3}$ major quàm x. Similitèr erit $n^3x = p^4 - x^4$, et consequentèr p^4 major quàm x^4, et p major quàm x, et $p^3 x$ major quàm x^4. Sed est prætereà $x^4 + n^3x = p^4$, ideóque $p^3x + n^3x$ major quàm p^4, et x major quàm $\frac{p^4}{p^3+n^3}$. Invenimus itaque, quòd radix æquationis propositæ est major quàm $\frac{p^4}{p^3+n^3}$, at minor quàm $\frac{p^4}{n^3}$ et p.

CAPUT

CAPUT VI.

De Æquationibus quatuor dimenſionum, in quibus tertius et quartus terminus deficiunt.

PROP. 1. $x^4 - lx^3 ** + p^4 = 0$.

Per tranſpoſitionem eſt $x^4 = + lx^3 - p^4$, ideóque x^3 major quàm $\frac{p^4}{l}$. At verò eſt etiàm $p^4 = lx^3 - x^4$, et conſequentèr l major quàm x. Invenimus igitùr, quòd unaquæque duarum radicum æquationis propoſitæ eſt major quàm $\surd C.\frac{p^4}{l}$, at minor quàm l. Hinc, quoniam l major eſt quàm x, et lx^3 major quàm p^4, habebitur $llxx$ major quàm p^4, et conſequentèr xx major quàm $\frac{p^4}{ll}$, et x major quàm $\frac{pp}{l}$.

PROP. 2. $x^4 - lx^3 ** - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - lx^3 = p^4$, ideóque x major quàm l. Similitèr eſt $x^4 - p^4 = lx^3$, et conſequentèr x^4 major quàm p^4, et x major quàm p, ac proinde px^3 major quàm p^4. Sed eſt etiàm $lx^3 + p^4 = x^4$. Ergò $lx^3 + px^3$ major erit quàm x^4, et $l + p$ major quàm x. Invenimus igitùr, quòd radix x æquationis propoſitæ major eſt quàm l et p, at minor quàm $l + p$.

PROP. 3. $x^4 + lx^3 ** - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 = p^4 - lx^3$, ideóque $\frac{p^4}{l}$ major quàm x^3. Similitèr eſt $lx^3 = p^4 - x^4$, ac per conſequens p major quàm x, et px^3 major quàm x^4. Atqui eſt etiàm $x^4 + lx^3 = p^4$. Ergò $lx^3 + px^3$ major erit quàm p^4, et x^3 major quàm $\frac{p^4}{l+p}$. Quarè invenimus, quòd radix x æquationis propoſitæ major

eſt

eſt quàm $\surd C.\frac{p^4}{l+p}$, ſed minor quàm p et $\surd C.\frac{p^4}{l}$. Facillimè verò evitantur extractiones radicum cubicarum, ſumendo terminos paulò majores aut minores, proùt neceſſitas requirit. Atque in hoc caſu, quoniam x^3 major eſt quàm $\frac{p^4}{l+p}$, et p major quàm x, erit pxx major quàm $\frac{p^4}{l+p}$, et xx major quàm $\frac{p^4}{lp+pp}$, et x major quàm $\surd\frac{p^4}{lp+pp}$. Prætereà, quandoquidem $\frac{p^4}{l}$ major eſt quàm x^3, erit $\frac{p^4x}{l}$ major quàm x^4, et $lppx$ major quàm lx^3, quia p major eſt quàm x. Atqui eſt $x^4 + lx^3 = p^4$. Ergò $\frac{p^4x}{l} + lppx$ major erit quàm p^4. Hinc, multiplicatâ utrâque parte per l, et divisâ per pp, habebitur $ppx + llx$ major quam lpp, et x major quàm $\frac{lpp}{ll+pp}$.

De æquationibus quatuor dimenſionum, in quibus ſecundus et quartus terminus deficiunt, nihil addimus: ſiquidem illæ ad quadratas referuntur, ita ut ipſarum limites eodem modo quo quadratarum inveniri poſſint.

CAPUT VII.

De Æquationibus quatuor dimenſionum, ſecundo termino carentibus.

PROP. I. $x^4 * - mmxx + n^3x - p^4 = 0$.

Per tranſpoſitionem erit $x^4 - mmxx = p^4 - n^3x$. Unde apparet, quòd, ſi fuerit xx æquale ipſi mm, hoc eſt, $x = m$, etiàm x ipſi $\frac{p^4}{n^3}$ ſit æqualis futura. Ideóque, ſi fuerit m æqualis ipſi $\frac{p^4}{n^3}$, hoc eſt, $mn^3 = p^4$, radix x æquationis propoſitæ æquabitur ſingulis terminorum m et $\frac{p^4}{n^3}$; et, ſi inæquales fuerint, una-

quæque radicum æquationis propoſitæ, ſive unam ſive tres habuerit, ſempèr erit inter duos hóſce terminos. Prætereà cognoſcitur, ſi duo hi termini fuerint æquales, hoc eſt, $mn^3 = p^4$, ſubſtituto mn^3 loco p^4 in æquatione propoſitâ, [quæ tunc fiet x^4 * $- mmxx + n^3x - mn^3 = o$,] eâque divisâ per $x - m$,* fore, ut non poſſit habere aliam radicem realem præter m.

PROP. 2. x^4 * $+ mmxx - n^3x - p^4 = 0$.

Per tranſpoſitionem habebimus $x^4 - n^3x = p^4 - mmxx$. Unde conſtat, ſi x æqualis fuerit ipſi n, fore etiàm $xx = \frac{p^4}{mm}$; hoc eſt, $x = \sqrt{\frac{p^4}{mm}}$ aut $\frac{pp}{m}$; et, ſi fuerit $n = \sqrt{\frac{p^4}{mm}}$ aut $\frac{pp}{m}$, tunc radicem æquationis fore æqualem cuilibet horum terminorum; et, ſi inæquales fuerint, tunc eam neceſſariò futuram inter hóſce duos. Idem demonſtrabitur de duobus reliquis p et $\frac{n^3}{mm}$; nempe, ſi fuerint æquales, radix æquationis propoſitæ æquabitur unicuique illorum duorum;

* Hæc Diviſio perfici poteſt modo ſequente.

$$
\begin{array}{l}
x - m \;)\; x^4 \quad * \quad - mmxx + n^3x - mn^3 \;(\; x^3 + mx^2 + n^3 \\
\qquad\quad x^4 - mx^3 \\
\qquad\quad \overline{\qquad\qquad} \\
\qquad\quad * \quad + mx^3 - mmxx \\
\qquad\quad + mx^3 - mmxx \\
\qquad\quad \overline{\qquad\qquad\qquad} \\
\qquad\qquad\quad * \qquad * \quad + n^3x - mn^3 \\
\qquad\qquad\qquad\quad + n^3x - mn^3 \\
\qquad\qquad\qquad\quad \overline{\qquad\qquad} \\
\qquad\qquad\qquad\qquad * \qquad *
\end{array}
$$

Patet ergò quòd quantitas quadrinomia $x^4 - mmxx + n3x - mn^3$ dividi poteſt accuratè, ſive ſine ullo reſiduo, per quantitatem binomiam $x - m$, et quòd quotiens inde orta erit quantitas trinomia $x^3 + mx^2 + n^3$. Sequitur igitùr è converſo, quòd quantitas iſta quadrinomia poteſt generari ex multiplicatione quantitatis binomiæ $x - m$ in quantitatem trinomiam $x^3 + mx^2 + n^3$. Sed quantitas quadrinomia x^4 * $- mmxx + n^3x - mn^3$ eſt æqualis nihilo. Ergò neceſſe eſt ut una ſaltèm ex duabus quantitatibus $x - m$ et $x^3 + mx^2 + n^3$, (ex quarum multiplicatione quantitas quadrinomia x^4 * $- mmxx + n^3x - mn^3$ orietur,) ſit etiàm æqualis nihilo. Sed quantitas trinomia $x^3 + mx^2 + n^3$ (quæ eſt ſumma trium quantitatum x^3, mx^2, et n^3,) non poteſt eſſe æqualis nihilo; quantitas verò $x - m$ (quæ eſt differentia duarum quantitatum x et m,) poteſt eſſe æqualis nihilo, nempe, ſi quantitas ablata m fuerit æqualis quantitati x, à quâ aufertur. Ergò quantitas binomia $x - m$ erit æqualis nihilo, et pro-inde quantitas x erit æqualis quantitati m; hoc eſt, quantitas m erit radix æquationis x^4 * $- mmxx + n^3x - mn^3 = 0$, et hæc æquatio, ſeu æquatio propoſita x^4 * $- mmxx + n^3x - p^4 = o$, in hoc caſu, in quo mn^3 eſt $= p^4$, nullam aliam radicem habebit præter ipſam m. Q. E. D.

F. M.

ſin inæquales, neceſſariò conſtituetur inter duos, tranſpoſitâ, ſcilicet, æquatione in hunc modum, $x^4 - p^4 = n^3x - mmxx$ *.

PROP. 3. x^4 * $- mmxx - n^3x - p^4 = 0$.

Per tranſpoſitionem erit $x^4 - mmxx = n^3x + p^4$, ideóque xx major quàm mm, et x major quàm m, et mx^3 major quàm $mmxx$. Sed eſt etiàm $x^4 - n^3x = mmxx + p^4$, ideóque x^3 major quàm n^3, et x major quàm n, et nx^3 major quàm n^3x. Eodem modo eſt $x^4 - p^4 = mmxx + n^3x$, et conſequentèr x^4 major quàm p^4, et x major quàm p, et px^3 major quàm p^4. Atqui, per tranſpoſitionem, eſt quoque $mmxx + n^3x + p^4 = x^4$. Quarè $mx^3 + nx^3 + px^3$ major erit quàm x^4, et $m + n + p$ major quàm x. Conſimili ratione demonſtrabitur, quòd $mmxx + nnxx + ppxx$

* Quoniam enim $x^4 - p^4$ eſt $= n^3x - mmxx$, ſequitur quòd, ſi x^4 ſit æqualis p^4, et pro-inde quantitas binomia $x^4 - p^4$ ſit æqualis nihilo, quantitas binomia $n^3x - mmxx$ erit etiàm æqualis nihilo, et pro-inde n^3x erit $mmxx$, et $n^3 = mmx$, et $x = \frac{n^3}{mm}$; hoc eſt, ſi x ſit $= p$, erit etiàm $= \frac{n^3}{mm}$, ideóque $\frac{n^3}{mm}$ erit $= p$. Ergò, è converſo, ſi $\frac{n^3}{mm}$ ſit $= p$, vel n^3 ſit $= mmp$, radix x æquationis propoſitæ $x^4 + mmxx - n^3x - p^4 = 0$ erit $= p$, vel $\frac{n^3}{mm}$.

Porrò, quoniam $x^4 - p^4$ eſt $= n^3x - mmxx$, ſequitur quòd, ſi x^4 ſit major quàm p^4, erit n^3x major quàm $mmxx$, ideóque, ſi x ſit major quàm p, erit $\frac{n^3}{mm}$ major quàm x; hoc eſt, radix x erit mediæ cujuſdam magnitudinis inter p et $\frac{n^3}{mm}$, ſeu p et $\frac{n^3}{mm}$ erunt ejuſdem limites, ut ab auctore affirmatur.

Denique, ſi x fuerit minor quàm p, et pro-inde x^4 ſit minor quàm p^4, quantitates binomiæ $x^4 - p^4$ et $n^3x - mmxx$ fient impoſſibiles, ſed earum loco habebimus quantitates binomias, $p^4 - x^4$ et $mmxx - n^3x$, quæ erunt inter ſe æquales. Nam, ſi in hoc caſu addamus utrinque æquationi propoſitæ $x^4 + mmxx - n^3x - p^4 = 0$ quantitatem binomiam $p^4 - x^4$, habebimus æquationem $mmxx - n^3x = p^4 - x^4$, in quâ n^3x eſt minor quàm $mmxx$, et x^4 eſt minor quàm p^4; et pro-inde $\frac{n^3}{mm}$ erit minor quàm x, et x erit minor quàm p; hoc eſt, radix x eſt mediæ cujuſdam magnitudinis inter $\frac{n^3}{mm}$ et p, ſeu $\frac{n^3}{mm}$ et p erunt ejuſdem limites, ut ab auctore affirmatur.

Q. E. D.

F. M.

major erit quàm x^4, * et consequentèr $mm + nn + pp$ major quàm xx. Invenimus ergò, quòd radix x propositæ æquationis major est quàm m, n, et p, at minor quàm $m + n + p$, et $\surd \overline{mm + nn + pp}$.

PROP. 4. x^4 * $+ mmxx + n^3x - p^4 = 0$.

Per transpositionem est $x^4 + mmxx = p^4 - n^3x$, ideóque $\frac{p^4}{n^3}$ major quàm x. Similitèr est $x^4 + n^3x = p^4 - mmxx$, ac per consequens $\frac{p^4}{mm}$ major quàm xx, hoc est, $\frac{pp}{m}$ major quàm x. Atqui est quoque $mmxx + n^3x = p^4 - x^4$, et consequentèr p^4 major quàm x^4, ac p major quàm x, et $p^3 x$ major quàm x^4, nec non $mmpx$ major quàm $mmxx$. Sed est etiàm $x^4 + mmxx + n^3x = p^4$. Quarè $p^3x + mmpx + n^3x$ major est quàm p^4, ac per consequens, divisâ utrâque parte per $p^3 + mmp + n^3$, erit radix x propositæ æquationis major quàm $\frac{p^4}{p^3 + mmp + n^3}$, at minor quàm $\frac{p^4}{x^3}$, $\frac{pp}{m}$, et p.

PROP. 5. x^4 * $- mmxx + n^3x + p^4 = 0$.

Per transpositionem est $n^3x + p^4 = mmxx - x^4$, ideóque mm major quàm xx, et m major quàm x. Similitèr est $x^4 + p^4 = mmxx - n^3x$, et consequentĕr x major quàm $\frac{n^3}{mm}$. Prætereà est $x^4 + n^3x = mmxx - p^4$, ac per consequens xx major quàm $\frac{p^4}{mm}$, hoc est, x major quàm $\frac{pp}{m}$. Invenimus ergò quamlibet radicum æquationis propositæ majorem esse quàm $\frac{n^3}{mm}$ et $\frac{pp}{m}$, at minorem quàm m.

* Quoniam enim x est major quàm n, erit $nnxx$ major quàm n^3x; et, quoniam x est major quàm p, erit $ppxx$ major quàm p^4. Ergò quantitas trinomia $mmxx + nnxx + ppxx$ erit major quantitate trinomiâ $mmxx + n^3x + p^4$, atque ideò major quàm x^4. Q. E. D.

F. M.

PROP.

PROP. 6. $x^4 * + mmxx - n^3x + p^4 = 0$.

Per tranſpoſitionem eſt $x^4 + mmxx = n^3x - p^4$, ideóque x major quàm $\frac{p^4}{n^3}$. Sed eſt etiàm $x^4 + p^4 = n^3x - mmxx$, ac per conſequens $\frac{n^3}{mm}$ major quàm x. Similitèr eſt $x^4 + mmxx + p^4 = n^3x$, ideóque n^3 major quàm x^3, et n major quàm x. Invenimus ergò quamlibet duarum radicum æquationis propoſitæ majorem eſſe quàm $\frac{p^4}{n^3}$, at minorem quàm $\frac{n^3}{mm}$ et n.

PROP. 7. $x^4 * - mmxx - n^3x + p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - mmxx = n^3x - p^4$. Unde patet, ſi xx æquatur ipſi mm, hoc eſt, $x = m$, eandem x fore æqualem ipſi $\frac{p^4}{n^3}$; ac per conſequens, ſi fuerint termini hi m et $\frac{p^4}{n^3}$ æquales, erit una radicum æquationis propoſitæ æqualis ſingulis ipſorum; et ſi inæquales fuerint, neutra duarum radicum æquationis propoſitæ poterit eſſe inter illos duos m et $\frac{p^4}{n^3}$.

Eodem modo, per tranſpoſitionem, eſt $x^4 - n^3x = mmxx - p^4$. Unde ſimilitèr diſcimus, ſi x^3 æquatur ipſi n^3, hoc eſt, $x = n$, fore etiàm $xx = \frac{p^4}{m^4}$, hoc eſt, $x = \frac{pp}{m}$. Ideóque, ſi hi termini n et $\frac{pp}{m}$ fuerint æquales, una ex radicibus æquationis propoſitæ æquabitur ſingulis eorundem terminorum; ſin verò inæquales fuerint, nulla radicum æquationis propoſitæ inter illos duos conſtituta erit.

Prætereà, per tranſpoſitionem, eſt $x^4 + p^4 = mmxx + n^3x$, ideóque $mmxx + n^3x$ major quàm x^4, et $mmx + n^3$ major quàm x^3. Porrò, ſi propoſita æquatio eſt realis, erit x realis, et æqualis, vel major, vel minor quàm n. Si fuerit æqualis vel major, erit $mmx + nnx$ major quàm x^3, et $mm + nn$ major quàm xx, hoc eſt, x minor erit quàm $\sqrt{mm + nn}$. Si fuerit x minor quàm n, minor etiàm erit quàm $\sqrt{mm + nn}$. Quarè patet, quamlibet radicum æquationis propoſitæ neceſſariò

fariò minorem eſſe quàm $\sqrt{mm + nn}$. Denique, exiſtente $x^4 + p^4 = mmxx + n^3x$, erit ſimilitèr $mmxx + n^3x$ major quàm p^4. Et, quia inventa eſt $\sqrt{mm + nn}$ major quàm x, erit conſequentèr $mmx\sqrt{mm + nn}$ major quàm $nmxx$, ideóque $mmx\sqrt{mm + nn} + n^3x$ major quàm p^4, et x major quàm $\frac{p^4}{mm\sqrt{mm + nn}, + n^3}$. Quarè inventus eſt terminus unus major, et alter minor, quàm quælibet duarum radicum æquationis propoſitæ. Atque ita, modo † [ſequenti capite †] obſervato propoſitione ſeptimâ, demonſtrari poteſt, quòd x major eſt quàm minor horum terminorum n et $\frac{p^4}{mmn + n^3}$ *.

† Potiùs legendum videtur *in præcedente capite.*

* Quoniam $x^4 + p^4$ eſt $= mmxx + n^3x$, quantitas binomia $mmxx + n^3x$ erit major quàm p^4.

Ponamus quòd radix x ſit minor quàm n, et quæramus ejus limites in hoc caſu.

Quoniam x eſt minor quàm n, erit $mmxx$ minor quàm $mmnx$, et quantitas binomia $mmxx + n^3x$ minor quàm quantitas binomia $mmnx + n^3x$. Sed quantitas $mmxx + n^3x$ eſt major quàm p^4. Ergò, à fortiori, quantitas $mmnx + n^3x$ erit major quàm p^4; et pro-inde x erit major quàm fractio $\frac{p^4}{mmn + n^3}$. Eſt igitùr x in hoc caſu minor quàm n, ſed major quàm fractio $\frac{p^4}{mmn + n^3}$, ſeu hæ duæ quantitates $\frac{p^4}{mmn + n^3}$ et n ſunt ejuſdem limites, ut in auctoris textu affirmatur.

Q. E. D.

F. M.

CAPUT

CAPUT VIII.

De Æquationibus quatuor dimenſionum, penultimo termino carentibus.

PROP. I. $x^4 - lx^3 + mmxx * - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - lx^3 = p^4 - mmxx$. Unde conſtat, ſi x eſt æqualis ipſi l, etiàm xx æquari $\frac{p^4}{mm}$, hoc eſt, $x = \frac{pp}{m}$; ideóque, ſi l æqualis eſt ipſi $\frac{pp}{m}$, hoc eſt, $lm = pp$, erit radix æquationis propoſitæ æqualis ſingulis terminorum l et $\frac{pp}{m}$; et, ſi fuerint inæquales, unaquæque radicum æquationis propoſitæ, ſive unam, ſive tres habuerit, ſempèr erit inter hos terminos; ſed, ſi fuerint æquales, hoc eſt, $lm = pp$, et $llmm = p^4$, ſubſtituto $llmm$ loco p^4 in æquatione propoſitâ, eâque divisâ per $x - l$, * cognoſcemus in hoc caſu non haberi aliam radicem veram præter l.

* Hæc diviſio erit hujuſmodi.

$$
\begin{array}{rl}
x - l) & x^4 - lx^3 \quad + mmxx * - llmm \; (x^3 + mmx + lmm \\
 & x^4 - lx^3 \quad\quad mmxx * - llmm \\
\hline
 & * \quad * \quad + mmxx \quad * \\
 & \quad\quad + mmxx - lmmx \\
\hline
 & \quad\quad\quad * \quad + lmmx - llmm \\
 & \quad\quad\quad\quad + lmmx - llmm \\
\hline
 & \quad\quad\quad\quad\quad * \quad\quad * \\
\end{array}
$$

Patet igitùr quòd quantitas quadrinomia $x^4 - lx^3 + mmxx * - llmm$ dividi poteſt accuratè, ſive ſine ullo reſiduo, per quantitatem binomiam $x - l$, et quòd quotiens inde orta erit quantitas trinomia $x^3 + mmx + lmm$. Sequitur igitùr, è converſo, quòd quantitas iſta quadrinomia poteſt generari ex multiplicatione quantitatis binomiæ $x - l$ in quantitatem trinomiam $x^3 + mmx + lmm$. Sed quantitas quadrinomia $x^4 - lx^3 + mmxx * - llmm$ eſt æqualis nihilo. Ergò neceſſe eſt ut una ſaltèm ex duabus quantitatibus $x - l$ et $x^3 + mmx + lmm$, (ex quarum multiplicatione quantitas quadrinomia $x^4 - lx^3 + mmxx * - llmm$ orietur,) ſit etiàm æqualis nihilo. Sed quantitas trinomia $x^3 + mmx + lmm$ (quæ eſt ſumma trium quantitatum x^3, mmx, et lmm,) non poteſt eſſe æqualis nihilo; quantitas verò binomia $x - l$ (quæ eſt differentia duarum quantitatum x et l,) poteſt eſſe æqualis nihilo, nempe, ſi quantitas ablata l fuerit æqualis quantitati x, à quâ aufertur. Ergò quantitas binomia $x - l$ erit æqualis nihilo, et pro-inde quantitas x erit æqualis quantitati l; hoc eſt, quantitas l erit radix æquationis $x^4 - lx^3 + mmxx * - llmm = 0$, et hæc æquatio, ſeu æquatio propoſita $x^4 - lx^3 + mmxx * - p^4 = 0$ in hoc caſu, in quo $llmm$ eſt $= p^4$, nullam aliam radicem habebit præter ipſam l. Q. E. D.

F. M.

PROP.

PROP. 2. $x^4 + lx^3 - mmxx * - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - mmxx = p^4 - lx^3$. Unde conſtat, ſi fuerit $x = m$, etiàm $\surd$ C. $\frac{p^4}{l}$ æquari ipſi x; ideóque, ſi duo termini m et $\surd$ C. $\frac{p^4}{l}$ ſint æquales, erit radix æquationis æqualis ſingulis horum terminorum; ſin verò inæquales fuerint, erit illa neceſſariò inter duos. Similitèr, per tranſpoſitionem, eſt $x^4 - p^4 = mmxx - lx^3$. Unde diſcimus, quòd ſi x æqualis eſt ipſi p, fore quoque eam æqualem ipſi $\frac{mm}{l}$: ideóque, ſi termini p et $\frac{mm}{l}$ æquantur, erit radix æquationis æqualis unicuique illorum; ſed, ſi inæquales fuerint, erit illa neceſſariò inter utroſque conſtituta.

PROP. 3. $x^4 - lx^3 - mmxx * - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - lx^3 = mmxx + p^4$, ideóque x major quàm l. Sed et, per tranſpoſitionem, eſt $x^4 - mmxx = lx^3 + p^4$, ideóque x major quàm m, et mx^3 major quàm $mmxx$. Similitèr, per tranſpoſitionem, eſt $x^4 - p^4 = lx^3 + mmxx$, ideóque x major quàm p, et px^3 major quàm p^4. Prætereà, per tranſpoſitionem propoſitionis, eſt $lx^3 + mmxx + p^4 = x^4$, ideóque $lx^3 + mx^3 + px^3$ major quàm x^4, et $l + m + p$ major quàm x. Quarè invenimus radicem x æquationis propoſitæ majorem eſſe quàm l, m, et p, at minorem quàm $l + m + p$. Denique, per tranſpoſitionem, eſt $x^4 = lx^3 + mmxx + p^4$, ideóque x^4 major quàm $lx^3 + mmxx$, et xx major quàm $lx + mm$. Atqui demonſtratum eſt ſuperiùs x majorem eſſe quàm l, ac pro-inde ll minorem eſſe quàm lx. Multò igitùr magìs xx major erit quàm $ll + mm$, et x major quàm $\surd\overline{ll + mm}$. Non diſſimili ratione demonſtrabitur, quòd x major eſt quàm $\surd\surd\overline{l^4 + p^4}$, $\surd\surd\overline{m^4 + p^4}$, et $\surd\surd\overline{mmpp + p^4}$ *.

* Hæ concluſiones demonſtrari poterunt hoc modo.

Imprimis demonſtrandum eſt, quòd x erit major quàm $\surd\surd\overline{l^4 + p^4}$, ſeu radix quadratica radicis quadraticæ quantitatis binomiæ $\overline{l^4 + p^4}$. Hoc verò ita demonſtrari poteſt.

Oſtenſum eſt ſuprà quòd x^4 eſt = quantitati trinomiæ $lx^3 + mmxx + p^4$. Erit ergò major quàm quantitas binomia $lx^3 + p^4$. Sed x eſt major quàm l. Ergò lx^3 erit major quàm $l \times l^3$, ſeu l^4, et $lx^3 + p^4$ erit major quàm $l^4 + p^4$. Ergò quantitas x^4, quæ major eſt quantitate binomiâ $lx^3 + p^4$, erit, à fortiori, major quàm quantitas binomia $l^4 + p^4$, et pro-inde x erit major quàm

$\surd$

PROP. 4. $x^4 + lx^3 + mmxx * - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 + lx^3 = p^4 - mmxx$, ideóque erit $\frac{p^4}{mm}$ major quàm xx, et $\frac{pp}{m}$ major quàm x. Similitèr eſt $x^4 + mmxx = p^4 - lx^3$, ac per conſequens p major quàm x, et $ppxx$ major quàm x^4, et $llpp$ major quàm lx^3. Sed, per tranſpoſitionem propoſitionis, eſt etiàm $x^4 + lx^3 + mmxx = p^4$. Quarè $ppxx + lpxx + mmxx$ major erit quàm p^4, et xx major quàm $\frac{p^4}{pp + lp + mm}$, et x major quàm $\surd \frac{p^4}{pp + lp + mm}$. At verò, exiſtente $x^4 + lx^3 + mmxx = p^4$, erit quoque p^4 major quàm lx^3, ideóque $\frac{p^4}{l}$ major quàm x^3, et $\surd$ C. $\frac{p^4}{l}$ major quàm x.

$\surd^4\overline{(l^4 + p^4}$, ſeu quarta radix quantitatis binomiæ $l^4 + p^4$, ſeu, ſecundùm notationem auctoris in textu, quàm $\surd\surd\overline{(l^4 + p^4}$, ſeu radix quadratica radicis quadraticæ dictæ quantitatis $l^4 + p^4$. Q. E. D.

Secundo loco demonſtrandum eſt, quòd x erit major quàm $\surd\surd\overline{(m^4 + p^4}$, ſeu radix quadratica radicis quadraticæ quantitatis binomiæ $m^4 + p^4$. Hoc verò ita demonſtrari poteſt.

Quoniam x^4 eſt æqualis quantitati trinomiæ $lx^3 + mmxx + p^4$, erit major quàm quantitas binomia $mmxx + p^4$. Sed x eſt major quàm m. Ergò $mmxx$ erit major quàm $mm \times mm$, ſeu m^4, et pro-inde $mmxx + p^4$ erit major quàm $m^4 + p^4$. Ergò quantitas x^4, quæ major eſt quàm $mmxx + p^4$, erit, à fortiori, major quàm $m^4 + p^4$; et pro-inde x erit major quàm $\surd^4\overline{(m^4 + p^4}$, ſeu quarta radix quantitatis binomiæ $m^4 + p^4$, ſeu, ſecundùm notationem auctoris in textu, quam radix quadratica radicis quadraticæ dictæ quantitatis $m^4 + p^4$. Q. E. D.

Tertio loco demonſtrandum eſt, quòd x erit major quàm $\surd\surd\overline{(mmpp + p^4}$, ſeu radix quadratica radicis quadraticæ quantitatis binomiæ $mmpp + p^4$. Hoc verò ita demonſtrari poteſt.

Quoniam x^4 eſt æqualis quantitati trinomiæ $lx^3 + mmxx + p^4$, erit major quàm quantitas binomia $mmxx + p^4$: Sed x eſt major quàm p. Ergò $mmxx$ erit major quàm $mmpp$, et pro-inde quantitas binomia $mmxx + p^4$ erit major quàm quantitas binomia $mmpp + p^4$. Ergò quantitas x^4, quæ major eſt quàm $mmxx + p^4$, erit, à fortiori, major quàm $mmpp + p^4$; et pro-inde x erit major quàm $\surd^4\overline{(mmpp + p^4}$, ſeu quarta radix quantitatis binomiæ $mmpp + p^4$, ſeu, ſecundùm auctoris notationem in textu, quàm $\surd\surd\overline{(mmpp + p^4}$, ſeu radix quadratica radicis quadraticæ dictæ quantitatis $mmpp + p^4$. Q. E. D.

F. M.

Inventa igitùr eſt radix x æquationis propoſitæ major quàm $\surd \frac{p^4}{pp+lp+mm}$, at minor quàm $\frac{pp}{m}$, $\surd$ C. $\frac{p^4}{l}$, et p.

PROP. 5. $x^4 - lx^3 + mmxx * + p^4 = 0$.

Per tranſpoſitionem eſt $mmxx + p^4 = lx^3 - x^4$, ideóque l major quàm x. Deinde eſt $x^4 + p^4 = lx^3 - mmxx$, ac per conſequens x major quàm $\frac{mm}{l}$. Prætereà eſt $x^4 + mmxx = lx^3 - p^4$, ac pro-inde x^3 major quàm $\frac{p^4}{l}$, et x major quàm $\surd$ C. $\frac{p^4}{l}$. Invenimus igitùr, unamquamque duarum radicum æquationis propoſitæ majorem eſſe quàm $\frac{mm}{l}$ et $\surd$ C. $\frac{p^4}{l}$, at minorem quàm l.

PROP. 6. $x^4 + lx^3 - mmxx * + p^4 = 0$.

Per tranſpoſitionem eſt $x^4 + p^4 = mmxx - lx^3$, ideóque $\frac{mm}{l}$ major quàm x. Deinde eſt $lx^3 + p^4 = mmxx - x^4$, ac pro-inde m major quàm x. Prætereà eſt $x^4 + lx^3 = mmxx - p^4$, et conſequentèr xx major quàm $\frac{p^4}{mm}$, hoc eſt, x major quàm $\frac{pp}{m}$. Invenimus ergò, unamquamque duarum radicum æquationis propoſitæ majorem eſſe quàm $\frac{pp}{m}$, at minorem quàm $\frac{mm}{l}$ et m.

PROP. 7. $x^4 - lx^3 - mmxx * + p^4 = 0$.

Per tranſpoſitionem habebimus $x^4 - lx^3 = mmxx - p^4$. Unde patet, ſi x æqualis eſt ipſi l, ipſam x quoque fore æqualem ipſi $\frac{pp}{m}$; et per conſequens, ſi fuerint termini l et $\frac{pp}{m}$ æquales, erit una radicum æquationis propoſitæ æqualis ſingulis illorum; et, ſi fuerint inæquales, neutra duarum radicum æquationis propoſitæ poterit eſſe inter illos duos conſtituta. Eodem modo per tranſpoſi-

tionem

tionem eſt $x^4 - mmxx = lx^3 - p^4$. Unde ſimilitèr conſtat, ſi fuerit x æqualis ipſi m, fore quoque x æqualem ipſi $\surd$ C. $\frac{p^4}{l}$; ideóque, ſi æquales fuerint m et $\surd$ C. $\frac{p^4}{l}$, una radicum æquationis propoſitæ æqualis erit cuilibet horum terminorum æqualium; et, ſi fuerint inæquales, nulla radicum æquationis propoſitæ erit inter illos duos conſtituta. Porrò, per tranſpoſitionem, eſt quoque $x^4 + p^4 = lx^3 + mmxx$; unde $lx^3 + mmxx$ major erit quàm x^4, et $lx + mm$ major quàm xx. Jam, ſi fuerit æquatio propoſita realis, erit x vel æqualis, vel major, vel minor quàm m, et $l + m$ major quàm x. Quòd ſi fuerit x minor quàm m, multo magìs ipſa minor erit quàm $l + m$.

Deinde ex eâdem æquatione $x^4 + p^4 = lx^3 + mmxx$ etiàm conſtat, quòd $lx^3 + mmxx$ major eſt quàm p^4. Atqui inventa eſt $l + m$ major quàm x. Ergò $llxx + lmxx$ major erit quàm lx^3, et $llxx + lmxx + mmxx$ major quàm p^4, ideóque xx major quàm $\frac{p^4}{ll + lm + mm}$. Hinc, cum $llxx + 2lmxx + mmxx$ multò major ſit quàm p^4, erit quoque per conſequens $lx + mx$ major quàm pp, et x major quàm $\frac{pp}{l+m}$. Quarè invenimus quamlibet duarum radicum æquationis propoſitæ majorem eſſe quàm $\surd \frac{p^4}{ll + lm + mm}$ et $\frac{pp}{l+m}$, at minorem quàm $l + m$.

Cæterùm, quoniam invenimus, quòd x neceſſariò eſt minor quàm $l + m$; patet, ſi x ſupponitur major quàm m, eam fore inter hos terminos $l + m$ et m. Quòd ſi m fuerit æqualis aut major quàm x; quoniam $lx^3 + mmxx$ major eſt quàm p^4, erit et $lmxx + mmxx$ major quàm p^4, et xx major quàm $\frac{p^4}{lm + mm}$, et x major quàm $\surd \frac{p^4}{lm + mm}$. Quarè unaquæque duarum radicum æquationis propoſitæ major erit quàm minor duorum terminorum m et $\surd \frac{p^4}{lm + mm}$, at minor quàm $l + m$.

CAPUT IX.

De limitibus Æquationum quatuor dimensionum, tertio termino carentium.

PROP. 1. $x^4 - lx^3 * + n^3x - p^4 = 0$.

Per transpositionem est $x^4 - lx^3 = p^4 - n^3x$. Unde patet, quòd si x æqualis est ipsi l, fore quoque x æqualem ipsi $\frac{p^4}{n^3}$; ideóque, si fuerit l æqualis ipsi $\frac{p^4}{n^3}$, hoc est, $ln^3 = p^4$, radix æquationis propositæ æqualis erit singulis terminorum l et $\frac{p^4}{n^3}$; et, si fuerint inæquales, unaquæque radicum æquationis propositæ, sive unam, sive tres habuerit, semper erit inter hoc terminos. Præterea cognoscimus, quòd, si fuerint hi ultimi termini æquales, hoc est, $ln^3 = p^4$, substituto in æquatione propositâ ln^3 loco p^4, et, divisâ æquatione per $x - l$*, ipsa non possit aliam habere veram radicem quàm l.

* Hæc divisio erit hujusmodi.

$$
\begin{array}{r|l}
x - l & x^4 - lx^3 * + n^3x - ln^3 \;(x^3 + n^3 \\
 & x^4 - lx^3 \\
\hline
 & * \quad * \quad + n^3x - ln^3 \\
 & \qquad\quad + n^3x - ln^3 \\
\hline
 & \qquad\quad * \quad *
\end{array}
$$

Patet ergò quòd quantitas quadrinomia $x^4 - lx^3 * + n^3x - ln^3$ dividi potest accuratè, sive sine ullo residuo, per quantitatem binomiam $x - l$, et quòd quotiens inde orta erit quantitas binomia $x^3 + n^3$. Sequitur igitùr quòd quantitas ista quadrinomia potest generari ex multiplicatione quantitatis binomiæ $x - l$ in quantitatem binomiam $x^3 + n^3$. Sed quantitas quadrinomia $x^4 - lx^3 * + n^3x - ln^3$ est æqualis nihilo. Ergò necesse est ut una saltèm ex duabus quantitatibus binomiis $x - l$ et $x^3 + n^3$, (ex quarum multiplicatione quantitas quadrinomia $x^4 - lx^3 * + n^3x - ln^3$ orietur,) sit etiàm æqualis nihilo. Sed quantitas binomia $x^3 + n^3$ (quæ est summa duarum quantitatum x^3 et n^3,) non potest æqualis esse nihilo; quantitas verò $x - l$ (quæ est differentia duarum quantitatum x et l,) potest esse æqualis nihilo, nempe, si quantitas x fuerit æqualis quantitati l, quæ ab illa auferenda est. Ergò quantitas binomia $x - l$ erit æqualis nihilo, et pro-inde quantitas x erit æqualis quantitati l; hoc est, quantitas l erit radix æquationis $x^4 - lx^3 * + n^3x - ln^3 = 0$, et dicta æquatio, seu æquatio proposita $x^4 - lx^3 * + n^3x - p^4 = 0$ in hoc casu (in quo l est $= \frac{p^4}{n^3}$, seu ln^3 est $= p^4$,) nullam aliam radicem habebit præter ipsam l. Q. E. D.

F. M.

PROP.

PROP. 2. $x^4 + lx^3 * - n^3x - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - n^3x = p^4 - lx^3$. Unde conſtat, ſi x æqualis eſt ipſi n, fore quoque $x^3 = \frac{p^4}{l}$, hoc eſt, $x = \surd$ C. $\frac{p^4}{l}$; et, ſi fuerit n æqualis ipſi $\surd$ C. $\frac{p^4}{l}$, radix æquationis æquabitur ſingulis horum terminorum ; et, ſi fuerint inæquales, erit neceſſariò inter duos. Deinde, per tranſpoſitionem, eſt $x^4 - p^4 = n^3x - lx^3$. Unde patet, ſi fuerit x æqualis ipſi p, fore quoque $xx = \frac{n^3}{l}$, hoc eſt, $x = \surd \frac{n^3}{l}$; ideóque, ſi fuerit p æqualis ipſi $\surd\frac{n^3}{l}$, radix æquationis propoſitæ æquabitur ſingulis horum terminorum, et, ſi fuerint inæquales, erit neceſſariò inter utroſque.

PROP. 3. $x^4 - lx^3 * - n^3x - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - lx^3 = n^3x + p^4$, ideóque x major quàm l. Eodem modo eſt $x^4 - n^3x = lx^3 + p^4$, ac pro-inde x major quàm n, et nx^3 major quàm n^3x. Similitèr eſt $x^4 - p^4 = lx^3 + n^3x$, ideóque x major quàm p, et px^3 major quàm p^4. Sed, per tranſpoſitionem, eſt quoque $lx^3 + n^3x + p^4 = x^4$. Quarè $lx^3 + nx^3 + px^3$ major erit quàm x^4, et $l + n + p$ major quàm x. Ergò invenimus radicem x æquationis propoſitæ majorem eſſe quàm l, n, et p, at minorem quàm $l + n + p$. Porrò ex hâc æquatione $lx^3 + n^3x + p^4 = x^4$ etiàm conſtat, quod $lx^3 + n^3x$ eſt minor quàm x^4 : et, quandoquidem invenimus l minorem eſſe quàm x, erit l^3x minor quàm lx^3, ideóque $l^3x + n^3x$ multò minor quam x^4, et x^3 major quàm $l^3 + n^3$. Non diſſimili ratione demonſtrabitur, quòd x major eſt quàm $\surd\surd\overline{l^4 + p^4}$ et $\surd\surd\overline{ln^3 + p^4}$.*

* Eſt enim x^4 æqualis quantitati trinomiæ $lx^3 + n^3x + p^4$, et pro-inde eſt major quàm $lx^3 + p^4$. Sed x eſt major quàm l. Ergò lx^3 erit major quàm $l \times l^3$, ſeu l^4, et pro-inde $lx^3 + p^4$ erit major quàm $l^4 + p^4$. Ergò quantitas x^4, (quæ eſt major quàm $lx^3 + p^4$,) erit, à fortiori, major quàm $l^4 + p^4$, et pro-inde x erit major quàm $\surd^4\overline{l^4 + p^4}$, ſeu, ſecundùm notationem auctoris in textu, quàm $\surd\surd\overline{l^4 + p^4}$, ſeu radix quadratica radicis quadraticæ quantitatis binomiæ $l^4 + p^4$. Q. E. D.

Porrò eſt etiàm quantitas x^4 major quàm $n^3x + p^4$. Sed x eſt major quàm l. Ergò n^3x eſt major quàm n^3l, ſeu ln^3, et $n^3x + p^4$ eſt major quàm $ln^3 + p^4$. Ergò quantitas x^4, (quæ major eſt quàm $n^3x + p^4$,) erit, à fortiori, major etiàm quàm $ln^3 + p^4$ et pro-inde x erit major quàm $\surd^4\overline{ln^3 + p^4}$, ſeu, ſecundùm notationem auctoris in textu, quàm $\surd\surd\overline{ln^3 + p^4}$, ſeu radix quadratica radicis quadraticæ quantitatis binomiæ $ln^3 + p^4$. Q. E. D.

F. M.

PROP.

PROP. 4. $x^4 + lx^3 * + n^3x - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 + lx^3 = p^4 - n^3x$, ideóque $\frac{p^4}{n^3}$ major quàm x. Eodem modo eſt $x^4 + n^3x = p^4 - lx^3$, ac pro-inde $\frac{p^4}{l}$ major quàm x^3. Similitèr eſt $lx^3 + n^3x = p^4 - x^4$, et per conſequens p major quàm x, et p^3x major quàm x^4, ac $lppx$ major quàm lx^3. Sed, per tranſpoſitionem propoſitionis, eſt quoque $x^4 + lx^3 + n^3x = p^4$. Quarè $p^3x + lppx + n^3x$ major erit quàm p^4, et x major quàm $\frac{p^4}{p^3 + lpp + n^3}$. Et ſic inventa eſt radix x æquationis propoſitæ major quàm $\frac{p^4}{p^3 + lpp + n^3}$, at minor quàm $\frac{p^4}{n^3}$, $\surd$ C. $\frac{p^4}{l}$ et p.

PROP. 5. $x^4 - lx^3 * + n^3x + p^4 = 0$.

Per tranſpoſitionem eſt $n^3x + p^4 = lx^3 - x^4$, ideóque l major quàm x. Deinde eſt $x^4 + p^4 = lx^3 - n^3x$; quarè erit xx major quàm $\frac{n^3}{l}$, hoc eſt, x major quàm $\surd\frac{n^3}{l}$. Sed eſt quoque $x^4 + n^3x = lx^3 - p^4$, ideóque x^3 major quàm $\frac{p^4}{l}$, et x major quàm $\surd$ C. $\frac{p^4}{l}$. Quarè invenimus, quòd quælibet duarum radicum æquationis propoſitæ neceſſariò major eſt quàm $\surd\frac{n^3}{l}$ et $\surd$ C. $\frac{p^4}{l}$, at minor quàm l.

PROP. 6. $x^4 + lx^3 * - n^3x + p^4 = 0$.

Per tranſpoſitionem eſt $x^4 + p^4 = n^3x - lx^3$, ideóque $\frac{n^3}{l}$ major quàm xx, hoc eſt, x minor quàm $\surd\frac{n^3}{l}$. Deinde eſt $x^4 + lx^3 = n^3x - p^4$, ideóque x major quàm $\frac{p^4}{n^3}$. Prætereà eſt $lx^3 + p^4 = n^3x - x^4$, et idcircò n^3 major quàm x^3, hoc eſt, x minor quàm n. Ergò invenimus, unamquamque duarum radicum x æquationis propoſitæ majorem eſſe quàm $\frac{p^4}{n^3}$, at minorem quàm $\surd\frac{n^3}{l}$ et n.

PROP.

PROP. 7. $x^4 - lx^3 * - n^3x + p^4 = 0$.

Per tranſpoſitionem habebimus $x^4 - lx^3 = n^3x - p^4$. Unde patet, ſi x æqualis eſt ipſi l, fore quoque x æqualem ipſi $\frac{p^4}{n^3}$; et per conſequens, ſi fuerint hi termini l et $\frac{p^4}{n^3}$ æquales, hoc eſt, $ln^3 = p^4$, una ex radicibus æquationis propoſitæ æquales erit ſingulis horum terminorum æqualium l et $\frac{p^4}{n^3}$; et, ſi inæquales fuerint, neutra duarum radicum æquationis propoſitæ poterit eſſe inter ipſos.

Deinde, per tranſpoſitionem, eſt $x^4 - n^3x = lx^3 - p^4$. Unde ſimili modo patet, ſi x æquatur ipſi n, ipſam x quoque æquari ipſi $\surd$ C. $\frac{p^4}{l}$; ideóque, ſi termini hi n et $\surd$ C. $\frac{p^4}{l}$ æquales fuerint, una radicum æquationis propoſitæ æquabitur ſingulis horum terminorum æqualium; et, ſi fuerint inæquales, nulla radicum æquationis propoſitæ erit inter utroſque.

Porrò, per tranſpoſitionem, eſt quoque $x^4 + p^4 = lx^3 + n^3x$, ideóque $lx^3 + n^3$ major quàm x^4, et $lxx + n^3$ major quàm x^3. Jam, ſi fuerit propoſita æquatio realis, erit x realis, et vel æqualis, vel major vel minor quàm m. Quòd ſi fuerit æqualis vel major, erit $lxx + nxx$ major quàm x^3. Sin verò minor ſit, erit x multò minor quàm $l + n$. Quarè utraque duarum radicum propoſitæ æquationis neceſſariò minor erit quàm $l + n$.

Quìn et exiſtente $x^4 + p^4 = lx^3 + n^3x$, erit quoque $lx^3 + n^3x$ major quàm p^4. Atqui invenimus $l + n$ majorem eſſe quàm x, ac pro-inde $ll + nn + 2ln$ majorem quàm xx, et $l^3 + lnn + 2lln$ majorem quàm lxx, nec non $l^3x + lnnx + 2llnx$ majorem quàm lx^3. Ergò $l^3x + lnnx + 2llnx + n^3x$ major erit quàm p^4, et x major quàm $\frac{p^4}{l^3 + lnn + 2lln + n^3}$. Et, quandoquidem cubus ex $l + n$ major eſt quàm $l^3 + lnn + 2lln + n^3$, multo magìs erit x major quàm p^4 diviſum per cubum ex $l + n$. Invenimus itaque quòd quælibet duarum radicum æquationis propoſitæ major eſt quàm p^4 diviſum per cubum ex $l + n$, ut et major quàm $\frac{p^4}{l^3 + lnn + 2lln + n^3}$, at minor quàm $l + n$. Prætereà, quoniam $l + n$ major eſt quàm x, ſi fuerit x major quàm n, erit neceſſariò

fariò inter hos terminos $l + n$ et n. Quòd ſi verò n fuerit vel æqualis vel major quàm x, quia $lx^3 + n^3x$ major eſt quàm p^4, erit et $lnnx + n^3x$ major quàm p^4, et x major quàm $\frac{p^4}{lnn + n^3}$. Ac pro-inde quælibet radicum æquationis propoſitæ major erit quàm minor horum terminorum n et $\frac{p^4}{lnn + n^3}$, at minor quàm $l + n$.

CAPUT X.

De limitibus Æquationum quatuor dimenſionum, in quibus nullus terminus deeſt.

PROP. I. $x^4 - lx^3 + mmxx - n^3x + p^4 = 0$.

Per tranſpoſitionem eſt $lx^3 + n^3x = x^4 + mmxx + p^4$, ideóque $lx^3 + n^3x$ major quàm x^4, et $lxx + n^3$ major quàm x^3. Jàm, ſi fuerit propoſita æquatio realis, erit et x realis, et vel æqualis vel major vel minor quàm n. Quòd ſi fuerit æqualis vel major, erit $lxx + nxx$ major quàm x^3, hoc eſt, $l + n$ major quàm x, et x minor quàm $l + n$. Quod ſi x fuerit minor quàm n, multo igitùr magìs minor erit quàm $l + n$. Ergò x neceſſariò ſempèr minor erit quàm $l + n$. Deinde ex eâdem æquatione $lx^3 + n^3x = x^4 + mmxx + p^4$ conſtat, eſſe $lx^3 + n^3x$ majorem quàm p^4. Sed inventa eſt $l + n$ major quàm x, ac per conſequens $ll + nn + 2ln$ major quàm xx, et $l^3x + lnnx + 2llnx$ major quàm lx^3. Quarè erit $l^3x + lnnx + 2llnx + n^3x$ major quàm p^4, et x major quàm $\frac{p^4}{l^3 + lnn + 2lln + n^3}$; et, quandoquidem cubus ex $l + n$ major eſt quàm $l^3 + lnn + 2lln + n^3$, multo magìs erit x major quàm p^4 diviſum per cubum ex $l + n$. Inventus eſt itaque terminus unus major et alter minor quàm unaquæque radicum æquationis propoſitæ, ſive hæc duas ſive quatuor radices habuerit. Præterea, quoniam invenimus, quòd $l + n$ ſempèr major eſt quàm x, ſi ponatur x quoque major quàm n; manifeſtum eſt eam eſſe inter duos terminos $l + n$ et

n.

n. Quòd ſi autem x fuerit æqualis vel minor quàm n, quoniam eſt $lx^3 + n^3x$ major quàm p^4; erit $lnnx + n^2x$ major quàm p^4, ideóque x major quàm $\frac{p^4}{lnn+n^3}$. Ergò unaquæque radicum propoſitæ æquationis, (ſive duas, ſive tres habuerit,) major erit quàm minor horum duorum terminorum n et $\frac{p^3}{lnn+n^3}$, at minor quàm $l + n$.

PROP. 2. $x^4 - lx^3 + mmxx + n^3x + p^4 = 0$.

Per tranſpoſitionem eſt $mmxx + n^3x + p^4 = lx^3 - x^4$, ideóque l major quàm x. Similitèr eſt $x^4 + n^3x + p^4 = lx^3 - mmxx$, ac idcircò x major quàm $\frac{mm}{l}$. Prætereà eſt $x^4 + mmxx + p^4 = lx^3 - n^3x$, ac per conſequens xx major quàm $\frac{n^3}{l}$. Denique eſt $x^4 + mmxx + n^3x = lx^3 - p^4$, et conſequentèr x^3 major quàm $\frac{p^4}{l}$. Quarè invenimus, unamquamque duarum radicum æquationis propoſitæ majorem eſſe quàm $\frac{mm}{l}$, $\sqrt{\frac{n^3}{l}}$, et $\sqrt{\text{C}.\frac{p^4}{l}}$ at minorem quàm l.

PROP. 3. $x^4 - lx^3 - mmxx - n^3x + p^4 = 0$.

Per tranſpoſitionem eſt $lx^3 + mmxx + n^3x = x^4 + p^4$, ideóque $lx^3 + mmxx + n^3x$ major quàm x^4. Jam, ſi propoſita æquatio fuerit realis, erit x realis, et vel æqualis, vel major, vel minor, quàm maxima duarum m et n. Quòd ſi fuerit æqualis vel major, erit $lx^3 + mx^3 + nx^3$ major quàm x^4, et $l + m + n$ major erit quàm x; et magìs, ſi fuerit x minor quàm maxima duarum m et n. Quarè $l + m + n$ erit neceſſariò major quàm x.

Prætereà m erit aut æqualis, aut major, aut minor quàm n. Quòd ſi fuerit æqualis aut major, et quidem x major quàm m, erit radix æquationis propoſitæ inter hóſce terminos $l + m + n$ et m. Quòd ſi, exiſtente m æquali aut majore quàm n, etiàm m ſit æqualis vel major quàm x; erit et $lmmx + m^3x + n^3x$ æqualis aut major quàm $lx^3 + mmxx + n^3x$, ac per conſequens major quàm p^4; ideóque x major quàm $\frac{p^4}{lmm + m^3 + n^3}$. Unde, ſi fuerit m vel æqualis vel major quàm n, erit x neceſſariò major quàm minor horum duorum terminorum m et $\frac{p^4}{lmm + m^3 + n^3}$; et, ſi n fuerit major quàm m, conſimili ratione demonſtrabitur x etiàm neceſ-

fariò majorem eſſe minore horum duorum terminorum n et $\frac{p^4}{lnn + mmn + n^3}$*. Invenimus ergò unamquamque duarum radicum æquationis propoſitæ majorem eſſe minore horum terminorum m et $\frac{p^4}{lmm + m^3 + n^3}$, ſi m vel æqualis vel major fuerit quàm n; aut majorem minore duorum n et $\frac{p^4}{lnn + mmn + n^3}$, ſi n major ſit quàm m; at verò ſempèr minorem quàm $l + m + n$.

PROP. 4. $x^4 - lx^3 - mmxx + n^3x + p^4 = 0$.

Per tranſpoſitionem eſt $lx^3 + mmxx = x^4 + n^3x + p^4$, ideóque $lx^3 + mmxx$ major quàm x^4, et $lx + mm$ major quàm xx. Jam, ſi fuerit propoſita æquatio realis, erit et x realis, et vel æqualis, vel major, vel minor, quàm m. Quòd ſi fuerit æqualis vel major, erit $lx + mx$ major quàm xx, et $l + m$ major quàm x; et multo magìs, ſi fuerit x minor quàm m. Ergò x neceſſariò minor erit quàm $l + m$. Unde, ſi fuerit x major quàm m, erit inter hóſce terminos $l + m$ et m.

Quòd ſi x fuerit vel æqualis vel minor quàm m, quandoquidem et $lx^3 + mmxx$ major eſt quàm p^4, erit $lmxx + mmxx$ major quàm p^4; ideóque xx major quàm $\frac{p^4}{lm + mm}$, et x major quàm $\surd\frac{p^4}{lm + mm}$. Quarè quælibet duarum radicum æquationis propoſitæ major erit quàm minor duorum terminorum m et $\surd\frac{p^4}{lm + mm}$, at minor quàm $l + m$.

PROP. 5. $x^4 + lx^3 + mmxx - n^3x + p^4 = 0$.

Demonſtrabitur ex tranſpoſitionibus requiſitis n^3x fore majorem quàm x^4, ideóque n majorem quàm x; et n^3x fore majorem quàm lx^3, ac pro-inde $\frac{n^3}{l}$ majorem quàm xx; et denique n^3x fore majorem quàm p^4, et per conſequens x majorem quàm $\frac{p^4}{n^3}$. Invenimus itaque terminum unum minorem ſingulis radicum æquationis propoſitæ, at verò duos alios majores.

* Quoniam enim quantitas trinomia $lx^3 + mmxx + n^3x$ eſt major quàm p^4, ſequitur quòd, ſi m ſit major quàm x, et n ſit major quàm m, et pro-inde $lnnx$ major quàm $lmmx$ et, à fortiori, major quàm lx^3, et $mmnx$ major quàm $mmmx$, et, à fortiori, major quàm $mmxx$, quantitas trinomia $lnnx + mmnx + n^3x$ erit major quàm quantitas trinomia $lx^3 + mmxx + n^3x$, et, à fortiori, major quàm p^4, et pro-inde quantitas x erit major quàm fractio $\frac{p^4}{lnn + mmn + n^3}$. Q. E. D.

PROP.

PROP. 6. $x^4 + lx^3 - mmxx - n^3x + p^4 = 0$.

Per tranſpoſitionem eſt $mmxx + n^3x = x^4 + lx^3 + p^4$, ideóque $mmxx + n^3x$ major quàm x^4, et $mmx + n^3$ major quàm x^3. Jam, ſi fuerit propoſita æquatio realis, erit x realis, et vel æqualis, vel major, vel minor, quàm n. Quòd ſi fuerit æqualis vel major, erit $mmx + nnx$ major quàm x^3, et $\sqrt{mm + nn}$ major quàm x; et multo magìs, ſi fuerit x minor quàm n. Quarè erit $\sqrt{mm + nn}$ ſempèr major quàm x, et x erit inter terminos $\sqrt{mm + nn}$ et n, ſi major eſt quàm n. Quòd ſi fuerit æqualis aut minor quàm n, quoniam eſt $mmxx + n^3x$ major quàm p^4, erit quoque $mmnx + n^3x$ major quàm p^4, ideóque x major quàm $\frac{p^4}{mmn + n^3}$. Ergò quælibet duarum radicum æquationis propoſitæ major erit quàm minor horum duorum terminorum n et $\frac{p^4}{mmn + n^3}$, at minor quàm $\sqrt{mm + nn}$.

PROP. 7. $x^4 + lx^3 - mmxx + n^3x + p^4 = 0$.

Factis tranſpoſitionibus requiſitis, demonſtrabitur eſſe x minorem quàm m et $\frac{mm}{l}$, at majorem quàm $\frac{n^3}{mm}$ et $\frac{pp}{m}$ *.

* Nam, ſi addatur utrinque æquationi propoſitæ $x^4 + lx^3 - mmxx + n^3x + p^4 = o$ quantitas $mmxx$, habebimus æquationem $x^4 + lx^3 + n^3x + p^4 = mmxx$. Ergò $mmxx$ erit major quàm x^4; et pro-inde mm erit major quàm xx, et m erit major quàm x.

Sed et $mmxx$ erit major quàm lx^3; et pro-inde mm erit major quàm lx, et $\frac{mm}{l}$ erit major quàm x.

Eſt autèm $mmxx$ major etiàm quàm n^3x; et pro-inde mmx erit major quàm n^3, et x erit major quàm $\frac{n^3}{mm}$.

Denique $mmxx$ eſt etiàm major quàm p^4; et pro-inde xx erit major quàm $\frac{p^4}{mm}$, et x erit major quàm $\frac{pp}{m}$.

Erit igitûr x minor quàm m et $\frac{mm}{l}$, at major quàm $\frac{n^3}{mm}$ et $\frac{pp}{mm}$, ut in auctoris textu declaratur. Q. E. D.

F. M.

PROP. 8. $x^4 - lx^3 + mmxx - n^3x - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - lx^3 = n^3x + p^4 - mmxx$, ideóque, ſi fuerit $x^4 = lx^3$, erit $x = l$, et $ln^3 = n^3x$, et $n^3x + p^4 = mmxx$, ac pro-inde $ln^3 + p^4 = mmxx$, et $x = \sqrt{\frac{ln^3 + p^4}{mm}}$. Unde patet, ſi fuerit $l = \sqrt{\frac{ln^3 + p^4}{mm}}$, hoc eſt, ſi habeatur $llmm = ln^3 + p^4$; radicem æquationis propoſitæ fore æqualem ſingulis terminorum æqualium l et $\sqrt{\frac{ln^3 + p^4}{mm}}$: ac idcircò, ſubſtituto in hoc caſu in æquatione propoſitâ valore ipſius p^4, nempe $llmm - ln^3$, ipſam eſſe diviſibilem per $x - l$.

Quòd ſi fuerit x^4 major quàm lx^3, hoc eſt, x major quàm l, erit quoque $n^3x + p^4$ major quàm $mmxx$; et, ſi fuerit lx^3 major quàm x^4, hoc eſt, l major quàm x, erit et $mmxx$ major quàm $n^3x + p^4$. Jam, quandoquidem æquatio propoſita eſt realis, erit x realis, et vel æqualis, vel major, vel minor, quàm p. Quòd ſi fuerit æqualis vel major quàm p, ſítque major quàm l, quoniam tunc $n^3x + p^4$ quoque major eſt quàm $mmxx$, erit et $n^3x + p^3x$ major quàm $mmxx$, et $\frac{n^3 + p^3}{mm}$ major quàm x. Ergò in hoc caſu erit x major quàm l, et minor quàm $\frac{n^3 + p^3}{mm}$.

Quòd ſi autem, x minori exiſtente quàm l, ipſa ſit æqualis vel major quàm p, quoniam et tunc $n^3x + p^4$ minor eſt quàm $mmxx$, erit ſimilitèr $n^3x + p^3x$ minor quàm $mmxx$, et conſequentèr $\frac{n^3 + p^3}{mm}$ minor quàm x. Igitùr in hoc caſu erit x minor quàm l, et major quàm $\frac{n^3 + p^3}{mm}$. Quarè univerſaliter apparet, æquationem propoſitam non habere præter unam radicem realem ipſi l æqualem, cùm eſt $llmm = ln^3 + p^4$; modò quælibet radicum, (ſive unam, ſive tres habuerit,) fuerit ſemper neceſſariò inter maximum et minimum trium terminorum l, $\frac{n^3 + p^3}{mm}$, et $\sqrt{\frac{n^3p + p^4}{mm}}$ *.

PROP.

* Hæc Propoſitio fiet, ut opinor, magìs perſpicua ſi pleniùs explicetur hoc modo.

Æquationi propoſitæ $x^4 - lx^3 + mmxx - n^3x - p^4 = o$ addatur utrinque quantitas binomia $n^3x + p^4$. Et habebimus æquationem $x^4 - lx^3 + mmxx = n^3x + p^4$ quæ poteſt habere aut unam

PROP. 9. $x^4 - lx^3 + mmxx + n^3x - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - lx^3 = p^4 - mmxx - n^3x$, ideóque ſi fuerit $x^4 = lx^3$, hoc eſt, $x = l$, erit $mmxx = -n^3x + p^4$, et per conſequens $mmxx = -n^3l + p^4$,

unam ſolummodò radicem, aut tres radices, ſecundùm varias magnitudines quantitatum l, m, n, et p.

Ponamus, primò, radicem x, ſi ſit unica, vel maximam ex tribus radicibus, ſi ſint tres, eſſe æqualem quantitati l. Et quantitas trinomia $x^4 - lx^3 + mmxx$ fiet in hoc caſu $= o + mmxx$, ſeu ſoli termino $mmxx$, et pro-inde ſolus iſte terminus $mmxx$ erit æqualis quantitati binomiæ $n^3x + p^4$. Erit autèm in hoc caſu $n^3x = n^3l$, ſeu ln^3, et pro-inde $n^3x + p^4$ erit $= ln^3 + p^4$. Ergò $mmxx$ (quæ eſt æqualis ipſi $n^3x + p^4$,) erit etiàm æqualis ipſi $ln^3 + p^4$, et pro-inde xx erit $= \frac{ln^3 + p^4}{mm}$, et x erit $= \sqrt{\frac{ln^3 + p^4}{mm}}$; hoc eſt, quando x eſt æqualis l, erit etiàm æqualis $\sqrt{\frac{ln^3 + p^4}{mm}}$, et pro-inde l erit $= \sqrt{\frac{ln^3 + p^4}{mm}}$, et ll erit $= \frac{ln^3 + p^4}{mm}$, et $llmm$ erit $= ln^3 + p^4$. Ergò, è converſo, ſi $llmm$ fuerit $= ln^3 + p^4$, ſeu ſi $llmm - ln^3$ fuerit $= p^4$, radix x æquationis propoſitæ $x^4 - lx^3 + mmxx - n^3x - p^4 = o$ erit æqualis l. Ergò, ſi ſubſtituamus in hâc æquatione quantitatem $llmm - ln^3$ pro quantitate p^4, æquatio quæ inde orietur, nempe, æquatio $x^4 - lx^3 + mmxx - n^3x - llmm + ln^3 = o$, erit diviſibilis per quantitatem binomiam $x - l$. Hoc verò patebit ex ipſâ diviſione, quæ erit hujuſmodi.

$$
\begin{array}{l|l}
x - l) \; x^4 - lx^3 + mmxx - n^3x - llmm + ln^3 \; (x^3 + mmx + lmm - n^3 \\
\quad x^4 - lx^3 \\
\hline
\quad * \quad * \quad + mmxx - n^3x \\
\quad\quad\quad + mmxx - lmmx \\
\hline
\quad\quad\quad * \quad + lmmx - n^3x - llmm + ln^3 \\
\quad\quad\quad\quad + lmmx - n^3x - llmm + ln^3 \\
\hline
\quad\quad\quad\quad * \quad * \quad * \quad *
\end{array}
$$

Patet ex hâc diviſione quòd quantitas ſextinomia $x^4 - lx^3 + mmxx - n^3x - llmm + ln^3$ dividi poteſt accuratè, ſive ſine ullo reſiduo, per quantitatem binomiam $x - l$, et quòd quotiens inde ortà erit quantitas quadrinomia $x^3 + mmx + lmm - n^3$. Sequitur ergò, è converſo, quòd quantitas iſta ſextinomia poteſt generari ex multiplicatione quantitatis binomiæ $x - l$ in quantitatem quadrinomiam $x^3 + mmx + lmm - n^3$. Sed quantitas ſextinomia $x^4 - lx^3 + mmxx - n^3x - llmm + ln^3$ eſt æqualis nihilo. Ergò neceſſe eſt ut una ſaltèm ex duabus quantitatibus $x - l$ et $x^3 + mmx + lmm - n^3$ (ex quarum multiplicatione quantitas ſextinomia $x^4 - lx^3 + mmxx - n^3x - llmm + ln^3$ orietur,) ſit etiàm æqualis nihilo. Sed quantitas quadrinomia $x^3 + mmx + lmm - n^3$ (quæ eſt ſumma trium quantitatum x^3, mmx, et quantitatis binomiæ $lmm - n^3$, ſeu exceſſûs quantitatis lmm ſupèr quantitatem n^3, quæ eſt ipſâ minor,) non poteſt eſſe æqualis nihilo; quantitas verò binomia $x - l$ (quæ eſt differentia duarum quantitatum x et l,) poteſt eſſe æqualis nihilo, nempe, ſi quantitas ablata l fuerit æqualis quantitati x à quâ aufertur. Ergò quantitas binomia $x - l$ erit æqualis nihilo, et pro-inde quantitas x erit æqualis quantitati l; hoc eſt, quantitas l erit radix

æquationis

$+ p^4$, et $xx = \frac{p^4 - ln^3}{mm}$, et $x = \sqrt{\frac{p^4 - ln^3}{mm}}$. Quòd ſi ergò l æqualis eſt $\sqrt{\frac{p^4 - ln^3}{mm}}$, hoc eſt, ſi fuerit $lln.m = p^4 - ln^3$, radix æquationis propoſitæ

æquationis $x^4 - lx^3 + mmxx - n^3x - llmm + ln^3 = 0$, et hæc æquatio, ſeu æquatio propoſita $x^4 - lx^3 + mmxx - n^3x - p^4 = 0$, in hoc caſu, in quo $llmm - ln^3$ eſt $= p^4$, nullam aliam radicem habebit prætèr ipſam l. Hic, ut opinor, eſt ſenſus auctoris in primis octò lineis hujus propoſitionis uſque ad verba, "*ipſam eſſe diviſibilem per* $x - l$."

Ponamus jàm ſecundò, quòd x^4 ſit major quàm lx^3, et pro-inde x major quàm l, et quæramus limites quantitatis x in hoc caſu.

Oſtenſum eſt ſuprà quòd $x^4 - lx^3 + mmxx$ eſt $= n^3x + p^4$. Ergò, cum x eſt major quàm l, et x^4 major quàm lx^3, quantitas trinomia $x^4 - lx^3 + mmxx$ erit major quàm ſola quantitas $mmxx$, et pro-inde quantitas $n^3x + p^4$ (quæ eſt æqualis quantitati $x^4 - lx^3 + mmxx$,) erit etiàm major quàm $mmxx$.

Ponamus jam tertiò, quòd x ſit minor quàm l, et pro-inde quòd x^4 ſit minor quàm lx^3. Et quantitas trinomia $x^4 - lx^3 + mmxx$ erit minor quàm ſola quantitas $mmxx$, et pro-inde quantitas $n^3x + p^4$ (quæ eſt æqualis $x^4 - lx^3 + mmxx$,) erit etiàm minor quàm $mmxx$.

Jam verò conferamus inter ſe quantitates x et p; quod fieri poteſt hoc modo. Quoniam æquatio propoſita $x^4 - lx^3 + mmxx - n^3x - p^4 = 0$ ſupponitur eſſe æquatio realis, et non æquatio impoſſibilis, et pro-inde quantitas x eſt quantitas realis et non quantitas impoſſibilis, neceſſe eſt ut quantitas x ſit aut æqualis quantitati p, aut ipſâ major, aut ipſâ minor. Si x fuerit æqualis ipſi p, vel ipſâ major, et ſit major etiàm quàm l, quantitas p^3x erit vel æqualis quantitati p^4, vel ipſâ major; et pro-inde quantitas binomia $n^3x + p^3x$ erit vel æqualis quantitati binomiæ $n^3x + p^4$ vel ipſâ major, ideóque erit vel æqualis quantitati trinomiæ $x^4 - lx^3 + mmxx$ (quæ eſt $= n^3x + p^4$,) vel ipſâ major. Eſt verò in hoc caſu (ob x majorem quàm l,) quantitas trinomia $x^4 - lx^3 + mmxx$ major quàm $mmxx$. Erit igitùr quantitas binomia $n^3x + p^4$ major quàm $mmxx$, et pro-inde quantitas binomia $n^3x + p^3x$ erit major quàm $mmxx$, et $n^3 + p^3$ erit major quàm mmx, et $\frac{n^3 + p^3}{mm}$ erit major quàm x. Ergò in hoc caſu, ſeu cum x eſt aut æqualis aut major quàm p, et major quàm b erit minor quam $\frac{n^3 + p^3}{mm}$, et pro-inde l et $\frac{n^3 + p^3}{mm}$ erunt ejus limites.

Q. E. I.

Si verò x ſit minor quàm l, et æqualis quantitati p, quantitas trinomia $x^4 - lx^3 + mmxx$ erit minor ſolâ quantitate $mmxx$, et pro-inde quantitas binomia $n^3x + p^4$ (quæ eſt æqualis quantitati trinomiæ $x^4 - lx^3 + mmxx$,) erit etiàm minor quàm $mmxx$. Sed, ob $x = p$, quantitas p^3x erit $= p^4$, et quantitas $n^3x + p^3x$ erit $= n^3x + p^4$. Ergò quantitas $n^3x + p^3x$ erit etiàm minor quàm $mmxx$, et quantitas $n^3 + p^3$ erit minor quàm mmx, et $\frac{n^3 + p^3}{mm}$ erit minor quàm x. Ergo in hoc caſu, ſeu cum x eſt minor quàm l ſed æqualis p, limites ejus erunt $\frac{n^3 + p^3}{mm}$ et l.

Q. E. I.

Sed auctor in textu dicit etiàm quòd, ſi x fuerit minor quàm l, et major quàm p, erit etiàm major quàm $\frac{n^3 + p^3}{mm}$, ut fit cum x eſt æqualis ipſi p. Hoc autèm non mihi demonſtratum eſſe videtur.

F. M.

æqualis

æqualis erit unicuique terminorum æqualium l et $\checkmark \frac{p^4 - ln^3}{mm}$: ideóque, ſi in hoc caſu in æquatione propoſitâ loco p^4 ſubſtituatur ejus valor, nempe $llmm + ln^3$, apparebit ipſam dividi poſſe per $x - l$, atque nullam aliam radicem veram admittere præter l *. Si verò x^4 fuerit major quàm lx^3, hoc eſt x major quàm l, erit et p^4 major quàm $mmxx + n^3x$; et contrà, ſi fuerit l major quàm x, erit etiàm $mmxx + n^3x$ major quàm p^4. Jam, ſi æquatio propoſita eſt realis, erit x realis, et vel æqualis, vel major, vel minor, quàm n. Eſto igitùr, quòd x (major quàm l) ſit vel æqualis vel major quàm n; quarè, cum & p^4 tunc major ſit quàm $mmxx + n^3x$, erit quoque p^4 major quàm $mmnx + n^3x$; ideóque $\frac{p^4}{mmn + n^3}$ major quàm x. Quarè in hoc caſu erit x major quàm l, et minor quàm $\frac{p^4}{mmn + n^3}$.

* Nam, ſi in æquatione propoſitâ $x^4 - lx^3 + mmxx + n^3x - p^4 = 0$ quantitas binomia $llmm + ln^3$ ſubſtituatur loco termini p^4, hæc æquatio fiet $x^4 - lx^3 + mmxx + n^3x - llmm - ln^3 = 0$, quæ dividi poterit accuratè, ſeu ſine ullo reſiduo, per quantitatem binomiam $x - l$ hoc modo.

$$
\begin{array}{l}
x - l) \; x^4 - lx^3 + mmxx + n^3x - llmm - ln^3 \; (x^3 + mmx + n^3 + lmm \\
\quad x^4 - lx^3 \\
\quad \hline \\
\quad * \quad * \quad + mmxx + n^3x \\
\quad\quad\quad + mmxx - lmmx \\
\quad\quad\quad \hline \\
\quad\quad\quad\quad * \quad + n^3x + lmmx - llmm - ln^3 \\
\quad\quad\quad\quad \hline \\
\quad\quad\quad\quad\quad * \quad * \quad * \quad *
\end{array}
$$

Patet ex hâc diviſione quòd quantitas ſextinomia $x^4 - lx^3 + mmxx + n^3x - llmm - ln^3$ dividi poteſt accuratè, ſive ſine ullo reſiduo, per quantitatem binomiam $x - l$, et quòd quotiens indè orta erit quantitas quadrinomia $x^3 + mmx + n^3 + lmm$. Sequitur ergò, è converſo, quòd quantitas iſta ſextinomia poteſt generari ex multiplicatione quantitatis binomiæ $x - l$ in quantitatem quadrinomiam $x^3 + mmxx + n^3 + lmm$. Sed quantitas ſextinomia $x^4 - lx^3 + mmxx + n^3x - llmm - ln^3$ eſt æqualis nihilo. Ergò neceſſe eſt ut una ſaltèm ex duabus quantitatibus $x - l$ et $x^3 + mmx + n^3 + lmm$ (ex quarum multiplicatione quantitas ſextinomia $x^4 - lx^3 + mmxx + n^3x - llmm - ln^3$ orietur,) ſit etiàm æqualis nihilo. Sed quantitas quadrinomia $x^3 + mmx + n^3 + lmm$ (quæ eſt ſumma quatuor quantitatum x^3, mmx, n^3, et lmm,) non poteſt eſſe æqualis nihilo; quantitas verò binomia $x - l$ (quæ eſt differentia duarum quantitatum x et l,) poteſt eſſe æqualis nihilo, nempe, ſi quantitas ablata l fuerit æqualis quantitati x, à quâ aufertur. Ergò quantitas binomia $x - l$ erit æqualis nihilo, et pro-inde quantitas x erit æqualis quantitati l; hoc eſt, quantitas l erit radix æquationis $x^4 - lx^3 + mmxx + n^3x - llmm - ln^3 = 0$, et hæc æquatio, ſeu æquatio propoſita $x^4 - lx^3 + mmxx + n^3x - p^4 = 0$ in hoc caſu, in quo $llmm + ln^3$ eſt $= p^4$, nullam aliam radicem habebit prætèr ipſam l; ut in textu auctoris declaratur. Q. E. D.

F. M.

Quòd

Quòd ſi x, cùm major eſt quàm l, minor fuerit quàm n, erit et p^4 major quàm $mmxx + n^3x$, ideóque multo major quàm $mmxx + nnxx$, et conſequentèr $\frac{p^4}{mm+nn}$ major quàm xx, et $\surd\frac{p^4}{mm+nn}$ major quàm x. Quarè in hoc caſu x erit major quàm l, et minor quàm $\surd\frac{p^4}{mm+nn}$.

Quòd ſi verò x, cùm minor eſt quàm l, vel æqualis fuerit vel major quàm n, erit $mmxx + n^3x$ major quàm p^4, ideóque $mmxx + nnxx$ major quàm p^4, et xx major quàm $\frac{p^4}{mm+nn}$, et x major quàm $\surd\frac{p^4}{mm+nn}$. Ergò in hoc caſu erit x minor quàm l, et major quàm $\surd\frac{p^4}{mm+nn}$.

Poſtremò, cùm x, minor quàm l, etiàm minor ſit quàm n, erit $mmxx + n^3x$ major quàm p^4, et $mmnx + n^3x$ multo major quàm p^4, et per conſequens x major quàm $\frac{p^4}{mmn+n^3}$. Igitùr x in hoc caſu, minor erit quàm l, et major quàm $\frac{p^4}{mmn+n^3}$.

Quæ cum ita ſint, conſtat univerſalitèr, æquationem propoſitam non habere niſi unam veram radicem, quæ æqualis eſt ipſi l, quando eſt $llmm = p^4 - ln^3$, modò unaquæque radicum, (ſive unam tantùm, ſive tres, habuerit,) fuerit ſempèr neceſſariò inter maximum et minimum trium terminorum l, $\frac{p^4}{mmn+n^3}$, et $\surd\frac{p^4}{mm+nn}$.

PROP.

PROP. 10. $x^4 - lx^3 - mmxx - n^3x - p^4 = 0$.

Factis neceſſariis tranſpoſitionibus, demonſtrabitur, quòd x major eſt quàm l, m, n, et p. Deinde erit quoque, per tranſpoſitionem, $lx^3 + mmxx + n^3x + p^4 = x^4$, et per conſequens $lx^3 + mx^3 + nx^3 + px^3$ major quàm x^4, et $l + m + n + p$ major quàm x. Porrò, quoniam eſt $x^4 = lx^3 + mmxx + n^3x + p^4$, erit x^4 major quàm $lx^3 + mmxx$, et xx major quàm $lx + mm$, ideóque multo magìs x major erit quàm $\sqrt{ll + mm}$ et $\sqrt{lm + mm}$. Similitèr, cum x^3 major ſit quàm $lxx + mmx + n^3$, erit multo magìs major quàm $l^3 + m^3 + n^3$, $l^3 + lmm + n^3$, et $2lmm + n^3$, et ſic de reliquis terminis, quos ſubſtituere licet loco x^3, minores quàm x^3. Sic x^4 major eſt quàm $l^4 + m^4 + n^4$, et quàm $m^4 + n^4 + p^4$, et ſic de reliquis. Prætereà, quoniam x major eſt quàm n et p, et $lx^3 + mmxx + n^3x + p^4$ eſt $= x^4$, erit $lx^3 + mmxx + nnxx + ppxx$ major quàm x^4, ideóque $lx + mm + nn + pp$ major quàm xx aliquâ quantitate; quæ quidem quantitas, etiàmſi ſit incognita, ſi appelletur zz, habebitur $lx + mm + nn + pp = xx + zz$ *. Quantitas autem hæc incognita zz neceſſariò minor erit quàm $mm + nn +$

$nn +$

* In hâc Propoſitione, ſi mentem auctoris rectè ſum aſſecutus, correctiones quædam ſunt neceſſariæ, et ratiocinia quæ continentur in lineis 14tâ, 15tâ, 16tâ, et ſequentibus uſque ad finem propoſitionis, incipiendo a verbis hiſce, *Quantitas autèm hæc incognita zz*, poterunt pleniùs et clariùs exhiberi verbis ſequentibus.

Quantitas autèm hæc incognita zz neceſſariò minor erit quàm quantitas trinomia $mm + nn + pp$. Nam, ſi eſſet æqualis quantitati iſti trinomiæ, auferendo ex duabus partibus oppoſitis æquationis præcedentis $lx + mm + nn + pp = xx + zz$ quantitates æquales $mm + nn + pp$ et zz, haberemus æquationem $lx = xx$; quod fieri non poteſt, quia ſuprâ eſt oſtenſum quòd x eſt major quàm l. Si verò zz eſſet major quàm quantitas trinomia $mm + nn + pp$, auferendo ex duabus partibus oppoſitis æquationis præcedentis $lx + mm + nn + pp = xx + zz$, quantitates inæquales, $mm + nn + pp$ et zz, nempe, quantitatem minorem $mm + nn + pp$ à primâ ejus parte, et quantitatem majorem zz à ſecundâ ejus parte, reſiduum xx ex ſecundâ ſubtractione profectum foret minus quàm reſiduum lx profectum ex primâ ſubtractione; quod fieri non poteſt, ut jam obſervatum eſt, quoniam ſuprà eſt oſtenſum quòd x eſt major quàm l. Ergò zz eſt neceſſariò minor quàm quantitas trinomia $mm + nn + pp$. Auferatur jàm utrinque in æquatione $lx + mm + nn + pp = xx + zz$ quantitas lx; et habebimus æquationem $mm + nn + pp = xx + zz - lx$. Jam verò, quoniam oſtenſum eſt quod zz eſt minor quàm quantitas trinomia $mm + nn + pp$, auferatur utrinque in æquatione $mm + nn + pp = xx + zz - lx$; et habebimus æquationem $mm + nn + pp - zz = xx - lx$, ſeu $xx - lx = mm + nn + pp - zz$. Ergò, addendo utrinque quantitatem $\frac{ll}{4}$, erit quantitas trinomia $xx - lx + \frac{ll}{4}$ = quantitati quinquinomiæ $\frac{ll}{4} + mm + nn + pp - zz$, et pro-inde quantitas binomia $x - \frac{l}{2}$ erit $= \sqrt{\frac{ll}{4} + mm + nn + pp - zz}$, et quantitas x erit $= \frac{l}{2}$

$nn + pp$; nam aliàs, ablatis ex duabus partibus æquationis præcedentis quantitatibus æqualibus, aut ablatâ minori quantitate ex primâ et majori ex ſecundâ, eſſet reliqua lx aut æqualis aut major quàm xx. Quòd foret abſurdum, quandoquidem x demonſtrata eſt eſſe major quàm l. Quarè habemus hanc æquationem $xx = lx + mm + nn + pp - zz$, quæ erit realis, erítque $x = \frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm + nn + pp}$. Manifeſtum verò eſt, quòd $\frac{1}{4}ll + mm + nn + pp$ major eſt quàm $\frac{1}{4}ll + mm + nn + pp - zz$. Ergò $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm + nn + pp}$ major erit quàm x, ideóque $\sqrt{ll + mm + nn + pp}$ multo magìs major erit quàm x; ita ut radix propoſitæ æquationis neceſſariò ſit inter $\sqrt{ll + mm}$ et $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm + nn + pp}$.

PROP. 11. $x^4 - lx^3 - mmxx + n^3x - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - lx^3 - mmxx = p^4 - n^3x$. Unde, ſi fuerit $x^4 - lx^3 - mmxx = 0$, hoc eſt, omnibus per xx diviſis, $xx - lx - mm = 0$, erit quoque $p^4 - n^3x = 0$. Hoc eſt, ſi fuerit $xx = lx + mm$, vel $x = \frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$; erit et $p^4 = n^3x$, vel $x = \frac{p^4}{n^3}$. Quarè conſtat, ſi fuerit $\frac{p^4}{n^3} = \frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$, radicem æquationis propoſitæ fore æqualem ſingulis terminorum æqualium $\frac{p^4}{n^3}$ et $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$.

$= \frac{l}{2} + \sqrt{\frac{ll}{4} + mm + nn + pp - zz}$. Sed quantitas $\frac{ll}{4} + mm + nn + pp$ eſt major quàm quantitas $\frac{ll}{4} + mm + nn + pp - zz$, et pro-inde $\sqrt{\frac{ll}{4} + mm + nn + pp}$ eſt major quàm $\sqrt{\frac{ll}{4} + mm + nn + pp - zz}$, et $\frac{l}{2} + \sqrt{\frac{ll}{4} + mm + nn + pp}$ eſt major quàm $\frac{l}{2} + \sqrt{\frac{ll}{4} + mm + nn + pp - zz}$, atque ideò major eſt quàm x. Eſt igitùr x, ſeu radix æquationis propoſitæ $x^4 - lx^3 - mmxx - n^3x - p^4 = 0$, major quàm $\sqrt{ll + mm}$, ſed minor quàm $\frac{l}{2} + \sqrt{\frac{ll}{4} + mm + nn + pp}$, ut in auctoris textu declaratur.

F. M.

Quòd

Quòd ſi fuerit x^4 major quàm $lx^3 + mmxx$, hoc eſt, xx major quàm $lx + mm$; erit p^4 etiàm major quàm n^3x, hoc eſt, $\frac{p^4}{n^3}$ major quàm x. Jam, exiſtente xx majori quàm $lx + mm$, erit $xx = lx + mm$ plus aliquâ quantitate. Hæc autèm quantitas, etiamſi ſit incognita, vocetur zz: et tunc habebitur $xx = lx + mm + zz$, et $x = \frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm + zz}$, erítque x major quàm $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$. Quarè in hoc caſu erit x minor quàm $\frac{p^4}{n^3}$, et major quàm $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$.

Quòd ſi fuerit x^4 minor quàm $lx^3 + mmxx$, hoc eſt, xx minor quàm $lx + mm$; erit et p^4 minor quàm n^3x, hoc eſt, x major quàm $\frac{p^4}{n^3}$. Hinc, exiſtente xx minori quàm $lx + mm$, erit $xx = lx + mm$ minus aliquâ quantitate quæ ſi nominetur zz, habebitur $xx = lx + mm - zz$, hoc eſt, $x = \frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm - zz}$, erítque x minor quàm $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$. Ergò in hoc caſu x erit major quàm $\frac{p^4}{n^3}$, et minor quàm $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$.

Quarè univerſalitèr patet, radicem æquationis propoſitæ æqualem eſſe ipſi $\frac{p^4}{n^3}$ et $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$, quando $\frac{p^4}{n^3}$ æquatur ipſi $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$; Sìn ſecùs, quamlibet radicum, (ſive unam tantùm, ſive tres habuerit,) ſemper eſſe inter hóſce terminos $\frac{p^4}{n^3}$ et $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$.

PROP. 12. $x^4 + lx^3 + mmxx - n^3x - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - p^4 = n^3x - lx^3 - mmxx$; ideóque, ſi fuerit $x = p$, erit quoque $n^3x = lx^3 + mmxx$, et $x = \frac{n^3}{lp + mm}$. Unde conſtat, ſi $lpp + mmp$ æquetur n^3, radicem æquationis fore æqualem ſingulis terminorum æqualium p et $\frac{n^3}{lp + mm}$.

Quòd ſi fuerit x^4 major quàm p^4, hoc eſt, x major quàm p, erit quoque n^3x major quàm $lx^3 + mmxx$. Jam, ſi æquatio propoſita eſt realis, erit et x realis, et vel æqualis, vel major, vel minor quàm m. Quòd ſi fuerit

æqualis vel major quàm m, et eadem quantitas x etiàm major sit quàm p, quandoquidem et tunc n^3x major est quam $lx^3 + mmxx$, erit n^3x major quàm $lmxx + mmxx$, et $\frac{n^3}{lm+mm}$ major quàm x. Quarè in hoc casu erit x major quàm p, et minor quàm $\frac{n^3}{lm+mm}$.

Quòd si, existente x majore quàm p, ipsa minor sit quàm m, erit n^3x major quàm $lx^3 + mx^3$, et $\frac{n^3}{l+m}$ major quàm xx. Ergò in hoc casu erit x major quàm p, et minor quàm $\surd\frac{n^3}{l+m}$.

Quòd si verò, x minori existente quàm p, ipsa sit major quàm m, vel eidem æqualis, quandoquidem et tunc n^3x minor est quàm $lx^3 + mmxx$, erit n^3x minor quàm $lx^3 + mx^3$, hoc est, xx major quàm $\frac{n^3}{l+m}$. Quarè in hoc casu erit x minor quàm p, et major quàm $\surd\frac{n^3}{l+m}$.

Denique, cùm fuerit x minor quàm p, et ipsa etiàm minor quàm m, quoniam et tunc n^3x minor est quàm $lx^3 + mmxx$, erit n^3x minor quàm $lmxx + mmxx$, et x major quàm $\frac{n^3}{lm+mm}$.

Unde constat universalitèr, radicem æquationis propositæ esse æqualem singulis terminorum æqualium p et $\frac{n^3}{lp+mm}$, cùm est $lpp + mmp$ æqualis ipsi n^3; sed, cùm hæ quantitates inæquales sunt, esse radicem æquationis propositæ necessariò inter maximam et minimam trium terminorum p, $\surd\frac{n^3}{l+m}$, et $\frac{n^3}{lm+mm}$

PROP. 13. $x^4 + lx^3 + mmxx + n^3x - p^4 = 0$.

Factis necessariis transpositionibus, demonstrabitur x fore minorem quàm p, $\surd$ C.$\frac{p^4}{l}$, $\surd\frac{p^4}{mm}$, et $\frac{p^4}{n^3}$; at verò majorem quàm $\frac{p^4}{p^3+lpp+mmp+n^3}$*.

PROP.

* Addatur enim utrinque æquationi propositæ $x^4 + lx^3 + mmxx + n^3x - p^4 = 0$ quantitas p^4; et habebimus æquationem $x^4 + lx^3 + mmxx + n^3x = p^4$: Ergò x^4 erit minor quàm p^4, et pro-inde x erit minor quàm p; et lx^3 erit minor quàm p^4, et pro-inde x^3 erit minor quàm $\frac{p^4}{l}$, et x erit

PROP. 14. $x^4 + lx^3 - mmxx - n^3x - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 - n^3x = mmxx + p^4 - lx^3$; ideóque, ſi fuerit $x^4 = n^3x$, hoc eſt, $x = n$, erit $mmxx + p^4 = lx^3$, hoc eſt, $\frac{mmmn + p^4}{l} = x^3$. Quòd ſi fuerit x major quàm n, erit et $mmxx + p^4$ major quàm lx^3: ſìn minor fuerit, erit $mmxx + p^4$ minor quàm lx^3. Jam, ſi x major eſt quàm n, et etiàm vel æqualis vel major quàm p, quandoquidem et tunc $mmxx + p^4$ major eſt quàm lx^3, multo magìs erit $mmxx + ppxx$ major quàm lx^3, hoc eſt, $\frac{mm + pp}{l}$ major quàm x. Ergò in hoc caſu erit x major quàm n, et minor quàm $\frac{mm + pp}{l}$.

Quòd ſi x major fuerit quàm n, et etiàm minor quàm p, quoniam et tunc $mmxx + p^4$ major eſt quàm lx^3, erit quoque $mmpp + p^4$ major quàm lx^3, hoc eſt, x^3 minor quàm $\frac{mmpp + p^4}{l}$, et x minor quàm $\surd C \frac{mmpp + p^4}{l}$. Quarè in hoc caſu erit x major quàm n, et minor quàm $\surd C \frac{mmpp + p^4}{l}$.

x erit minor quàm $\surd^3 \frac{p^4}{l}$, ſeu radix cubica quantitatis $\frac{p^4}{l}$; et $mmxx$ erit minor quàm p^4, et pro-inde xx erit minor quàm $\frac{p^4}{mm}$, et x erit minor quàm $\surd \frac{p^4}{mm}$, ſeu quàm $\frac{pp}{m}$; et n^3x erit minor quàm p^4, et pro-inde x erit minor quàm $\frac{p^4}{n^3}$. Eſt igitùr x minor utrâvis harum quatuor quantitatum, p, $\surd C. \frac{p^4}{l}$, ſeu $\surd^3 \frac{p^4}{l}$, $\frac{p^4}{mm}$ ſeu $\frac{pp}{m}$, et $\frac{p^4}{n^3}$; ut in textu auctoris declaratur.

Porrò, quoniam x eſt minor quàm p, erit quantitas quadrinomia $x^4 + lx^3 + mmxx + n^3x$ minor quàm quantitas quadrinomia $p^3x + lppx + mmpx + n^3x$. Sed quantitas quadrinomia $x^4 + lx^3 + mmxx + n^3x$ eſt æqualis quantitati p^4. Ergò quantitas p^4 erit minor quàm quantitas quadrinomia $p^3x + lppx + mmpx + n^3x$, et pro-inde fractio $\frac{p^4}{p^3 + lpp + mmp + n^3}$ erit minor quàm x, et x erit major quàm fractio $\frac{p^4}{p^3 + lpp + mmp + n^3}$; ut in auctoris textu declaratur. Q. E. D.

F. M.

Quòd

Quòd ſi x minor fuerit quàm n, et vel æqualis vel major quàm p, quandoquidem et tunc $mmxx + p^4$ minor eſt quàm lx^3, erit quoque $mmpp + p^4$ minor quàm lx^3, et x major quàm $\sqrt{\text{C.}\frac{mmpp + p^4}{l}}$. Ergò in hoc caſu erit x minor quàm n, et major quàm $\sqrt{\text{C.}\frac{mmpp + p^4}{l}}$.

Quòd ſi verò x minor fuerit quàm n, et ipſa etiàm minor ſit quàm p, quoniam et tunc $mmxx + p^4$ minor eſt quàm lx^3; erit quoque $mmxx + ppxx$ minor quam lx^3, et $\frac{mm + pp}{l}$ minor quàm x. Quarè in hoc caſu, erit x minor quàm n, et major quàm $\frac{mm + pp}{l}$.

Unde univerſalitèr apparet, radicem æquationis propoſitæ neceſſariò eſſe inter maximum et minimum trium terminorum n, $\frac{mm + pp}{l}$, et $\sqrt{\text{C.}\frac{mmpp + p^4}{l}}$.

PROP. 15. $x^4 + lx^3 - mmxx + n^3x - p^4 = 0$.

Per tranſpoſitionem eſt $x^4 + lx^3 - mmxx = p^4 - n^3x$; ideóque, ſi fuerit $x^4 + lx^3 - mmxx = 0$, ſeu, (diviſis omnibus terminis per xx,) $xx + lx - mm = 0$, erit quoque $p^4 - n^3x = 0$, hoc eſt, ſi eſt $xx = -lx + mm$, vel $x = -\frac{1}{2}l + \sqrt{\frac{1}{2}ll + mm}$, erit $p^4 = n^3x$, ſeu $x = \frac{p^4}{n^3}$. Unde patet, ſi $\frac{p^4}{n^3}$ eſt æqualis ipſi $-\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$, radicem æquationis propoſitæ eſſe æqualem ſingulis terminorum æqualium $\frac{p^4}{n^3}$ et $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$.

Quòd ſi fuerit $x^4 + lx^3$ major quàm $mmxx$, hoc eſt, $xx + lx$ major quàm mm, erit p^4 quoque major quam n^3x, hoc eſt, $\frac{p^4}{n^3}$ major quàm x. Ac pro-inde, cum $xx + lx$ major ſit quàm mm, erit $xx + lx$ major quàm mm aliquâ quantitate. Quantitas autem hæc, licèt ſit incognita, vocetur zz; et tunc erit $xx + lx = mm + zz$, ſeu $xx = -lx + mm + zz$, hoc eſt, $x = -\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm + zz}$, ideóque x major quàm $-\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$. Ergò in hoc caſu x minor erit quàm $\frac{p^4}{n^3}$, et major quàm $-\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$, ſeu $\sqrt{\frac{1}{4}ll + mm} - \frac{l}{2}$.

Quòd

Quòd ſi fuerit $x^4 + lx^3$ minor quàm $mmxx$, hoc eſt, $xx + lx$ minor quàm mm, erit quoque p^4 minor quàm n^3x, hoc eſt, x major quàm $\frac{p^4}{n^3}$. Hinc, cum $xx + lx$ minor ſit quàm mm, erit $xx + lx$ minor quàm mm aliquâ quantitate. Vocetur quantitas hæc (quamvis incognita) zz; et erit $xx + lx + zz = mm$, vel $xx = -lx + mm - zz$, hoc eſt, $x = -\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm - zz}$, ideóque x minor erit quàm $-\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$. Quarè in hoc caſu erit x major quàm $\frac{p^4}{n^3}$, et minor quàm $-\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$.

Atque ita in genere perſpicuum eſt, cùm $\frac{p^4}{n^3}$ æquatur ipſi $-l + \sqrt{\frac{1}{4}ll + mm}$, radicem æquationis propoſitæ æqualem eſſe ſingulis terminorum æqualium $\frac{p^4}{n^3}$ et $-\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$; ſìn minus, quamlibet radicum æquationis propoſitæ, (ſive unam tantùm radicem ſive tres habuerit,) neceſſariò eſſe inter hóſce terminos $\frac{p^4}{n^3}$ et $-\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$, ſeu $\sqrt{\frac{1}{4}ll + mm} - \frac{l}{2}$.

Finis tractatûs FLORIMONDI DE BEAUNE *de Limitibus radicum Æquationum adfectarum, Quadraticarum, Cubicarum, et Biquadraticarum.*

RECENSIO

RECENSIO LIMITUM

Intrà quos radices veræ Æquationum Quadraticarum, Cubicarum, et Biquadraticarum, debent offendi,

In præcedente tractatu FLORIMONDI DE BEAUNE inventorum.

CAPUT I.

De Æquationum Quadraticarum, seu duarum dimensionum Limitibus.

PROP. 1.

De æquatione $xx - lx + mm = 0$, seu $lx - xx = mm$.

Hæc æquatio habet duas radices veras, quarum utraque est major quàm $\frac{mm}{l}$, sed minor quàm l.

PROP. 2.

De æquatione $xx - lx - mm = 0$, seu $xx - lx = mm$.

Hæc æquatio habet tantùm unam radicem veram. Et hæc radix unica est major quàm maxima harum duarum quantitatum, $\sqrt{ll + mm}$ et $\sqrt{lm + mm}$, sed minor quàm $l + m$.

PROP. 3.

De æquatione $xx + lx - mm = 0$, ſeu $xx + lx = mm$.

Hæc etiàm æquatio habet tantùm unam radicem veram. Et hæc radix unica eſt major quàm $\frac{mm}{l+m}$, at minor quàm utravis harum duarum quantitatum, m et $\frac{mm}{l}$.

CAPUT II.

De Limitibus Æquationum Cubicarum, ſeu trium dimenſionum, ſecundo termino carentium.

PROP. 1.

De æquatione $x^3 * - mmx + n^3 = 0$, ſeu $mmx * - x^3 = n^3$.

Hæc æquatio habet duas radices veras, quarum utraque eſt major quàm $\frac{n^3}{mm}$, ſed minor quàm m.

PROP. 2.

De æquatione $x^3 * - mmx - n^3 = 0$, ſeu $x^3 * - mmx = n^3$.

Hæc æquatio habet tantùm unam radicem veram. Et hæc radix unca erit major quàm utravis harum duarum quantitatum m et n, ſed minor quàm $\sqrt{mm + nn}$.

PROP. 3.

De æquatione $x^3 * + mmx - n^3 = 0$, ſeu $x^3 * + mmx = n^3$.

Hæc æquatio habet tantùm unam radicem veram. Et hæc radix unica eſt major quàm $\frac{n^3}{mm+nn}$, ſed minor quàm utravis harum duarum quantitatum, $\frac{n^3}{mm}$ et n.

CAPUT III.

De Limitibus Æquationum Cubicarum, penultimo termino carentium.

PROP. 1.

De æquatione $x^3 - lxx * + n^3 = 0$, ſeu $lxx - x^3 = n^3$.

Hæc æquatio habet duas radices veras, quarum utraque eſt major quàm $\sqrt{\frac{n^3}{l}}$, ſed minor quàm l.

PROP. 2.

De æquatione $x^3 - lxx * - n^3 = 0$, ſeu $x^3 - lxx * = n^3$.

Hæc æquatio habet tantùm unam radicem veram. Et hæc radix unica eſt major quàm utravis duarum quantitatum l et n, ſed minor quàm $l + n$.

PROP. 3.

De æquatione $x^3 + lxx * - n^3 = 0$, ſeu $x^3 + lxx * = n^3$.

Hæc æquatio habet tantùm unam radicem veram. Et hæc radix unica eſt major quàm $\sqrt{\frac{n^3}{l+n}}$, ſed minor quàm utravis harum duarum quantitatum, $\sqrt{\frac{n^3}{l}}$ et l.

CAPUT IV.

De Æquationibus Cubicis, in quibus omnes termini extant.

PROP. 1.

De æquatione $x^3 - lxx + mmx - n^3 = 0$, ſeu $x^3 - lxx + mmx = n^3$.

Hæc æquatio habebit in quibuſdam caſibus tantùm unam radicem veram, nempe, cum n^3 eſt aut æqualis lmm, aut ipſâ major; et in aliis caſibus habere poterit tres radices veras, nempe, cum n^3 eſt minor quàm lmm. Si n^3 ſit æqualis lmm, radix unica hujus æquationis erit æqualis quantitati l; et, ſi n^3 ſit major quàm lmm, hæc radix unica erit major quàm l, ſed minor quàm $\frac{n^3}{mm}$. Si verò n^3 ſit minor quàm lmm, et æquatio $x^3 - lxx + mmx = n^3$ habeat tres radices veras, utravis harum trium radicum erit major quàm $\frac{n^3}{mm}$, ſed minor quàm l.

PROP. 2.

De æquatione $x^3 + lxx - mmx - n^3 = 0$, ſeu $x^3 + lxx - mmx = n^3$.

Hæc æquatio habet tantùm unam radicem veram. Et, ſi m ſit æqualis $\sqrt{\frac{n^3}{l}}$, ſeu lmm ſit æqualis n^3, hæc radix unica erit $= m$, ſeu $\sqrt{\frac{n^3}{l}}$; et, ſi m et $\sqrt{\frac{n^3}{l}}$ non fuerint æquales, hæc radix unica erit major minore harum duarum quantitatum m et $\sqrt{\frac{n^3}{l}}$, ſed minor earundem majore. Et prætereà hæc radix unica erit major minore aliarum duarum quantitatum, ſcilicet, n et $\frac{mm}{l}$, ſed minor earundem majore.

PROP.

Prop. 3.

De æquatione $x^3 - lxx - mmx - n^3 = 0$, ſeu $x^3 - lxx - mmx = n^3$.

Hæc æquatio habet tantùm unam radicem veram. Et hæc radix unica eſt major quàm utravis trium quantitatum l, m, et n, ſed minor quàm earum ſumma, ſeu $l + m + n$.

Prop. 4.

De æquatione $x^3 + lxx + mmx - n^3 = 0$, ſeu $x^3 + lxx + mmx = n^3$.

Hæc etiàm æquatio habet tantum unam radicem veram. Et hæc radix unica eſt major quàm $\frac{n^3}{nn + ln + mm}$, ſed minor quàm quælibet trium quantitatum $\sqrt{\frac{n^3}{l}}$, $\frac{n^3}{mm}$, et n.

Prop. 5.

De æquatione $x^3 - lxx + mmx + n^3 = 0$, ſeu $lxx - x^3 - mmx = n^3$.

Hæc æquatio poteſt habere duas radices veras. Harum autèm quælibet erit major quàm $\frac{mm}{l}$, ſed minor quàm l.

Prop. 6.

De æquatione $x^3 + lxx - mmx + n^3 = 0$, ſeu $mmx - lxx - x^3 = n^3$.

Hæc æquatio poteſt etiàm habere duas radices veras. Et harum quælibet erit major quàm $\frac{n^3}{mm}$, ſed minor quàm quælibet harum duarum quantitatum $\frac{mm}{l}$ et m.

Prop. 7.

De æquatione $x^3 - lxx - mmx + n^3 = 0$, ſeu $mmx + lxx - x^3 = n^3$.

Hæc æquatio poteſt etiàm habere duas radices veras. Et, ſi lmm fuerit $= n^3$, una harum radicum erit æqualis quantitati l, et altera æqualis quantitati m.

Et, ſi una harum radicum ſit major quàm l, erit etiàm major quàm $\frac{n^3}{mm}$; et, ſi ſit minor quàm l, minor erit etiàm quàm $\frac{n^3}{mm}$.

Et, ſi una harum radicum ſit major quàm m, erit etiàm major quàm $\sqrt{\frac{n^3}{l}}$; et, ſi ſit minor quàm m, erit etiàm minor quàm $\frac{n^3}{l}$.

Et, ſi una harum radicum ſit major quàm m, erit tamen minor quàm $l+m$; et, ſi ſit minor quàm m, erit major quàm fractio $\frac{n^3}{lm+mm}$.

CAPUT V.

De Æquationibus quatuor dimenſionum, ſecundo et tertio termino carentibus.

Prop. 1.

De æquatione $x^4 * * - n^3x + p^4 = 0$, ſeu $n^3x - x^4 = p^4$.

Hæc æquatio poteſt habere duas radices veras. Harum autèm utraque erit major quàm $\frac{p^4}{n^3}$, ſed minor quàm n.

PROP. 2.

De æquatione $x^4 * * - n^3x - p^4 = 0$, ſeu $x^4 - n^3x = p^4$.

Hæc æquatio habet tantùm unam radicem veram. Et hæc radix unica erit major quàm alterutra duarum quantitatum n et p, ſed minor quàm $\sqrt{nn + pp}$, et pro-inde multo minor quàm $n + p$.

PROP. 3.

De æquatione $x^4 * * + n^3x - p^4 = 0$, ſeu $x^4 + n^3x = p^4$.

Hæc æquatio habet etiàm tantùm unam radicem veram. Et hæc radix uniea eſt major quàm quantitas $\frac{p^4}{p^3 + n^3}$, ſed minor quàm alterutra duarum quantitatum $\frac{p^4}{n^3}$ et p.

CAPUT VI.

De Æquationibus quatuor dimenſionum, in quibus tertius et quartus terminus deficiunt.

PROP. 1.

De æquatione $x^4 - lx^3 * * + p^4 = 0$, ſeu $lx^3 - x^4 = p^4$.

Hæc æquatio poteſt habere duas radices veras. Et harum radicum utraque erit major quàm $\sqrt[3]{\frac{p^4}{l}}$, et quàm $\frac{pp}{l}$, ſed minor quàm l.

PROP.

Prop. 2.

De æquatione $x^4 - lx^3 * * - p^4 = 0$, ſeu $x^4 - lx^3 = p^4$.

Hæc æquatio habet tantùm unam radicem veram. Et hæc radix unica eſt major quàm l et quàm p, ſed minor quàm $l + p$.

Prop. 3.

De æquatione $x^4 + lx^3 * * - p^4 = 0$, ſeu $x^4 + lx^3 = p^4$.

Hæc æquatio habet etiàm tantùm unam radicum veram. Et hæc radix unica eſt major quàm $\sqrt[3]{\frac{p^4}{l+p}}$, ſed minor quàm p, et quàm $\sqrt[3]{\frac{p^4}{l}}$.

CAPUT VII.

De Æquationibus quatuor dimenſionum, ſecundo termino carentibus.

Prop. 1.

De æquatione $x^4 * - mmxx + n^3x - p^4 = 0$, ſeu $x^4 - mmxx + n^3x = p^4$.

Hæc æquatio aliquandò habet tantùm unam radicem veram, ſed aliquandò habet tres radices veras, ſecundùm diverſas rationes quantitatum m, n, et p ad ſe invicèm. Si quantitas m fuerit æqualis quantitati $\frac{p^4}{n^3}$, ſeu ſi mn^3, fuerit $= p^4$, æquatio $x^4 - mmxx + n^3x = p^4$ habebit tantùm unam radicem; et hæc radix unica erit æqualis quantitati m, vel quantitati $\frac{p^4}{n^3}$. Et, ſi m fuerit minor

quàm

quàm $\frac{p^4}{n^3}$, ſeu mn^3 fuerit minor quàm p^4, hæc æquatio habebit etiàm tantùm unam radicem veram: et hæc radix unica erit major quàm m. Si verò m ſit major quàm $\frac{p^4}{n^3}$, ſeu mn^3 ſit major quàm p^4, hæc æquatio poterit habere tres radices veras: et unaquæque harum trium radicum erit major quàm $\frac{p^4}{n^3}$, ſed minor quàm m.

PROP. 2.

De æquatione $x^4 * + mmxx - n^3x - p^4 = 0$, ſeu $x^4 + mmxx - n^3x = p^4$.

Hæc æquatio habet tantùm unam radicem veram. Et, ſi n fuerit $= \sqrt{\frac{p^4}{mm}}$, ſeu $\frac{pp}{m}$, hæc radix unica erit æqualis ipſi n vel $\frac{pp}{m}$; ſi verò n fuerit major quàm $\frac{pp}{m}$, hæc radix unica erit etiàm major quàm $\frac{pp}{m}$, ſed minor quàm n; et, ſi n fuerit minor quàm $\frac{pp}{m}$, hæc radix unica erit major quàm n, ſed minor quàm $\frac{pp}{m}$.

Porrò, ſi p fuerit $= \frac{n^3}{mm}$, hæc radix unica erit æqualis quantitati p, ſeu quantitati $\frac{n^3}{mm}$; ſi verò p fuerit major quàm $\frac{n^3}{mm}$, hæc radix unica erit etiàm major quàm $\frac{n^3}{mm}$, ſed minor quàm p; et, ſi p fuerit minor quàm $\frac{n^3}{mm}$, hæc radix unica erit major quàm p, ſed minor quàm $\frac{n^3}{mm}$.

Ergò hæc radix unica aut eſt æqualis quantitati n, vel quantitati p, aut eſt mediæ cujuſdam magnitudinis inter quantitates n et $\frac{pp}{m}$, et inter quantitates p et $\frac{n^3}{mm}$.

Prop. 3.

De æquatione $x^4 * - mmxx - n^3x - p^4 = 0$, ſeu $x^4 - mmxx - n^3x = p^4$.

Hæc æquatio habet etiàm tantùm unam radicem. Et hæc radix unica eſt major quàm m, et quàm n, et quàm p, ſed minor quàm $m + n + p$, et quàm $\sqrt{mm + nn + pp}$.

Prop. 4.

De æquatione $x^4 * + mmxx + n^3x - p^4 = 0$, ſeu $x^4 + mmxx + n^3x = p^4$.

Hæc æquatio habet etiàm tantùm unam radicem veram. Et hæc radix unica eſt major quàm $\frac{p^4}{p^3+mmp+n^3}$, ſed minor quàm $\frac{p^4}{n^3}$, et quàm $\frac{pp}{m}$, et quàm p.

Prop. 5.

De æquatione $x^4 * - mmxx + n^3x + p^4 = 0$, ſeu $mmxx - x^4 - n^3x = p^4$.

Hæc æquatio poteſt habere duas radices veras. Harum autèm radicum utraque erit major quàm $\frac{n^3}{mm}$ et quàm $\frac{pp}{m}$, ſed minor quàm m.

Prop. 6.

De æquatione $x^4 * + mmxx - n^3x + p^4 = 0$, ſeu $n^3x - mmxx - x^4 = p^4$.

Hæc etiàm æquatio poteſt habere duas radices veras. Harum autèm radicum utraque erit major quàm $\frac{p^4}{n^3}$, ſed minor quàm $\frac{n^3}{mm}$ et quàm n.

Prop.

PROP. 7.

De æquatione $x^4 * - mmxx - n^3x + p^4 = 0$, ſeu $n^3x + mmxx - x^4 = p^4$.

Hæc etiàm æquatio poteſt habere duas radices veras. Harum autèm radicum utraque erit major quàm fractio $\frac{p^4}{mm \times \sqrt{mm + nn} + n^3}$ et quàm minor duarum quantitatum n et $\frac{p^4}{mmn + n^3}$, ſed minor quàm $\sqrt{mm + nn}$.

CAPUT VIII.

De Æquationibus quatuor dimenſionum, penultimo termino carentibus.

PROP. 1.

De æquatione $x^4 - lx^3 + mmxx * - p^4 = 0$, ſeu $x^4 - lx^3 + mmxx = p^4$.

Hæc æquatio quandóque habet tantùm unam radicem veram, quandóque autèm tres, proùt rationes quantitatum l, m, et p ad ſe invicèm fuerint diverſæ. Si l ſit $= \frac{pp}{m}$, ſeu ll ſit $= \frac{p^4}{mm}$, ſeu $llmm$ ſit $= p^4$, hæc æquatio habebit tantùm unam radicem, et hæc radix unica erit $= l$, ſeu $\frac{pp}{m}$; et, ſi l ſit minor quàm $\frac{pp}{m}$, ſeu $llmm$ ſit minor quàm p^4, æquatio paritèr habebit tantùm unam radicem, et hæc radix unica erit major quàm l, ſed minor quàm $\frac{pp}{m}$. Si verò l ſit major quàm $\frac{pp}{m}$, ſeu $llmm$ ſit major quàm p^4, æquatio poterit habere tres radices veras: et harum unaquæque erit major quàm $\frac{pp}{m}$, ſed minor quàm l.

Prop. 2.

De æquatione $x^4 + lx^3 - mmxx * - p^4 = o$, ſeu $x^4 + lx^3 - mmxx = p^4$.

Hæc æquatio habet tantùm unam radicem. Et, ſi m ſit $= \sqrt[3]{\frac{p^4}{l}}$, ſeu m^3 ſit $= \frac{p^4}{l}$, ſeu lm^3 ſit $= p^4$, hæc radix unica erit $= m$ ſeu $\sqrt[3]{\frac{p^4}{l}}$; et, ſi m non ſit $= \sqrt[3]{\frac{p^4}{l}}$, hæc radix unica erit minor minore duarum quantitatum m et $\sqrt[3]{\frac{p^4}{l}}$, ſed minor majore earundem.

Porrò, ſi p ſit $= \frac{mm}{l}$, hæc radix unica erit $= p$, ſeu $\frac{mm}{l}$; et, ſi quantitates p et $\frac{mm}{l}$ non fuerint æquales, hæc radix unica erit major minore, ſed minor majore, earundem.

Prop. 3.

De æquatione $x^4 - lx^3 - mmxx * - p^4 = o$, ſeu $x^4 - lx^3 - mmxx = p^4$.

Hæc etiàm æquatio habet tantùm unam radicem veram. Et hæc radix unica eſt major quàm l, et quàm m, et quàm p, et quàm $\sqrt{ll + mm}$, et quàm $\sqrt[4]{l^4 + p^4}$, et quàm $\sqrt[4]{m^4 + p^4}$, et quàm $\sqrt[4]{mmpp + p^4}$, ſed minor quàm $l + m + p$.

Prop. 4.

De æquatione $x^4 + lx^3 + mmxx * - p^4 = o$, ſeu $x^4 + lx^3 + mmxx = p^4$.

Hæc etiàm æquatio habet tantùm unam radicem veram. Et hæc radix unica erit major quàm $\sqrt{\frac{p^4}{pp + lp + mm}}$, ſeu $\frac{pp}{\sqrt{pp + lp + mm}}$, ſed minor quàm $\frac{pp}{m}$, et quàm $\sqrt[3]{\frac{p^4}{l}}$, et quàm p.

Prop.

PROP. 5.

De æquatione $x^4 - lx^3 + mmxx * + p^4 = 0$, ſeu $lx^3 - x^4 - mmxx = p^4$.

Hæc æquatio poteſt habere duas radices veras. Harum autèm radicum utraque erit major quàm $\frac{mm}{l}$ et quàm $\sqrt[3]{\frac{p^4}{l}}$, ſed minor quàm l.

PROP. 6.

De æquatione $x^4 + lx^3 - mmxx * + p^4 = 0$, ſeu $mmxx - lx^3 - x^4 = p^4$.

Hæc etiàm æquatio poteſt habere duas radices veras. Harum autèm radicum utraque erit major quàm $\frac{pp}{m}$, ſed minor quàm $\frac{mm}{l}$ et quàm m.

PROP. 7.

De æquatione $x^4 - lx^3 - mmxx * + p^4 = 0$, ſeu $mmxx + lx^3 - x^4 = p^4$.

Hæc etiàm æquatio poteſt habere duas radices veras. Et utraque harum radicum erit major quàm $\sqrt{\frac{p^4}{ll + lm + mm}}$, et quàm $\frac{pp}{l+m}$, et quàm $\sqrt{\frac{p^4}{lm + mm}}$, ſed minor quàm $l + m$.

CAPUT

CAPUT IX.

De Limitibus Æquationum quatuor dimenſionum, tertio termino carentium.

PROP. 1.

De æquatione $x^4 - lx^3 * + n^3x - p^4 = 0$, ſeu $x^4 - lx^3 + n^3x = p^4$.

Hæc æquatio habet quandóque tantùm unam radicem veram, ſed quandóque tres radices veras, proùt rationes quantitatum l, n, et p ad ſe invicèm fuerint diverſæ. Si l ſit $= \frac{p^4}{n^3}$, ſeu ln^3 ſit $= p^4$, hæc æquatio habebit tantùm unam radicem, et hæc radix unica erit $= l$, vel $\frac{p^4}{n^3}$; et, ſi l ſit minor quàm $\frac{p^4}{n^3}$, ſeu ln^3 ſit minor quàm p^4, hæc æquatio paritèr habebit tantùm unam radicem, et hæc radix unica erit major quàm l, ſed minor quàm $\frac{p^4}{n^3}$. Si verò l ſit major quàm $\frac{p^4}{n^3}$, ſeu ln^3 ſit major quàm p^4, hæc æquatio poterit habere tres radices veras; et unaquæque harum trium radicum erit major quàm $\frac{p^4}{n^3}$, ſed minor quàm l.

PROP. 2.

De æquatione $x^4 + lx^3 * - n^3x - p^4 = 0$, ſeu $x^4 + lx^3 - n^3x = p^4$.

Hæc æquatio habet tantùm unam radicem veram. Et, ſi n ſit $= \sqrt[3]{\frac{p^4}{l}}$, ſeu ln^3 ſit $= p^4$, hæc radix unica erit $= n$, vel $\sqrt[3]{\frac{p^4}{l}}$; et, ſi n ſit major quàm $\sqrt[3]{\frac{p^4}{l}}$, ſeu ln^3 ſit major quàm p^4, hæc radix unica erit major quàm $\sqrt[3]{\frac{p^4}{n^3}}$, ſed minor quàm n; et, ſi n ſit minor quàm $\sqrt[3]{\frac{p^4}{l}}$, ſeu ln^3 ſit minor quàm p^4, hæc radix unica erit major quàm n, ſed minor quàm $\sqrt[3]{\frac{p^4}{l}}$.

Porrò,

Porrò, ſi p fuerit $= \sqrt{\frac{n^3}{l}}$, ſeu lpp fuerit $= n^3$, hæc radix unica erit $= p$ vel $\sqrt{\frac{n^3}{l}}$. Et, ſi p ſit major quàm $\sqrt{\frac{n^3}{l}}$, ſeu lpp ſit major quàm n^3, hæc radix unica erit etiàm major quàm $\sqrt{\frac{n^3}{l}}$, ſed minor quàm p; et, ſi p ſit minor quàm $\sqrt{\frac{n^3}{l}}$, ſeu lpp ſit minor quàm n^3, hæc radix unica erit major quàm p, ſed minor quàm $\sqrt{\frac{n^3}{l}}$.

Prop. 3.

De æquatione $x^4 - lx^3 * - n^3x - p^4 = 0$, ſeu $x^4 - lx^3 - n^3x = p^4$.

Hæc etiàm æquatio habet tantùm unam radicem veram. Et hæc radix unica erit major quàm l, et quàm n, et quàm p, et quàm $\sqrt{^3 l^3 + n^3}$, et quàm $\sqrt{^4 l^4 + p^4}$, et quàm $\sqrt{^4 ln^3 + p^4}$, ſed minor quàm $l + n + p$.

Prop. 4.

De æquatione $x^4 + lx^3 * + n^3x - p^4 = 0$, ſeu $x^4 + lx^3 + n^3x = p^4$.

Hæc etiàm æquatio habet tantùm unam radicem veram. Et hæc radix unica erit major quàm $\frac{p^4}{p^3 + lpp + n^3}$, ſed minor quàm $\frac{p^4}{n^3}$, et quàm $\sqrt{^3\frac{p^4}{l}}$, et quàm p.

Prop. 5.

De æquatione $x^4 - lx^3 * + n^3x + p^4 = 0$, ſeu $lx^3 - x^4 - n^3x = p^4$.

Hæc æquatio poteſt habere duas radices veras. Et harum radicum utraque erit major quàm $\sqrt{\frac{n^3}{l}}$ et quàm $\sqrt{^3\frac{p^4}{l}}$, ſed minor quàm l.

PROP. 6.

De æquatione $x^4 + lx^3 * - n^3x + p^4 = 0$, ſeu $n^3x - x^4 - lx^3 = p^4$.

Hæc etiàm æquatio poteſt habere duas radices veras. Et utraque harum duarum radicum erit major quàm $\frac{p^4}{n^3}$, ſed minor quàm $\sqrt{\frac{n^3}{l}}$ et quàm n.

PROP. 7.

De æquatione $x^4 - lx^3 * - n^3x + p^4 = 0$, ſeu $n^3x + lx^3 - x^4 = p^4$.

Hæc etiàm æquatio poterit habere duas radices veras. Et utraque harum duarum radicum erit major quàm $\frac{p^4}{l^3 + 3l^2n + 3lnn + n^3}$, ſeu $\frac{p^4}{\overline{l+n}^3}$, et quàm $\frac{p^4}{l^3 + 2lln + lnn + n^3}$, et quàm minor horum duorum terminorum, n et $\frac{p^4}{lnn + n^3}$, ſed minor quàm $l + n$.

CAPUT X.

De Limitibus Æquationum quatuor dimenſionum, in quibus nullus terminus deeſt.

PROP. 1.

De æquatione $x^4 - lx^3 + mmxx - n^3x + p^4 = 0$, ſeu $n^3x - mmxx + lx^3 - x^4 = p^4$.

Hæc æquatio poteſt habere quatuor radices veras. Et quælibet harum radicum erit major quàm $\frac{p^4}{\overline{l+n}^3}$, ſeu $\frac{p^4}{l^3 + 3lln + 3lnn + n^3}$, et quàm $\frac{p^4}{l^3 + 2lln + lnn + n^3}$, et quàm minor horum duorum terminorum, n et $\frac{p^3}{lnn + n^3}$, ſed minor quàm $l + n$.

PROP.

PROP. 2.

De æquatione $x^4 - lx^3 + mmxx + n^3x + p^4 = 0$, ſeu $lx^3 - x^4 - mmxx - n^3x = p^4$.

Hæc æquatio poterit habere duas radices veras. Et utraque harum duarum radicum erit major quàm $\frac{mm}{l}$, et quàm $\sqrt{\frac{n^3}{l}}$, et quàm $\sqrt[3]{\frac{p^4}{l}}$, ſed minor quàm l.

PROP. 3.

De æquatione $x^4 - lx^3 - mmxx - n^3x + p^4 = 0$, ſeu $n^3x + mmxx + lx^3 - x^4 = p^4$.

Hæc etiàm æquatio poterit habere duas radices veras. Harum autèm utraque, ſi m ſit vel æqualis vel major quàm n, erit major minore duorum terminorum m et $\frac{p^4}{lmm + m^3 + n^3}$; et, ſi m ſit minor quàm n, erit major minore duorum terminorum n et $\frac{p^4}{lnn + mmn + n^3}$, ſed ſemper erit minor quàm $l + m + n$.

PROP. 4.

De æquatione $x^4 - lx^3 - mmxx + n^3x + p^4 = 0$, ſeu $lx^3 + mmxx - x^4 - n^3x = p^4$.

Hæc etiàm æquatio poterit habere duas radices veras. Et harum radicum utraque erit major quàm minor duorum terminorum m et $\sqrt{\frac{p^4}{lm + mm}}$, ſed minor quàm $l + m$.

PROP. 5.

De æquatione $x^4 + lx^3 + mmxx - n^3x + p^4 = 0$, ſeu $n^3x - mmxx - lx^3 - x^4 = p^4$.

Hæc etiàm æquatio poterit habere duas radices veras. Harum autèm radicum utraque erit major quàm $\frac{p^4}{n^3}$, ſed minor quàm n, et quàm $\sqrt{\frac{n^3}{l}}$.

Prop. 6.

De æquatione $x^4 + lx^3 - mmxx - n^3x + p^4 = 0$, ſeu $n^3x + mmxx - lx^3 - x^4 = p^4$.

Hæc etiàm æquatio poteſt habere duas radices veras. Et harum radicum utraque erit major quàm minor horum duorum terminorum n et $\frac{p^4}{mmn + n^3}$, ſed minor quàm $\sqrt{mm + nn}$.

Prop. 7.

De æquatione $x^4 + lx^3 - mmxx + n^3x + p^4 = 0$, ſeu $mmxx - n^3x - lx^3 - x^4 = p^4$.

Hæc etiàm æquatio poteſt habere duas radices veras. Et harum radicum utraque erit major quàm $\frac{n^3}{mm}$ et quàm $\frac{pp}{m}$, ſed minor quàm m et quàm $\frac{mm}{l}$.

Prop. 8.

De æquatione $x^4 - lx^3 + mmxx - n^3x - p^4 = 0$, ſeu $x^4 - lx^3 + mmxx - n^3x = p^4$.

Hæc æquatio quandóque habet tantùm unam radicem veram, quandóque autèm tres, proùt rationes quantitatum l, m, n, et p, ad ſe invicèm fuerint diverſæ. Si $llmm$ ſit $= ln^3 + p^4$, ſeu $llmm - ln^3$ ſit $= p^4$, hæc æquatio habebit tantùm unam radicem veram; et hæc radix unica erit $= l$. Et, ſi x ſit major quàm l et major quàm p, erit minor quàm $\frac{n^3 + p^3}{mm}$.

In aliis caſibus hujus æquationis $x^4 - lx^3 + mmxx - n^3x = p^4$ Limites ejus radicum non ſunt ab auctore inventi.

Prop. 9.

De æquatione $x^4 - lx^3 + mmxx + n^3x - p^4 = 0$, ſeu $x^4 - lx^3 + mmxx + n^3x = p^4$.

Hæc etiàm æquatio habet quandóque tantùm unam radicem veram, quandóque autèm tres, proùt rationes quantitatum l, m, n, et p, ad ſe invicèm fuerint

erint diverſæ. Si l ſit æqualis $\sqrt{\frac{p^4 - ln^3}{mm}}$, ſeu ll ſit $= \frac{p^4 - ln^3}{mm}$, ſeu $llmm$ ſit $= p^4 - ln^3$, ſeu p^4 ſit $= llmm + ln^3$, hæc æquatio habebit tantùm unam radicem, et hæc radix unica erit $= l$, vel $\sqrt{\frac{p^4 - ln^3}{mm}}$. Et, ſi p^4 ſit major quàm $llmm + ln^3$, hæc æquatio habebit etiàm tantùm unam radicem; et hæc radix unica erit major quàm l.

Et, ſi p^4 ſit major quàm $llmm + ln^3$, et æquatio propoſita $x^4 - lx^3 + mmxx + n^3x - p^4 = 0$, ſeu $x^4 - lx^3 + mmxx + n^3x = p^4$, habeat pro-inde tantùm unam radicem, et hæc radix unica eſt major quàm l; tunc, ſi hæc radix unica (quæ eſt major quàm l,) ſit etiàm major quàm n, erit tamen minor quàm $\frac{p^4}{mmn + n^3}$. Si verò (quanquàm major ſit quàm l,) ſit minor quàm n, erit etiàm minor quàm $\sqrt{\frac{p^4}{mm + nn}}$. Ergò Limites hujus radicis unicæ erunt aut l et $\frac{p^4}{mmn + n^3}$, aut l et $\sqrt{\frac{p^4}{mm + nn}}$, proùt hæc radix unica fuerit major vel minor quàm n.

Si verò p^4 ſit minor quàm $llmm + ln^3$, et pro-inde $p^4 - ln^3$ ſit minor quàm $llmm$, et $\frac{p^4 - ln^3}{mm}$ ſit minor quàm ll, et $\sqrt{\frac{p^4 - ln^3}{mm}}$ ſit minor quàm l, æquatio propoſita $x^4 - lx^3 + mmxx + n^3x - p^4 = 0$, vel $x^4 - lx^3 + mmxx + n^3x = p^4$, poterit habere tres radices veras. Et harum trium radicum unaquæque erit minor quàm l. Et, ſi x (licet minor quàm l,) ſit tamen major quàm n, erit etiàm major quàm $\sqrt{\frac{p^4}{mm + nn}}$; et, ſi x (quæ minor eſt quàm l,) ſit etiàm minor quàm n, erit tamen major quàm $\frac{p^4}{mmn + n^3}$; hoc eſt, unquæque harum trium radicum quæ fuerit major quàm n, erit etiàm major quàm $\sqrt{\frac{p^4}{mm + nn}}$, et unaquæque earum quæ fuerit minor quàm n, erit tamen major quàm $\frac{p^4}{mmn + n^3}$.

Prop. 10.

De æquatione $x^4 - lx^3 - mmxx - n^3x - p^4 = 0$, ſeu $x^4 - lx^3 - mmxx - n^3x = p^4$.

Hæc æquatio habet tantùm unam radicem veram. Et hæc radix unica eſt major quàm l, et quàm m, et quàm n, et quàm p, ſed minor quàm $l + m + n + p$. Eſt etiàm major quàm $\sqrt{ll + mm}$ et quàm $\sqrt{lm + mm}$, et quàm $\sqrt[3]{l^3 + m^3 + n^3}$, et quàm $\sqrt[3]{l^3 + lmm + n^3}$, et quàm $\sqrt[3]{2lmm + n^3}$, et quàm $\sqrt[4]{l^4 + m^4 + n^4}$, et quàm $\sqrt[4]{m^4 + n^4 + p^4}$, ſed minor quàm $\frac{l}{2} + \sqrt{\frac{ll}{4} + mm + nn + pp}$.

Prop. 11.

De æquatione $x^4 - lx^3 - mmxx + n^3x - p^4 = 0$, ſeu $x^4 - lx^3 - mmxx + n^3x = p^4$.

Hæc æquatio quandóque habet tantùm unam radicem veram, quandóque autèm tres, proùt rationes quantitatum l, m, n, et p, ad ſe invicèm fuerint diverſæ. Si $\frac{p^4}{n^3}$ ſit $= \frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$, hæc æquatio habebit tantùm unam radicem; et hæc radix unica erit $= \frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$, ſeu $\frac{p^4}{n^3}$. Et, ſi $\frac{p^4}{n^3}$ ſit major quàm $\frac{1}{2}l + \sqrt{\frac{1}{4}ll + mm}$, ſeu p^4 ſit major quàm $\frac{ln^3}{2} + \sqrt{\frac{lln^6}{4} + mmn^6}$, hæc æquatio habebit etiàm tantùm unam radicem; et hæc radix unica erit major quàm $\frac{l}{2} + \sqrt{\frac{ll}{4} + mm}$, ſed minor quàm $\frac{p^4}{n^3}$. Si verò $\frac{p^4}{n^3}$ ſit minor quàm $\frac{l}{2} + \sqrt{\frac{ll}{4} + mm}$, hæc æquatio poterit habere tres radices; et harum radicum unaquæque erit major quàm $\frac{p^4}{n^3}$, ſed minor quàm $\frac{l}{2} + \sqrt{\frac{ll}{4} + mm}$.

Prop.

Prop. 12.

De æquatione $x^4 + lx^3 + mmxx - n^3x - p^4 = 0$, ſeu $x^4 + lx^3 + mmxx - n^3x = p^4$.

Hæc æquatio habet tantùm unam radicem veram. Et, cùm p eſt $= \frac{n^3}{lp + mm}$, ſeu $lpp + mmp$ eſt $= n^3$, hæc radix unica erit æqualis p vel $\frac{n^3}{lp + mm}$. Et, cùm p non eſt æqualis $\frac{n^3}{lp + mm}$, ſeu $lpp + mmp$ non eſt æqualis n^3, hæc radix unica erit mediæ cujuſdem magnitudinis inter maximam et minimam harum trium quantitatum, p, $\surd \frac{n^3}{l + m}$, et $\frac{n^3}{lm + mm}$.

Prop. 13.

De æquatione $x^4 + lx^3 + mmxx + n^3x - p^4 = 0$, ſeu $x^4 + lx^3 + mmxx + n^3x = p^4$.

Hæc æquatio habet tantùm unam radicem veram. Et hæc radix unica eſt major quàm $\frac{p^4}{p^3 + lpp + mmp + n^3}$, ſed minor quàm p, et quàm $\surd^3\sqrt{\frac{p^4}{l}}$, et quàm $\surd\sqrt{\frac{p^4}{mm}}$, ſeu $\frac{pp}{m}$, et quàm $\frac{p^4}{n^3}$.

Prop. 14.

De æquatione $x^4 + lx^3 - mmxx - n^3x - p^4 = 0$, ſeu $x^4 + lx^3 - mmxx - n^3x = p^4$.

Hæc etiàm æquatio habet tantùm unam radicem veram. Et hæc radix unica erit mediæ cujuſdem magnitudinis intèr maximam et minimam harum trium quantitatum, n, $\frac{mm + pp}{l}$, et $\surd^3\sqrt{\frac{mmpp + p^4}{l}}$.

Prop. 15.

De æquatione $x^4 + lx^3 - mmxx + n^3x - p^4 = 0$, ſeu $x^4 + lx^3 - mmxx + n^3x = p^4$.

Hæc æquatio habebit quandóque tantùm unam radicem veram, quandóque autèm tres, proùt rationes quantitatum l, m, n, et p ad ſe invicèm fuerint diverſæ. Si $\frac{p^4}{n^3}$ ſit $= -\frac{l}{2} + \sqrt{\frac{1}{4}ll + mm}$, ſeu $\sqrt{\frac{ll}{4} + mm} - \frac{l}{2}$, ſeu ſi p^4 ſit $= n^3 \times \sqrt{\frac{ll}{4} + mm} - \frac{ln^3}{2}$, ſeu $\sqrt{\frac{lln^6}{4} + mmn^6} - \frac{ln^3}{2}$, hæc æquatio habebit tantùm unam radicem; et hæc radix unica erit $= \frac{p^4}{n^3}$, ſeu $\sqrt{\frac{ll}{4} + mm} - \frac{l}{2}$. Et, ſi $\frac{p^4}{n^3}$ ſit major quàm $\sqrt{\frac{ll}{4} + mm} - \frac{l}{2}$, ſeu p^4 ſit major quàm $\sqrt{\frac{lln^6}{4} + mmn^6} - \frac{ln^3}{2}$, hæc æquatio habebit etiàm in hoc caſu tantùm unam radicem; et hæc radix unica erit major quàm $\sqrt{\frac{ll}{4} + mm} - \frac{l}{2}$, ſed minor quàm $\frac{p^4}{n^3}$.

Si verò $\frac{p^4}{n^3}$ ſit minor quàm $\sqrt{\frac{ll}{4} + mm} - \frac{l}{2}$, ſeu p^4 ſit minor quàm $\sqrt{\frac{lln^6}{4} + mmn^6} - \frac{ln^3}{2}$, hæc æquatio poterit habere tres radices. Et harum radicum unaquæque erit mediæ cujuſdam magnitudinis inter quantitates $\frac{p^4}{n^3}$ et $\sqrt{\frac{ll}{4} + mm} - \frac{l}{2}$, hoc eſt, major quàm $\frac{p^4}{n^3}$, ſed minor quàm $\sqrt{\frac{ll}{4} + mm} - \frac{l}{2}$.

Scholium

Scholium.

Hi Limites inter quos jacent radices Æquationum Quadraticarum, Cubicarum, et Biquadraticarum ſuprà deſcriptarum, à Florimondo de Beaune inventi, valdè utiles erunt in Reſolutione iſtarum Æquationum per Methodos Approximationis, ſive eos ipſos limites uſurpemus pro primis valoribus radicum quas inveſtigare propoſitum eſt, ſive ab ipſis conjecturâ derivemus alias quantitates iis ipſis paulo majores aut minores, ut opus erit, quæ iis ipſis adhùc propiùs accedant ad veras magnitudines radicum quæſitarum. In his enim methodis reſolvendi æquationes per approximationem (quæ quidem ſunt optimæ methodi, et plerumque etiàm ſolæ, quibus æquationes ultrà quadratum quantitatis ignotæ aſcendentes reſolvi poſſunt,) magni intereſt ut quantitas primò aſſumpta pro valore propinquo radicis quæſitæ, et à quâ alius valor qui ſit primâ iſtâ quantitate aſſumptâ multo propinquior ipſi quantitati quæſitæ, mòx derivandus eſt, ſit veræ quantitati admodùm propinqua; ne approximatio ad veram magnitudinem quantitatis quæſitæ nimis lentè procedat. Et, ut hoc incommodum evitetur, optandum erit ſempèr, et maximâ curâ enitendum, ut quantitas aſſumpta pro valore primo et propinquo quantitatis quæſitæ à quo approximatio ulterior ad ejus veram magnitudinem ordiri debet, non differat à quantitate quæſitâ magìs quàm decimâ ejus parte, et, ſi id efficere poſſimus conjecturis et experimentis facilibus, (ſeu ſubſtitutionibus numerorum parvorum, non pluribus quàm duabus figuris decimalibus expreſſorum, loco quæſitæ quantitatis x, in æquatione reſolvendâ,) ut non magìs differat à quantitate quæſitâ quàm nonageſimâ, vel centeſimâ, ejus parte. Ad tales autèm primos valores radicum quæſitarum, veris ipſarum valoribus commodè propinquos, inveniendos, Limites earum à Florimondo de Beaune inventi, et in tractatu præcedente deſcripti, et hîc brevitèr recenſiti, ſæpiſſimè, ſed non ſempèr, magno nobis adjumento eſſe poterunt.

F. M.

April 14, 1804.

EXEMPLA

EXEMPLA

Quædam reſolutionis Æquationum numeralium, Cubicarum et Biquadraticarum, per Methodum Approximationis à Joſepho Raphſon traditam;

In quibus utilitas Limitum, intèr quos jacent earum radices, à FLORIMONDI DE BEAUNE inventorum, et in tractatu ſuperiore deſcriptorum, manifeſtè apparebit; ex Libello celeberrimo dicti Joſephi Raphſon cui titulus eſt *Analyſis Æquationum Univerſalis*, deſumpta.

Exemplum 1.

Reſolvenda ſit æquatio Cubica $x^3 + 24x = 587{,}914$.

Hæc æquatio eſt ejuſdem formæ ac æquatio $x^3 + mmx = n^3$ in Propoſitione 3^tiâ capitis ſecundi tractatûs ſuperioris; et *mm* in iſtâ propoſitione generali fit in hâc æquatione numerali $= 24$, et n^3 fit $= 587{,}914$, et pro-inde n fit $= \sqrt[3]{587{,}914}$, et $nn = \sqrt[3]{587{,}914}^2$. Ergò, per iſtam Propoſitionem tertiam, radix x hujus æquationis erit major quàm $\frac{n^3}{mm+nn}$, ſeu $\frac{587{,}914}{24+\sqrt[3]{587{,}914}^2}$, ſed minor quàm $\frac{n^3}{mm}$, ſeu $\frac{587{,}914}{24}$ et quàm n, ſeu $\sqrt[3]{587{,}914}$. Eſt verò (ut ex Tabulâ cuborum numerorum omnium naturalium 1, 2, 3, 4, 5, 6, 7, &c. uſque ad 100, patebit,) cubus proximè minor quàm 587,914, ſeu n^3, numerus 571,787, qui eſt cubus numeri 83; et cubus proximè major quàm 587,914 eſt numerus 592,704, qui eſt cubus numeri 84. Ergò n eſt major quàm 83, et minor quàm 84, et nn eſt major quàm $\sqrt{83}^2$, ſeu 6889, ſed minor quàm $\sqrt{84}^2$, ſeu 7056. Ergò $mm + nn$ erit propemodùm æqualis, ſed paulo minor quàm, 24 + 7056, ſeu 7080, et $\frac{n^3}{mm+nn}$ erit propemodùm $= \frac{587{,}914}{7080} = 83.0$. Eſt verò $\frac{n^3}{mm} = \frac{587{,}914}{24} = 24{,}496$. Erit igitùr (per dictam propoſitionem tertiam,) radix x hujus æquationis numeralis $x^3 + 24x = 587{,}914$ major quàm 83, ſed minor quàm 24,496, et etiàm quàm 84. Horum trium numerorum, 83, 24496, et 84, primus, 83, et tertius, 84, ſunt limites ſatìs arcti radicis x; nam

eorum

eorum differentia, ſcilicet, (84 — 83, ſeu) 1, eſt tantùm pars octogeſima et tertia minoris limitis 83, et pro-inde differentia inter 83 et veram radicem x erit adhùc minor quàm 1, aut pars 83^{ia} minoris limitis 83. Ex hoc ergò limite 83, (quem ope dictæ propoſitionis tertiæ tractatûs Florimondi de Beaune invenimus,) ordiri poſſumus proceſſum approximationis ulterioris ad hujus radicis magnitudinem cum bonâ ſpe eventûs proſperi, ſeu inveniendi ſecundum valorem ejuſdem qui ſit multo propior ejus veræ magnitudini quàm eſt iſte limes 83. Hic verò proceſſus erit hujuſmodi.

Quoniàm 83 differt paulùm à radice x, et, ut videtur, eſt ipſâ aliquanto minor, ſubſtituamus 83 pro x in quantitate binomiâ $x^3 + 24x$, ut inde fiamus omninò certi an ſit reverâ iſtâ radice minor an major. Hæc autèm ſubſtitutio ita fieri poterit. Si x ſit $= 83$, x^3 erit $= 571{,}787$, et $24x$ erit $(= 24 \times 83) = 1992$, et pro-inde quantitas binomia $x^3 + 24x$ erit $(= 571{,}787 + 1992) = 573{,}779$; qui numerus eſt minor quàm 587,914, ſeu quantitas binomia $x^3 + 24x$ in propoſitâ æquatione $x^3 + 24x = 587{,}914$. Ergò 83 eſt minor quàm vera magnitudo radicis x in iſtâ æquatione. Q. E. I.

Et, quoniam jam certum eſt quòd 83 eſt minor quàm radix x in æquatione $x^3 + 24x = 587{,}914$, ponamus z pro earundem differentiâ; et habebimus $x = 83 + z$. Ergò x^3 erit $= \overline{83 + z}^3$ $(= \overline{83}^3 + 3 \times \overline{83}^2 \times z + 3 \times 83 \times zz + z^3 = 83^3 + 3 \times 6889 \times z + 249\,zz + z^3) = 571{,}787 + 20{,}667\,z + 249\,zz + z^3$, et $24x$ erit $= 24 \times \overline{83 + z}$ $(= 24 \times 83 + 24z) = 1992 + 24z$, et pro-inde $x^3 + 24x$ erit $(= 571{,}787 + 20{,}667\,z + 249\,zz + z^3 + 1992 + 24z) = 573{,}779 + 20{,}691\,z + 249\,zz + z^3$.

Sed $x^3 + 24x$ eſt $= 587{,}914$.

Ergò $573{,}779 + 20{,}691\,z + 249\,zz + z^3$ erit etiàm $= 587{,}914$, et (auferendo utrinque 573,779,) erit $20{,}691\,z + 249\,zz + z^3 = 14{,}135$; et (negligendo, ſecundùm præcepta Raphſoni, quantitates valdè parvas $249\,zz$ et z^3,) $20{,}691 \times z$ erit $= 14{,}135$. Ergò z erit $= \frac{14{,}135}{20{,}691} = 0.68$, et x, ſeu $83 + z$, erit $= 83 + 0.68$, ſeu 83.68; hoc eſt, ſecundus valor propinquus radicis x æquationis $x^3 + 24x = 587{,}914$ erit $= 83.68$.

Q. E. I.

Hic valor 83.68 eſt paulo major quàm vera magnitudo radicis x, ſed eam ſuperat minore differentiâ quàm 0.01, ſeu $8368^{mâ}$ parte hujus valoris; nàm ſex primæ figuræ hujus radicis, ſi ejus inveſtigatio ulteriùs continuetur, invenientur eſſe 83.6776, ut ſequenti calculo patebit.

Subſtituatur numerus 83.68 pro x in quantitate binomiâ $x^3 + 24x$, ut appareat an ſit major verâ radice x in æquatione propoſitâ $x^3 + 24x = 587{,}914$, an eâdem minor.

Jam, ſi x ſit $=$ 83.68, xx erit $=$ 7002.3424, et x^3 erit $=$ 585,956.012,032, et $24x$ erit $(= 24 \times 83.68) =$ 2008.32, et pro-inde quantitas binomia $x^3 + 24x$ erit $(= 585{,}956.012{,}032 + 2008.32) = 587{,}964.332{,}032$; qui numerus eſt paulo major quàm 587,914, ſeu quàm valor quantitatis binomiæ $x^3 + 24x$ in æquatione propoſitâ $x^3 + 24x = 587{,}914$. Ergò 83.68 eſt paulo major quàm radix x in iſtâ æquatione. Q. E. I.

Ponatur jam z pro differentiâ inter x et 83.68. Et erit $x = 83.68 - z$. Ergò x^3 erit $= \overline{83.68 - z}^3$ $(= \overline{83.68}^3 - 3 \times \overline{83.68}^2 \times z + \&c = \overline{83.68}^3 - 3 \times 7002.3424 \times z + \&c) = 585{,}956.012{,}032 - 21{,}007.0272 \times z +$ &c, et $24x$ erit $(= 24 \times \overline{83.68 - z}) = 2008.32 - 24z$, et quantitas binomia $x^3 + 24x$ erit $(= 585{,}956.012{,}032 - 21{,}007.0272 \times z + 2008.32 - 24z) = 587{,}964.332{,}032 - 21{,}031.0272 \times z$.

Sed quantitas binomia $x^3 + 24x$ eſt $= 587{,}914$.

Ergò et quantitas $587{,}964.332{,}032 - 21{,}031.0272 \times z$ erit etiàm $= 587{,}914$; et pro-inde quantitas 587,964.332,032 erit $= 587{,}914 + 21{,}031.0272 \times z$, et quantitas $21{,}031.0272 \times z$ erit $(= 587{,}964.332{,}032 - 587{,}914) = 50.332{,}032$, et quantitas z erit $= \frac{50.332{,}032}{21{,}031.0272} = 0.002{,}393$. Ergò quantitas x, ſive $83.68 - z$, erit $(= 83.68 - 0.002{,}393) = 83.677{,}607$, hoc eſt, tertius valor propinquus radicis x in æquatione $x^3 + 24x = 587{,}914$ erit 83.677,607. Q. E. I.

Hæ concluſiones cum concluſionibus Raphſoni, in ſolutione ejus Problematis octavi, omninò conſentiunt.

Exemplum 2, à Raphſoni Problemate duodecimo deſumptum.

Reſolvenda ſit æquatio cubica numeralis $300x - x^3 = 1000$, quæ exprimit relationem inter chordam arcûs circularis qui eſt ſexta pars totius circumferantiæ, ſeu chordem arcûs ſexagintà graduum, (quæ chorda eſt æqualis circuli radio,

radio, ſeu ſemi-diametro,) et chordam tertiæ partis hujus arcûs, ſeu chordam arcûs viginti graduum, poſito quòd radius circuli vocetur 10, et chorda quæſita arcûs viginti graduum vocetur x.

Hæc æquatio eſt ejuſdem formæ ac æquatio $mmx - x^3 = n^3$ in Propoſitione 1ma Capitis ſecundi tractatûs ſuperioris; et mm in iſtâ Propoſitione generali fit in hâc æquatione numerali $= 300$, et n^3 fit $= 1000$. Habet autèm, per iſtam propoſitionem primam, æquatio $mmx - x^3 = n^3$ duas radices veras, quarum utraque eſt major quàm $\frac{n^3}{mm}$, ſed minor quàm m. Ergò æquatio numeralis $300x - x^3 = 1000$ habebit duas radices veras, quarum utraque erit major quàm $\frac{1000}{300}$, (ſeu quàm $\frac{10}{3}$,) ſeu quàm 3.333, &c. ſed minor quàm $\surd 300$, (ſeu quàm $\surd 3 \times \surd 100$, ſeu quàm $\surd 3 \times 10$, ſeu quàm 1.732 &c $\times$ 10,) ſeu quàm 17.32 &c. Harum autèm radicum minor eſt chorda arcûs viginti graduum, quam Raphſon in dicto Problemate duodecimo inveſtigavit. Hanc igitùr radicem minorem æquationis $300x - x^3 = 1000$ hîc nunc quæremus per Methodum Approximationis à Raphſono traditam, aſſumentes pro primo hujus radicis valore propinquo, ſeu pro fundamento approximationis, numerum 3.3, ſeu duas primas figuras numeri 3.333, &c, ſeu limitis ejuſdem radicis quem per dictam primam propoſitionem capitis ſecundi tractatûs ſuperioris invenimus.

Imprimis, ſubſtituemus pro x quantitate binomiâ $300x - x^3$, (quæ eſt prima pars æquationis reſolvendæ $300x - x^3 = 1000$,) numerum 3.3, ut hoc modo fiat omninò manifeſtum quòd radix minor in iſtâ æquatione eſt major quàm 3.3. Hæc ſubſtitutio erit hujuſmodi.

Si x ſit $= 3.3$, xx erit $= \overline{3.3}^2 = 10.89$, et x^3 erit $= \overline{3.3}^3$ $(= \overline{3.3}^2 \times 3.3 = 10.89 \times 3.3) = 35.937$, et $300x$ erit $= 300 \times 3.3 = 990$, et proinde quantitas binomia $300x - x^3$ erit $(= 990.000 - 35.937) = 954.063$. Hic autem numerus eſt minor quàm 1000, qui eſt æqualis quantitati binomiæ in datâ æquatione $300x - x^3 = 1000$. Ergò 3.3 eſt minor quàm radix minor iſtius æquationis. Q. E. D.

Ponatur jam z pro differentiâ inter x et 3.3. Et tunc x erit $= 3.3 + z$, et pro-inde x^3 erit $= \overline{3.3 + z}^3$ $(= \overline{3.3}^3 + 3 \times \overline{3.3}^2 \times z +$ &c $= \overline{3.3}^3 + 3 \times 10.89 \times z +$ &c) $= 35.937 + 32.67 \times z +$ &c, et $300x$ erit $= 300 \times \overline{3.3 + z} = 990.000 + 300 \times z$, et quantitas binomia $300x - x^3$ erit $(= 990.000 + 300 \times z - 35.937 - 32.67 \times z -$ &c) $= 954.063 + 267.33 \times z -$ &c.

Sed quantitas binomia $300x - x^3$ eſt $= 1000$.

Ergò quantitas $954.063 + 267.33 \times z -$ &c erit etiàm $= 1000$; et pro-inde $267.33 \times z -$ &c erit $(= 1000.000 - 954.063) = 45.937$, et z erit $= \frac{45.937}{267.33} = 0.17$. Ergò x, ſeu $3.3 + z$, erit $= 3.3 + 0.17 = 3.47$, hoc eſt, ſecundus valor propinquus radicis x, à minore ejus limite 3.3, in tractatu præcedente indicato, derivatus, erit 3.47; qui numerus eſt in omnibus figuris accuratus. Nam, per Raphſoni calculum, hujus radicis valor magìs accuratus eſt 3.472,963,553,338,607, cujus primæ tres figuræ, 3.47, ſunt eædem quas hîc invenimus.

Subſtituatur jam numerus 3.47 pro x in quantitate binomiâ $300x - x^3$, ut appareat an valor iſtius quantitatis inde oriundus erit major vel minor quàm 1000, ſeu quàm valor ejuſdem quantitatis in æquatione $300x - x^3 = 1000$, et pro-inde an numerus 3.47 ſit major, vel ſit minor, quàm radix minor x iſtius æquationis.

Jam verò, ſi x ſit $= 3.47$, xx erit $= 12,0409$, et x^3 erit $= 41.781,923$, et $300x$ erit $(= 300 \times 3.47) = 1041$, et pro-inde $300x - x^3$ erit $(= 1041.000,000 - 41.781.923) = 999.218,077$; qui numerus eſt minor quàm 1000. Ergò 3.47 eſt minor quàm radix minor x æquationis $300x - x^3 = 1000$.

Ponatur jam z pro differentiâ inter x et 3.47. Et x erit $= 3.47 + z$. Et pro-inde x^3 erit $= \overline{3.47 + x}^3$ $(= \overline{3.47}^3 + 3 \times \overline{3.47}^2 \times z +$ &c $= \overline{3.47}^3 + 3 \times 12.0409 \times z +$ &c$) = 41.781,923 + 36.1227 \times z +$ &c, et $300x$ erit $(= 300 \times \overline{3.47 + z} = 300 \times 3.47 + 300 \times z) = 1041 + 300z$. Ergò quantitas binomia $300x - x^3$ erit $(= 1041 + 300z - 41.781,923 - 36.1227 \times z -$ &c$) = 999.218,077 + 263.8773 \times z -$ &c.

Sed quantitas binomia $300x - x^3$ eſt $= 1000$.

Ergò quantitas $999.218,077 + 263.8773 \times z$ erit etiàm $= 1000$; et pro-inde $263.8773 \times z$ erit $(= 1000.000,000 - 999.218,077) = 0.781,923$, et z erit $= \frac{0.781,923}{263.8773} = 0.002,963$. Ergò x, ſeu $3.47 + z$, erit $= 3.47 + 0.002,963 = 3.472,963$; hoc eſt, tertius propinquus valor radicis x, ſeu chordæ arcûs viginti graduum, erit $= 3.472,963$. Q. E. I.

Subſtituatur

Subſtituatur jam numerus 3.472,963 loco x in quantitate binomiâ $300x - x^3$, ut appareat an valor hujus quantitatis inde oriundus erit major, vel erit minor, quàm 1000, ſeu valor ejuſdem quantitatis in æquatione $300x - x^3 = 1000$, et pro-inde an numerus 3.472,963 erit major, vel erit minor, quàm x, ſeu radix minor iſtius æquationis.

Jam verò, ſi x ſit = 3.472,963, xx erit = 12.061,471,999,369, et x^3 erit = 41.889,045,979,344,560,347, et $300x$ erit (= 300 × 3.472,963) 1041.888,900, et quantitas binomia $300x - x^3$ erit (= 1041.888,900 — 41.889,045,979,344,560,347) = 999.999,854,020,655,439,653; qui numerus eſt minor quàm 1000. Ergò numerus 3.472,963 eſt minor quàm x, ſeu radix minor æquationis $300x - x^3 = 1000$. Q. E. I.

Ponatur jam z pro differentiâ inter x et 3.472,963. Et x erit = 3.472,963 + z. Et pro-inde x^3 erit = $\overline{3.472,963 + z}^3$ (= $\overline{3.472,963}^3 + 3 \times \overline{3.472,963}^2 \times z$ + &c = $\overline{3.472,963}^3 + 3 \times 12.061,471,999,369 \times z$ + &c) = 41.889,045,979,344,560,347 + 36.184,415,998,107 × z + &c, et $300x$ erit = $300 \times \overline{3.472,963 + z}$ (= 300 × 3.472,963 + 300 × z) = 1041.888,900 + 300 × z. Ergò quantitas binomia $300x - x^3$ erit (= 1041.888,900 + 300 × z — 41.889,045,979,344,560,347 — 36.184,415,998,107 × z — &c) = 999.999,854,020,655,439,653 + 263.815,584,001,893 × z — &c.

Sed quantitas binomia $300x - x^3$ eſt = 1000.

Ergò quantitas 999.999,854,020,655,439,653 + 263.815,584,001,893 × z erit etiàm = 1000; et pro-inde 263.815,584,001,893 × z erit (= 1000 — 999 999,854,020,655,439,653) = 0.000,145,979,344,560,347, et z erit = $\frac{0.000,145,979,344,560,347}{263.815,584,001,893}$ = 0.000,000,553,338,594. Ergò x, ſeu 3.472,963 + z, erit = 3.472,963 + 0.000,000,553,338,594 = 3.472,963,553,338,594; hoc eſt, quartus propinquus valor radicis x, ſeu chordæ arcûs viginti graduum, erit = 3.472,963,553,338,594. Q. E. I.

Hic numerus 3.472,963,553,338,594 convenit cum numero 3.472,963,553,338,607, quem Raphſonus invenit pro valore x, ſeu chordæ arcûs viginti graduum, in primis tredecim figuris 3 472,963,553,338, quas pro-inde veras eſſe videtur maximè probabile. Si verò hoc omninò certum reddere velimus, id efficere poterimus hoc modo.

Reſumatur

Refumatur Inveſtigatio hujus quarti valoris propinqui radicis x, et in valore quantitatis x^3, ſeu $\overline{3.472,963 + z}^3$, retineatur tertius terminus hujus valoris, nempe, $3 \times 3.472,963 \times zz$, ſeu $10.418,889 \times zz$; et habebimus hanc æquationem, $300 \times 3.472,963 + 300 \times z - \overline{3.472,963}^3 - 3 \times \overline{3.472,963}^2 \times z - 3 \times 3.472,963 \times zz -$ &c $(= 300x - x^3) = 1000$, ſeu $1041.888,900 + 300 \times z - 41.889,045,979,344,560,347 - 36.184,415,998,107 \times z - 10.418,889 \times zz -$ &c $(= 300x - x^3) = 1000$, ſeu $999.999,854,020,655,439,653 + 263.815,584,001,893 \times z - 10.418,889 \times zz -$ &c $= 1000$, et pro-inde $263.815,584,001,893 \times z - 10.418,889 \times zz -$ &c erit $(= 1000 - 999.999,854,020,655,439,653) = 0.000,145,979,344,560,347$, et $263.815,584,001,893, \times z$ erit $= 0.000,145,979,344,560,347 + 10.418,889, \times zz +$ &c, et z erit $= \frac{0.000,145,979,344,560,347}{263.815,584,001,893} + \frac{10.418,889}{263.815,584,001,893} \times zz +$ &c $=$ $0.000,000,553,338,594 + \frac{10.418,889}{263.815,584,001,893} \times zz +$ &c.

Sed $\frac{10.418,889}{263.815,584,001,893} \times zz$ eſt minor quàm $\frac{10.418,889}{263} \times zz$, ſeu quàm $0.039, \times zz$, et pro-inde (ſubſtituendo pro z numerum $0.000,000,56$, qui eſt major quàm $0.000,000,553,338,594$, ſeu z,) erit minor quàm $0.039, \times \overline{0.000,000,56}^2$, ſeu quàm $0.039, \times 0.000,000,000,000,313,6$, et, à fortiori, minor quàm $0.04 \times 0.000,000,000,000,000,313,6$, ſeu quàm $0.000,000,000,000,012,544$; hoc eſt, terminus $\frac{10.418,889}{263.815,584,001,893} \times zz$, (qui addendus eſt numero $0.000,000,553,338,594$, ſeu valori ipſius z jam invento, ut ad ejus verum valorem propiùs accedamus,) erit minor quàm fractio decimalis $0.000,000,000,000,012,544$, cujus prima figura occupat locum decimum quartum fractionum decimalium, et pro-inde nullam efficiet mutationem in figuris numeri $0.000,000,553,338,594$, quæ ante penultimam figuram 9 in decimo quarto loco poſitam inveniuntur, niſi quòd, cum figura in decimo quarto loco fractionum decimalium poſita ſit 9, additio unitatis, ipſi facta ex hoc novo termino, convertet iſtam figuram 9 in cyphram 0, et faciet ut figura 5 in loco proximè ſuperiore, ſeu decimo-tertio fractionum decimalium loco, poſita augeatur unitate, et convertatur in figuram 6; ita ut accuratior valor quantitatis z fiet $0.000,000,553,338,6$, et pro-inde valor accuratior quantitatis x, ſeu $3.472,963, + z$, fiet $= 3.472,693 + 0.000,000,553,338,6$, ſeu $3.472,963,553,338,6$. Ergò jam certò concludere poſſumus quòd numeri ſuprà inventi pro quarto valore radicis x, ſcilicet, numeri $3.472,963,553,338,594$, primæ tredecim figuræ, ſcilicet, $3.472,963,553,338$, erunt veræ.

Q. E. I.

Exemplum

Exemplum 3.

Reſolvenda ſit æquatio Cubica $9xx - x^3 = 100$,
à Raphſoni Problemate 13^tio deſumpta.

Hæc æquatio eſt ejuſdem formæ ac æquatio generalis $lxx - x^3 = n^3$ in primâ Propoſitione Capitis tertii tractatûs ſuperioris de Limitibus æquationum. Et co efficiens l in iſtâ æquatione generali fit in hâc æquatione numerali $9xx - x^3 = 100$ æqualis numero 9, et n^3 fit æqualis numero 100. Sed, per iſtam Propoſitionem tertiam, æquatio $lxx - x^3 = n^3$ habet duas radices, quarum utraque eſt major quàm $\sqrt{\frac{n^3}{l}}$, ſed minor quàm l. Ergò æquatio numeralis $9xx - x^3 = 100$ habebit duas radices, quarum utraque erit major quàm $\sqrt{\frac{100}{9}}$, ſed minor quàm 9. Eſt verò $\frac{100}{9} = \frac{10 \times 10}{3 \times 3}$, et pro-inde $\sqrt{\frac{100}{9}}$ erit $= \frac{10}{3} = 3.333$, &c. Ergò utraque duarum radicum æquationis $9xx - x^3 = 100$ erit major quàm 3.333, &c. Si igitùr quæramus primò valorem minoris harum radicum, eligenda erit in primis quædam quantitas aliquanto major quàm 3.333, &c, quæ ſubſtituatur pro x in quantitate binomiâ $9xx - x^3$, ut videamus an valor iſtius quantitatis binomiæ inde oriundus ſit major, vel ſit minor, quàm 100, ſeu ejuſdem valor in æquatione propoſitâ $9xx - x^3 = 100$, et pro-inde an quantitas ità ſubſtituta pro x ſit major, vel ſit minor, quàm x, ſeu radix minor ejuſdem æquationis. Eligamus igitùr ad hunc finem quantitatem 3.4, et ſubſtituamus eam pro x in quantitate binomiâ $9xx - x^3$.

Jam, ſi x ſit $= 3.4$, erit $xx = 11.56$, et $x^3 = 39.304$, et $9xx$ erit $(= 9 \times 11.56) = 104.04$; et pro-inde quantitas binomia $9xx - x^3$ erit $(= 104.04 - 39.304) = 64.736$. Hæc autèm quantitas 64.736 eſt multo minor quàm 100, ſeu valor quantitatis binomiæ $9xx - x^3$ in æquatione propoſitâ. Ergò 3.4 eſt multo minor quàm verus valor x, ſeu radicis minoris dictæ æquationis.

Ponamus ergò hanc radicem eſſe $= 4$; et ſubſtituamus hanc novam quantitatem pro x in quantitate binomiâ $9xx - x^3$.

Jàm, ſi x ſit $= 4$, xx erit $= 16$, et $x^3 = 64$, et $9xx$ $(= 9 \times 16) = 144$, et $9xx - x^3$ $(= 144 - 64) = 80$. Hæc autèm quantitas 80 eſt minor quàm 100, et differentia eſt 20, ſeu quinta pars numeri 100. Ergò 4 erit etiàm minor quàm x, et differentia earum erit ſatìs magna.

Ponamus

Ponamus ergò, novâ conjecturâ, x esse $= 4\,5$, et substituamus 4.5 pro x in quantitate binomiâ $9xx - x^3$.

Jam, si x sit $= 4.5$, xx erit $(= 4.5^{\,2}) = 20.25$, et x^3 erit $(= xx \times x = 20.25 \times 4.5) = 91.125$, et $9xx$ erit $(= 9 \times 20.25) = 182.25$, et $9xx - x^3$ erit $(= 182.25 - 91.125) = 91.125$. Hæc autèm quantitas est minor quàm 100, et pro-inde 4.5 est minor quàm x. Sed tamen satìs propinqua videtur esse ipsi x, ut eam possimus commodè assumere pro primo ipsius x valore propinquo, seu pro fundamento processûs approximatorii, secundùm Methodum à Raphsono traditam, ad alium ipsius x valorem, vero ejusdem valori adhùc ipso 4.5 propinquiorem, investigandum.

Quoniam igitùr x est major quàm 4.5, ponatur z pro earum differentiâ; et erit $x = 4.5 + z$. Ergò xx erit $= \overline{4.5 + z}^2$ $(= \overline{4.5}^2 + 2 \times 4.5 \times z +$ &c$) = 20.25 + 9z +$ &c, et x^3 erit $= \overline{4.5 + z}^3$ $(= \overline{4.5}^3 + 3 \times \overline{4.5}^2 \times z +$ &c $= 4.5^{\,3} + 3 \times 20.25 \times z +$ &c$) = 91.125 + 60.75 \times z +$ &c, et $9xx$ erit $(= 9 \times 20.25 + 9z +$ &c $= 9 \times 20.25 + 9 \times 9z +$ &c$) =$ $182.25 + 81z +$ &c, et pro-inde quantitas binomia $9xx - x^3$ erit $(= 182.25 + 81z - 91.125 - 60.75 \times z) = 91.125 + 20.25 \times z$.

Sed quantitas binomia $9xx - x^3$ est $= 100$.

Ergò quantitas $91.125 + 20.25 \times z$ erit etiàm $= 100$; et pro-inde $20.25 \times z$ erit $(= 100 - 91.125) = 8.875$, et $20.25 \times z$ erit $= 8.875$, et z erit $= \frac{8.875}{20.25} = 0.438$. Ergò x, seu $4.5 + z$, erit $= 4.5 + 0.438 =$ 4.938.

Substituatur jam 4.938 pro x in quantitate binomiâ $9xx - x^3$. Et erit xx $(= \overline{4.938}^2) = 24.383{,}844$, et x^3 erit $(= \overline{4.938}^3) = 120.407{,}421{,}672$, et $9xx$ erit $(= 9 \times 24.383{,}844) = 219.454{,}596$, et quantitas binomia $9xx - x^3$ erit $(= 219.454{,}596 - 120.407{,}421{,}672) = 99.047{,}174{,}328$; quæ est paulò minor quàm 100. Ergò 4.938 est paulò minor quàm x, seu radix minor æquationis $9xx - x^3 = 100$.

Ut

Ut igitùr valorem hujus radicis obtineamus, qui fit accuratior quàm 4.938, ponatur z pro exceſſu hujus radicis ſupèr 4.938; et erit $x = 4.938 + z$. z. Ergò xx etit $\overline{4.938 + z}^2$ $(= \overline{4.938}^2 + 2 \times 4.938 \times z + \&c) =$ $24.383,844 + 9.876 \times z + \&c$, et $9xx$ erit $(= 9 \times \overline{4.938 + z}^2 = 9 \times$ $\overline{24.383,844 + 9.876 \times z + \&c} = 9 \times 24.383,844 + 9 \times 9.876 \times z +$ $\&c) = 219.454,596 + 88.884 \times z + \&c$, et x^3 erit $= \overline{4.938 + z}^3$ $(=$ $\overline{4.938}^3 + 3 \times \overline{4.938}^2 \times z + \&c = \overline{4.938}^3 + 3 \times 24.383,844 \times z +$ $\&c) = 120.407,421,672 + 73.151,532 \times z + \&c$; et quantitas binomia $9xx - x^3$ erit $=$

$$\left\{\begin{array}{l} 219.454,596,000 + 88.884,000 \times z \\ -120.407,421,672 - 73.151,532 \times z \end{array}\right\} =$$

$$99.047,174,328 + 15.732,468 \times z.$$

Sed quantitas binomia $9xx - x^3$ eſt $= 100$.

Ergò quantitas $99.047,174,328 + 15.732,468 \times z$ erit etiàm $= 100$; et pro-inde $15.732,468 \times z$ erit $(= 100.000,000,000 - 99.047,174,328) =$ $0.952,825,672$, et z erit $= \frac{0.952,825,672}{15.732,468} = 0.060,564$. Ergò x, ſeu $4.938 + z$, erit $(= 4.938 + 0.060,564) = 4.998,564$.

Hæc quantitas 4.998,564 diſtat à numero integro **5** tàm parvâ differentiâ 0.001,436 ut ſuſpicionem inducat quòd radix x ſit præciſè æqualis iſti numero integro; quòd quidèm invenietur eſſe verum. Nam, ſi x ſit præciſè $= 5$, xx erit $= 25$, et x^3 erit $= 125$, et $9xx$ erit $(= 9 \times 25) = 225$, et $9xx - x^3$ erit $(= 225 - 125) = 100$; quæ eſt æquatio propoſita. Ergò nunc manifeſtum eſt quòd radix minor æquationis $9xx - x^3 = 100$ eſt numerus integer 5. Q. E. I.

Inventâ autèm radice minore hujus æquationis, ejus radix altera, ſeu major, poterit inveniri per reſolutionem æquationis quadraticæ ad quam prædicta æquatio cubica nunc reduci poterit. Hoc autèm fiet hoc modo. Pone a pro radice minore jam inventâ, et deſignetur radix altera, ſeu major, per Litteram x. Et quoniam $9aa - a^3$ eſt $= 100$, et $9xx - x^3$ eſt etiàm $= 100$, erit $9xx - x^3 = 9aa - a^3$; et pro-inde (addendo utrinque quantitatem x^3,) erit $9xx = 9aa + x^3 - a^3$, et (auferendo utrinque $9aa$,) erit $9xx - 9aa = x^3 - a^3$, hoc eſt, $9 \times \overline{xx - aa}$ erit $= x^3 - a^3$. Ergò $9 \times \frac{xx - aa}{x - a}$ erit $= \frac{x^3 - a^3}{x - a}$;

hoc eſt, $9 \times \overline{x + a}$ erit $= xx + xa + aa$, ſeu $9x + 9a$ erit $= xx + ax + aa$; et (auferendo utrinque $9x$,) erit $9a = xx + ax - 9x + aa$, et (auferendo utrinque aa,) erit $9a - aa = xx + ax - 9x$, ſeu $xx + ax - 9x$ erit $= 9a - aa$, ſeu $xx - \overline{9 - a} \times x$ erit $= \overline{9 - a} \times a$, ſeu (quoniam a eſt $= 5$,) $xx - \overline{9 - 5} \times x$ erit $= \overline{9 - 5} \times 5$, ſeu $xx - 4x$ erit $= 4 \times 5$, ſeu $xx - 4x$ erit $= 20$. Ergò $xx - 4x + 4$ erit $(= 20 + 4) = 24$, et $x - 2$ erit $= \surd 24$, et x, ſeu radix major æquationis $9xx - x^3 = 100$, erit $(= 2 + \surd 24 = 2 + 4.898,979,485) = 6.898,979,485$; cujus numeri omnes figuræ ſunt veræ. Q. E. I.

Exemplum 4.

Reſolvenda ſit æquatio Cubica $x^3 + 74xx + 8729x = 560,783$, in quâ omnes termini extant.

Hæc æquatio eſt ejuſdem formæ ac æquatio generalis $x^3 + lxx + mmx = n^3$, in Propoſitione quartâ capitis quarti tractatûs ſuperioris; et co-efficiens l ſecundi termini lxx in iſtâ æquatione generali fit in hâc æquatione numerali $= 74$, et co-efficiens mm tertii termini mmx in iſtâ æquatione fit in hâc $= 8729$, et quantitas abſoluta n^3 in iſtâ æquatione fit in hâc $= 560,783$. Sed, per iſtam Propoſitionem quartam, æquatio $x^4 + lx^3 + mmx = n^3$ habet tantùm unam radicem veram; et hæc radix unica eſt major quàm $\frac{n^3}{nn + ln + mm}$, ſed minor quàm quælibet trium quantitatum $\surd \frac{n^3}{l}$, $\frac{n^3}{mm}$, et n. Ergò æquatio numeralis $x^3 + 74xx + 8729x = 560,783$ habebit etiàm tantùm unam radicem veram; et hæc radix unica erit major quàm $\frac{560,783}{\overline{560,783}^{\frac{2}{3}} + 74 \times \overline{560,783}^{\frac{1}{3}} + 8729}$, ſed minor quàm $\surd \frac{560,783}{74}$, et quàm $\frac{560,783}{8729}$, et quàm $\overline{560,783}^{\frac{1}{3}}$. Hos igitùr Limites nunc neceſſe eſt ut computemus, ſed non ad plures quàm duas figuras decimales. Hoc verò fieri poterit hoc modo.

Inſpectâ tabulâ cuborum qui oriuntur ex centùm primis numeris naturalibus 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, uſque ad 100, apparebit quòd cubus proximè minor quàm numerus 560,783, ſeu n^3, eſt 551,368, ſeu cubus numeri 82, et quòd cubus proximè major quàm 560,783, ſeu n^3, eſt 571,787, ſeu cubus numeri 83.

83. Ergò n eſt major quàm 82, ſed minor quàm 83. Ponamus quòd ſit $= 83$, quòd à veritate paulùm abeſt. Ergò nn, ſeu $\overline{560,783}^{\frac{2}{3}}$ erit propemodùm $(= \overline{83}^{2},) = 6889$, et $74 \times \overline{560,783}^{\frac{1}{3}}$ erit $(= 74 \times 83) = 6142$, et $\overline{560,783}^{\frac{2}{3}} + 74 \times \overline{560,783}^{\frac{1}{3}} + 8729$ erit $= 6889 + 6142 + 8729 = 21,060$, et pro-inde fractio $\frac{560,783}{\overline{560,783}^{\frac{2}{3}} + 74 \times \overline{560,783}^{\frac{1}{3}} + 8729}$ erit $= \frac{560,783}{21,060} = 26$.

Et $\frac{560,783}{74}$ eſt $= 7578$, et pro-inde $\surd\frac{560,783}{74}$ erit $(= \surd 7578) = 87$; et $\frac{560,783}{8729}$ eſt $= 64$. Ergò radix unica æquationis propoſitæ erit major quàm 26, ſed minor quàm 87, vel quàm 83, vel quàm 64, ideóque erit mediæ cujuſdam magnitudinis inter 26 et 64. Hi limites ſunt à ſe invicèm admodùm remoti, et non multum nos adjuvant in conjecturâ faciendâ de magnitudine radicis x. Ponamus tamèn quòd hæc radix ſit $= 40$, et ſubſtituamus 40 pro x in quantitate trinomiâ $x^3 + 74xx + 8729x$, ut videamus an valor ejuſdem quantitatis ex hâc ſubſtitutione oriundus erit major, vel erit minor, quàm numerus abſolutus, ſeu quantitas nota, 560,783 in propoſitâ æquatione, et inde ſciamus an numerus 40 ſit major, vel ſit minor, quàm vera radix iſtius æquationis.

Poſito autèm quòd x ſit $= 40$, xx erit $= 1600$, et $x^3 = 64,000$, et $74xx$ erit $(= 74 \times 1600) = 118,400$, et $8729x$ erit $(= 8729 \times 40) = 349,160$, et pro-inde quantitas trinomia $x^3 + 74xx + 8729x$ erit $(= 64,000 + 118,400 + 349,160) = 531,560$; qui numerus eſt minor quàm 560,783. Ergò numerus 40 eſt minor quàm x in æquatione $x^3 + 74xx + 8729x = 560,783$. Differentia verò inter 531,560 et 560,783 non eſt magna, et pro-inde numerus 40 non erit multo minor quàm radix x.

Faciamus igitùr ſecundam conjecturam de magnitudine radicis x, et ponamus quòd ſit $= 41$, et ſubſtituamus 41 pro x in quantitate trinomiâ $x^3 + 74xx + 8729x$.

Poſito itaque quòd x ſit $= 41$, xx erit $(= \overline{41}^{2}) = 1681$, et x^3 erit $(= \overline{41}^{3}) = 68,921$, et $74xx$ erit $(= 74 \times 1681) = 124,394$, et $8729x$ erit $(= 8729 \times 41) = 357,889$, et pro-inde quantitas trinomia $x^3 + 74xx + 8729x$ erit $(= 68,921 + 124,394 + 357,889) = 551,207$; qui numerus eſt minor quàm 560,783. Ergò numerus 41 eſt minor quàm radix x æquationis pro-

positæ $x^3 + 74xx + 8729x = 560,783$. Est autèm satìs propinquus radici x ut fiet fundamentum processûs approximatorii, per Methodum à Raphsono traditam, quo inveniamus novum valorem istius radicis qui erit vero ejusdem valori multo propinquior quàm 41.

Ponamus igitùr z pro differentiâ inter x et 41; et erit $x = 41 + z$. Ergò xx erit $= \overline{41 + z}^2$ $(= \overline{41}^2 + 2 \times 41 \times z +$ &c$) = 1681 + 82 \times z +$ &c, et x^3 erit $= \overline{41 + z}^3$ $(= \overline{41}^3 + 3 \times \overline{41}^2 \times z +$ &c $= \overline{41}^3 + 3 \times 1681 \times z +$ &c$) = 68,921 + 5043 \times z +$ &c, et $74xx$ erit $(= 74 \times \overline{1681 + 82 \times z +}$ &c $= 74 \times 1681 + 74 \times 82 \times z +$ &c$) = 124,394 + 6068 \times z +$ &c, et $8729x$ erit $(= 8729 \times \overline{41 + z} = 8729 \times 41 + 8729 \times z) = 357,889 + 8729 \times z$; et pro-inde quantitas trinomia $x^3 + 74xx + 8729x$ erit = quantitati multinomiæ

$$\left\{\begin{array}{l} 68,921 + 5043\,z + \&c \\ + \ 124,394 + 6068\,z + \&c \\ + \ 357,889 + 8729\,z \end{array}\right\} =$$

$$551,207 + 19,840 \times z + \&c.$$

Sed quantitas trinomia $x^3 + 74xx + 8729x$ est $= 560,783$.

Ergò quantitas $551,207 + 19,840 \times z$ erit etiàm $= 560,783$; et pro-inde $19,840 \times z$ erit $(= 560,703 - 551,207) = 9,576$, et z erit $= \frac{9,576}{19,840}$ $= 0.48$. Ergò x, seu $41 + z$, erit $(= 41 + 0.48) = 41.48$, hoc est, radix æquationis propositæ $x^3 + 74xx + 8729x = 560,783$ erit $= 41.48$.

Substituatur jàm 41.48 pro x in quantitate trinomiâ $x^3 + 74xx + 8729x$, ut videamus an valor istius quantitatis inde oriundus erit major, vel erit minor, quàm 560,783, et pro-inde ut sciamus an 41.48 erit major, vel erit minor, quàm radix x æquationis $x^3 + 74xx + 8729x = 560,783$.

Si x sit $= 41.48$, xx erit $(= \overline{41.48}^2) = 1720.5904$, et x^3 erit $(= \overline{41.48}^3)$ $= 71,370.089,792$, et $74xx$ erit $(= 74 \times 1720.5904) = 127,323.6896$, et $8729x$ erit $(= 8729 \times 41.48) = 362,078.92$, et pro-inde quantitas trinomia $x^3 + 74xx + 8729x$ erit $(= 71,370.089,792 + 127,323.6896 + 362,078.92)$ $= 560,772.699,392$; qui numerus est minor quàm 560,783. Ergò numerus 41.48 est minor quàm x.

Ponamus

Ponamus jam z pro differentiâ inter 41,48 et x; et x erit $= 41.48 + z$. Ergò xx erit $= \overline{41.48 + z}^2$ $(= \overline{41.48}^2 + 2 \times 41.48 \times z +$ &c$) =$ $1720.5904 + 82.96 \times z +$ &c, et x^3 erit $= \overline{41.48 + z}^3$ $(= \overline{41.48}^3 + 3 \times \overline{41.48}^2 \times z +$ &c $= \overline{41.48}^3 + 3 \times 1720.5904 \times z +$ &c$) = 71,370.089,792 + 5161.7712 \times z +$ &c, et $74xx$ erit $(= 74 \times \overline{1720.5904 + 82.96 \times z +}$ &c $= 74 \times 1720.5904 + 74 \times 82.96 \times z +$ &c$) = 127,323.6896 + 6139.04 \times z +$ &c, et $8729x$ erit $(= 8729 \times \overline{41.48 + z} = 8729 \times 41.48 + 8729 \times z) = 362,078.92 + 8729z$; et pro-inde quantitas trinomia $x^3 + 74xx + 8729x$ erit $=$ quantitati multinomiæ

$$\left\{\begin{array}{rrrrrr} 71,370.089,792 & + & 5161.7712 & \times & z & + \text{ &c} \\ + 127,323.689,600 & + & 6139.0400 & \times & z & + \text{ &c} \\ + 362,078.920,000 & + & 8729.0000 & \times & z & \end{array}\right\} =$$

$$560,772.699,392 + 20,029.8112 \times z +$$ &c.

Sed quantitas trinomia $x^3 + 74xx + 8729x$ eſt $= 560,783$.

Ergò quantitas $560,772.699,392 + 20,029.8112 \times z +$ &c erit etiàm $= 560,783$; et pro-inde $20,029.8112 \times z +$ &c erit $(= 560,783 - 560,772.699,392) = 10.300,608$, et $z +$ &c erit $= \frac{10.300,608}{20,029.8112} = 0.000,514,263$. Ergò x, ſeu $41.48 + z$, erit $(= 41.48 + 0.000,514,263) = 41.480,514,263$, hoc eſt, radix æquationis propoſitæ $x^3 + 74xx + 8729x = 560,783$ erit $= 41.480,514,263$. Q. E. I.

Hujus numeri 41.480,514,263 primæ novèm figuræ, nempe 41.480,514,2, ſunt, ut opinor, veræ. Sed, ut hoc pro certo ſciatur, reſumemus ultimum proceſſum approximationis præcedentis, cujus ope invenimus differentiam z inter x et 41.48 eſſe $=$ 0.000,514,263, ut inde determinare poſſimus in quo loco fractionum decimalium figura prima proximi termini in valore quantitatis z (in quo zz invenietur,) calculum intrabit; ut factum eſt ſuprà ad finem exempli ſecundi, ubi demonſtravimus quòd tredecìm primæ figuræ numeri 3.472,963,553,338,594, nempe, figuræ 3.472,963,553,338, erant veræ.

In inveſtigatione præcedente quantitatis z quantitas trinomia $x^3 + 74xx + 8729x$ fit, per ſubſtitutionem quantitatis binomiæ $41.48 + z$ in ejus terminis pro x, æqualis quantitati $\overline{41.48 + z}^3 + 74 \times \overline{41.48 + z}^2 + 8729 \times 41.48$

$\overline{41.48 + z} = \overline{41.48}^3 + 3 \times \overline{41.48}^2 \times z + 3 \times 41.48 \times zz + z^3 +$
$74 \times \overline{41.48}^2 + 2 \times 41.48 \times z + zz + 8729 \times \overline{41.48 + z} = \overline{41.48}^3$
$+ 3 \times \overline{41.48}^2 \times z + 124.44 \times zz + z^3 + 74 \times \overline{41.48}^2 + 74 \times 82.96$
$\times z + 74\, zz + 8729 \times 41.48 + 8729\, z = \overline{41.48}^3 + 3 \times \overline{41.48}^2 \times z$
$+ 198.44\, zz + z^3 + 74 \times \overline{41.48}^2 + 74 \times 82.96 \times z + 8729 \times 41.48 +$
$8729\, z$. Hæc æquatio eſt accuratè vera. Sed, ut faciliùs reſolvi poſſet, negleximus in præcedente inveſtigatione duos terminos $198.44\, zz$ et z^3. Nunc verò retineamus primum horum terminorum, ſcilicet, $198.44\, zz$, et tunc æquatio erit $x^3 + 74\, xx + 8729\, x = \overline{41.48}^3 + 3 \times \overline{41.48}^2 \times z + 198.44\, zz + 74$
$\times \overline{41.48}^2 + 74 \times 82.96 \times z + 8729 \times 41.48 + 8729\, z =$ (per calculum præcedentem,) $560,772.699,392 + 20,029.8112 \times z + 198.44\, zz$ quàm proximè.

Sed quantitas trinomia $x^3 + 74\, xx + 8729\, x$ eſt $= 560,783$.

Ergò quantitas $560,772.699,392 + 20,029.8112 \times z + 198.44 zz$ erit etiàm $= 560,783$; et pro-inde $20,029.8112 \times z + 198.44\, zz$ erit $(= 560,783 - 560,772.699,392) = 10.300,608$ quàm proximè, et $20,029.8112 \times z$ erit $= 10.300,608 - 198.44\, zz$ quàm proximè, et z erit $= \frac{10.300,608}{20,029.8112} - \frac{198.44\, zz}{20,029.8112}$ $= 0.000,514,263 - \frac{198.44\, zz}{20,029.8112}$ quàm proximè. Eſt igitùr z paulo minor quàm fractio decimalis 0.000, 514,263, nempe, exceſſui iſtius fractionis ſupèr quantitatem valdè parvam $\frac{198.44 \times zz}{20,029.8112}$. Manifeſtum eſt autèm quòd z (quæ eſt propemodùm æqualis fractioni decimali $0.000,514,263$) erit minor quàm $0.000,52$. Ergò $\overline{0.000,52}^2$, hoc eſt, fractio decimalis $0.000,000,270,4$, erit major quàm zz, et $\frac{198.44 \times 0.000,000,270,4}{20,029.8112}$ erit major quàm $\frac{198.44\, zz}{20,029.8112}$, et, à fortiori, $\frac{198.44}{20,000} \times 0.000,000,270,4$, ſeu $\frac{99.22}{10,000} \times 0.000,000,270,4$, erit major quàm $\frac{198.44\, zz}{20,029.8112}$, et, iterùm à fortiori, $\frac{100}{10,000} \times 0.000,000,270,4$, ſeu $\frac{1}{100} \times 0.000,000,270,4$, ſeu $0.000,000,002,704$, erit major quàm $\frac{198.44\, zz}{20,029.8112}$.

Ergò

Ergò prima figura termini $\frac{198.44\,zz}{20,029,8112}$, (qui eſt ſubtrahendus à fractione decimali 0.000,514,263 ut valor quantitatis z accuratiùs obtineatur,) non erit major quàm 2 in nono fractionum decimalium loco poſita, et pro-inde, ſi à fractione decimali 0.000,514,263 ſubtraheretur, nullam mutationem efficeret in eâ niſi in ultimâ ejus figurâ 3, qui poſſet converti in figuram 1; ideóque omnes præcedentes figuræ ejuſdem fractionis decimalis, nempe, figuræ 0.000,514,26, immutatæ manerent; et pro-inde x, ſeu 41.48 + z, ſeu 41.480,514,263, erit accurata, non ſolùm in novèm figuris 41.480,514,2, ut opinatus fueram, ſed in decèm figuris 41.480,514,26. Q. E. I.

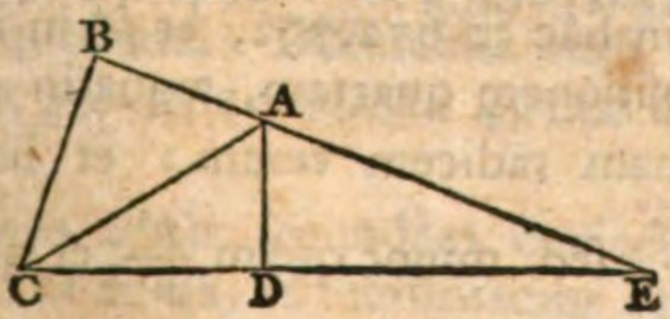
Exemplum 5.

Reſolvenda ſit æquatio $x^4 + 6808\,xx + 672{,}792\,x = 43{,}507{,}216$.

Hæc æquatio derivatur ex Problemate quodam Geometrico, quòd à Raphſono enunciatum eſt, ſed non ſolutum. Eſt autèm hujuſmodi.

Problema Geometricum.

Sit ACE triangulum habens angulum obtuſum A, ſeu CAE. Et à puncto A ducatur recta linea AD occurrens baſi CE ad rectos angulos in puncto D. Et à puncto C ducatur recta linea CB occurrens lateri EA, producto, ad rectos angulos in puncto B. Et ſint datæ magnitudines rectarum DE, CB, et BA; nempe, ſit recta DE = 97, et CB = 68, et BA = 51. Requiritur invenire longitudinem rectæ AD, quæ eſt perpendicularis ad baſim CE et cadit intrà triangulum CAE.



Solutio.

Solutio.

Ponatur x pro rectâ quæſitâ AD, et d pro DE, et s pro CB, et b pro BA.

Et quoniam triangula CBE et ADE ſunt rectangula, et habent angulum AED utrique communem, erunt inter ſe ſimilia. Ergò BE erit ad CB ut DE ad AD, hoc eſt, BE erit ad s ut d ad x. Sed BE eſt $=$ BA $+$ AE $=$ BA $+ \sqrt{\text{DE}q + \text{AD}q} = b + \sqrt{dd + xx}$. Ergò $b + \sqrt{dd + xx}$ erit ad s ut d ad x; et pro-inde $bx + x \times \sqrt{dd + xx}$ erit $= ds$, et $x \times \sqrt{dd + xx}$ erit $= ds - bx$, et $xx \times \overline{dd + xx}$ erit $(= \overline{ds - bx}^2) = ddss - 2bdsx + bbxx$, hoc eſt, $ddxx + x^4$ erit $= ddss - 2bdsx + bbxx$; et pro-inde $x^4 + ddxx + 2bdsx$ erit $= ddss + bbxx$, et $x^4 + ddxx - bbxx + 2bdsx$ erit $= ddss$; ut à Raphſono declaratur.

Eſt autèm $d = 97$, et $b = 51$, et $s = 68$. Ergò dd erit $(= \overline{97}^2) = 9409$, et bb erit $(= \overline{51}^2) = 2601$, et $dd - bb$ erit $(= 9409 - 2601) = 6808$, et $2bds$ erit $(= 2 \times 51 \times 97 \times 68 = 102 \times 97 \times 68) = 672,792$, et ss erit $(= \overline{68}^2) = 4624$, et $ddss$ erit $(= 9409 \times 4624) = 43,507,216$. Ergò æquatio litteralis $x^4 + ddxx - bbxx + 2bdsx = ddss$ jam convertetur in æquationem numeralem $x^4 + 6808\,xx + 672,792\,x = 43,507,216$; ut à Raphſono declaratur.

Hanc igitùr æquationem numeralem $x^4 + 6808\,xx + 672,792\,x = 43,507,216$ nunc reſolvemus ope limitum in tractatu ſuperiore Florimondi de Beaune inventorum.

Hæc æquatio numeralis eſt ejuſdem formæ ac æquatio generalis $x^4 + mmxx + n^3x = p^4$ in Propoſitione quartâ Capitis ſeptimi tractatûs præcedentis; et co-efficiens litteralis mm in iſtâ æquatione generali fit in hâc æquatione numerali $=$ 6808, et n^3 in iſtâ æquatione fit in hâc $=$ 672,792, et p^4 in iſtâ fit in hâc $=$ 43,507,216. Sed, per iſtam Propoſitionem quartam, æquatio generalis $x^4 + mmxx + n^3x = p^4$ habet tantùm unam radicem veram: et hæc radix unica eſt major quàm fractio $\frac{p^4}{p^3 + mmp + n^3}$, ſed minor quàm $\frac{p^4}{n^3}$, et quàm $\frac{pp}{m}$, ſeu $\sqrt{\frac{p^4}{mm}}$, et quàm p. Ergò æquatio numeralis $x^4 + 6808\,xx + 672,792\,x = 43,507,216$

$= 43,507,216$ habebit etiàm tantùm unam radicem ; et hæc radix unica erit major quàm fractio $\frac{43,507,216}{\overline{43,507,216}^{\frac{3}{4}} + 6808 \times \overline{43,507,216}^{\frac{1}{4}} + 672,792}$, ſed minor quàm $\frac{43,507,216}{672,792}$, et quàm $\sqrt{\frac{43,507,216}{6808}}$, et quàm $\sqrt[4]{43,507,216}$. Hos igitùr limites nunc neceſſe eſt ut computemus, ſed non ad plures quàm duas figuras decimales. Hoc verò fieri poteſt hoc modo.

Quoniàm p^4 eſt $= 43,507,216$, pp erit $= 6500$, et p erit $= 80$, et p^3 erit $= 512,000$, et mmp erit $(= 6808 \times 80) = 544,640$, et $\overline{43,507,216}^{\frac{3}{4}} + 6808 \times \overline{43,507,216}^{\frac{1}{4}} + 672,792$ erit $(= 512,000 + 544,640 + 672,792) = 1,729,432$, et $\frac{43,507,216}{\overline{43,507,216}^{\frac{3}{4}} + 6808 \times \overline{43,507,216}^{\frac{1}{4}} + 672,792}$ erit $\left(= \frac{43,507,216}{1,729,432}\right) = 25$, et $\frac{43,507,216}{672,792} = 63$, et $\frac{43,507,216}{6808}$ erit $= 6390$, et $\sqrt{\frac{43,507,216}{6808}}$ erit $(= \sqrt{6390}) = 79$. Ergò radix unica æquationis $x^4 + 6808\,xx + 672,792\,x = 43,507,216$ erit major quàm 25, ſed minor quàm 63, et quàm 79, et quàm 80, ideóque erit mediæ cujuſdam magnitudinis inter 25 et 63. Hi limites ſunt à ſe invicèm admodùm remoti, et non multum nos adjuvant in conjecturâ faciendâ de magnitudine radicis x. Ponamus tamèn quòd hæc radix ſit $= 40$, et ſubſtituamus 40 pro x in quantitate trinomiâ $x^4 + 6808\,xx + 672,792\,x$, ut videamus an valor ejuſdem quantitatis ex hâc ſubſtitutione oriundus erit major, vel erit minor, quàm numerus abſolutus, ſeu quantitas nota, 43,507,216 in propoſitâ æquatione, et inde ſciamus an numerus 40 ſit major, vel ſit minor, quàm vera radix iſtius æquationis.

Poſito autèm quòd x ſit $= 40$, xx erit $= 1600$, et x^4 erit $= 2,560,000$, et $6808xx$ erit $(= 6808 \times 1600) = 10,892,800$, et $672,792\,x$ erit $(= 672,792 \times 40) = 26,911,680$, et pro-inde quantitas trinomia $x^4 + 6808\,xx + 672,792x$ erit $(= 2,560,000 + 10,892,800 + 26,911,680) = 40,364,480$; qui numerus eſt minor quàm 43,507,212, ſeu terminus abſolutus æquationis $x^4 + 6808\,xx + 672,792\,x = 43,507,216$. Ergò numerus 40 eſt minor quàm x in iſtâ æquatione. Differentia verò inter 40,364,480 et 43,507,216 non eſt magna, et pro-inde numerus 40 non erit multo minor quàm radix x.

Faciamus igitùr ſecundam conjecturam de magnitudine radicis x, et ponamus quòd ſit $= 42$, et ſubſtituamus 42 pro x in quantitate trinomiâ $x^4 + 6808\,xxx + 672,792\,x$.

Posito itaque quòd x sit $= 42$, xx erit $(= \overline{42}^2) = 1764$, et x^4 erit $(= \overline{42}^4) = 3,111,696$, et $6808\,xx$ erit $(= 6808 \times 1764) = 12,009,312$, et $672,792x$ erit $(= 672,792 \times 42) = 28,257,264$, et pro-inde quantitas trinomia $x^4 + 6808\,xx + 672,792\,x$ erit $(= 3,111,696 + 12,009,312 + 28,257,264) = 43,378,272$; qui numerus est minor quàm 43,507,216. Ergò numerus 42 est minor quàm radix x æquationis propositæ $x^4 + 6808\,xx + 672,792\,x = 43,507,216$. Est autèm satìs propinquus radici x út fiat fundamentum processûs approximatorii, per Methodum à Raphsono traditam, quo inveniamus novum valorem istius radicis qui erit vero ejusdem valori multo propinquior quàm 42.

Ponamus igitùr z pro differentiâ inter x et 42; et erit $x = 42 + z$. Ergò xx erit $= \overline{42 + z}^2 (= \overline{42}^2 + 2 \times 42 \times z +$ &c$) = 1764 + 84 \times z +$ &c, et x^4 erit $= \overline{42 + z}^4 (= \overline{42}^4 + 4 \times \overline{42}^3 \times z +$ &c $= \overline{42}^4 + 4 \times 74,088 \times z +$ &c$) = 3,111,696 + 296,352 \times z +$ &c, et $6808\,xx$ erit $= 6808 \times \overline{1764 + 84z +}$ &c $(= 6808 \times 1764 + 6808 \times 84\,z +$ &c$) = 12,009,312 + 571,872 \times z +$ &c, et $672,792\,x$ erit $= 672,792 \times \overline{42 + z}$ $(= 672,792 \times 42 + 672,792 \times z) = 28,257,264 + 672,792\,z$; et pro-inde quantitas trinomia $x^4 + 6808\,xx + 672,792\,x$ erit æqualis quantitati multinomiæ

$$\left\{\begin{array}{r} 3,111,696 + 296,352 \times z +\text{&c} \\ + 12,009,312 + 571,872 \times z +\text{&c} \\ + 28,257,264 + 672,792 \times z \end{array}\right\} =$$

$$43,378,272 + 1,541,016 \times z +\text{&c.}$$

Sed quantitas trinomia $x^4 + 6808\,xx + 672,792x$ est $= 43,507,216$.

Ergò quantitas $43,378,272 + 1,541,016 \times z +$ &c erit etiàm $= 43,507,216$; et pro-inde $1,541,016 \times z +$ &c erit $(= 43,507,216 - 43,378,272) = 128,944$, et $z +$ &c erit $= \frac{128,944}{1,541,016} = 0.083$. Ergò x, seu $42 + z$, erit $(= 42 + 0.083) = 42.083$ quàm proximè, hoc est, radix æquationis propositæ $x^4 + 6808\,xx + 672,792\,x = 43,507,216$ erit $= 42.083$ quàm proximè. Q. E. I.

Substituatur

Subſtituatur jam numerus 42.083 pro x in quantitate trinomiâ $x^4 + 6808\,xx + 672{,}792\,x$, ut videamus an valor iſtius quantitatis inde oriundus erit major, vel erit minor quàm 43,507,216, et pro-inde ut ſciamus an 42.083 erit major, vel erit minor, quàm radix æquationis $x^4 + 6808\,xx + 672{,}792\,x = 43{,}507{,}216$. Hæc ſubſtitutio erit hujuſmodi.

Si x ſit $= 42.083$, xx erit $(= \overline{42.083}^2) = 1770.978{,}889$, et x^3 erit $(= \overline{42.083}^3) = 74{,}528.104{,}585{,}787$, et x^4 erit $(= \overline{42.083}^4) = 3{,}136{,}366.225{,}283{,}674{,}321$, et $6808\,xx$ erit $(= 6808 \times 1770.978{,}889) = 12{,}056{,}824.276{,}312$, et $672{,}792\,x$ erit $(= 672{,}792 \times 42.083) = 28{,}313{,}105.736$, et pro-inde quantitas trinomia $x^4 + 6808\,xx + 672{,}792\,x$ erit $(= 3{,}136{,}366.225{,}283{,}674{,}321 + 12{,}056{,}824.276{,}312 + 28{,}313{,}105.736) = 43{,}506{,}296.237{,}595{,}674{,}321$; qui numerus eſt paulò minor quàm 43,507,216, ideóque numerus 42.083 erit paulò minor quàm x, ſeu radix æquationis $x^4 + 6808\,xx + 672{,}792\,x = 43{,}507{,}216$.

Ponamus jam z pro differentiâ inter 42.083 et x; et erit $x = 42.083 + z$. Ergò xx erit $(= \overline{42.083 + z}^2 = \overline{42.083}^2 + 2 \times 42.083 \times z + zz) = 1770.978{,}889 + 84.166 \times z + zz$, et x^4 erit $= \overline{42.083 + z}^4 = \overline{42.083}^4 + 4 \times \overline{42.083}^3 \times z + 6 \times \overline{42.083}^2 \times zz + \&c = \overline{42.083}^4 + 4 \times 74{,}528.104{,}585{,}787 \times z + 6 \times 1770.978{,}889 \times zz + \&c) = 3{,}136{,}366.225{,}283{,}674{,}321 + 298{,}112.418{,}343{,}148 \times z + 10{,}625.873{,}334 \times zz + \&c$, et $6808\,xx$ erit $(= 6808 \times \overline{1770.978{,}889 + 84.166 \times z + zz} = 6808 \times 1770.978{,}889 + 6808 \times 84.166 \times z + 6808 \times zz) = 12{,}056{,}824.276{,}312 + 573{,}002.128 \times z + 6808\,zz$, et $672{,}792\,x$ erit $(= 672{,}792 \times \overline{42.083 + z} = 672{,}792 \times 42.083 + 672{,}792 \times z) = 28{,}313{,}105.736 + 672{,}792 \times z$; et pro-inde quantitas trinomia $x^4 + 6808\,xx + 672{,}792\,x$ erit æqualis quantitati multinomiæ

$$\left\{\begin{array}{l} 3{,}136{,}366.225{,}283{,}674{,}321 + 298{,}112.418{,}343{,}148 \times z \\ \qquad + 10{,}625.873{,}334 \times zz + \&c, \\ + 12{,}056{,}824.276{,}312{,}000{,}000 + 573{,}002.128{,}000{,}000 \times z \\ \qquad + 6{,}808.000{,}000 \times zz \\ + 28{,}313{,}105.736{,}000{,}000{,}000 + 672{,}792.000{,}000{,}000 \times z \end{array}\right\}$$

$= 43{,}506{,}296.237{,}595{,}674{,}321 + 1{,}543{,}906.546{,}343{,}148 \times z + 17{,}433.873{,}334 \times zz + \&c$

Sed quantitas trinomia $x^4 + 6808\,xx + 672{,}792\,x$ eſt $= 43{,}507{,}216$.

Ergò quantitas $43{,}506{,}296.237{,}595{,}674{,}321 + 1{,}543{,}906.546{,}343{,}148 \times z + 17{,}433.873{,}334 \times zz$ +&c erit etiàm $= 43{,}507{,}216$; et pro-inde quantitas $1{,}543{,}906.546{,}343{,}148 \times z$ erit $(= 43{,}507{,}216 - 43{,}506{,}296.237{,}595{,}674{,}321 - 17{,}433.873{,}334 \times zz) = 919.762{,}404{,}325{,}679 - 17{,}433.873{,}334 \times zz$, et z erit $= \frac{919.762{,}404{,}325{,}679}{1{,}543{,}906.546{,}343{,}148} - \frac{17{,}433.873{,}334 \times zz}{1{,}543{,}906.546{,}343{,}148} = 0.000{,}595{,}737$ — &c. Ergò x, ſeu $42.083 + z$, erit $(= 42{,}083 + 0.000{,}595{,}737 -$ &c$) = 42.083{,}595{,}737 -$, hoc eſt, radix æquationis $x^4 + 6808\,xx + 672{,}792\,x = 43{,}507{,}216$ erit propè æqualis numero $42.083{,}595{,}737$, ſed eodem aliquantulum minor. Q. E. I.

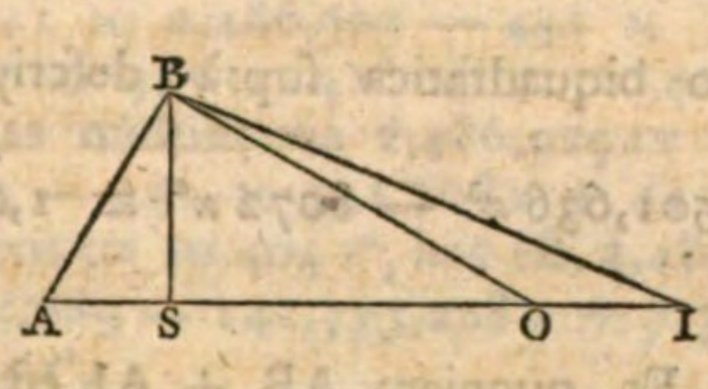
Ut verò ſciamus quot figuræ hujus numeri $42.083{,}595{,}737$ ſunt veræ, ſubſtituamus pro fractione $\frac{17{,}433.873{,}334 \times zz}{1{,}543{,}906.546{,}343{,}148}$ (quæ ſubtrahenda eſt à fractione $0.000{,}595{,}737$, ſeu valore quantitatis z jam invento, ut ejus valor accuratiùs obtineatur,) fractionem $\frac{17{,}434 \times \overline{0.000{,}6}^2}{1{,}543{,}906}$, ſeu $\frac{17{,}434 \times 0.000{,}000{,}36}{1{,}543{,}906}$, ſeu $\frac{0.006{,}276{,}24}{1{,}543{,}906}$, ſeu $0.000{,}000{,}004$, quæ major eſt priore fractione; et à valore z jàm invento, ſcilicet, $0.000{,}595{,}737$, ſubtrahamus hunc numerum $0.000{,}000{,}004$. Et numerus reſiduus inde ortus, nempe $0.000{,}595{,}733$, erit minor quàm verus valor quantitatis z. Ergò iſte verus valor erit major quàm $0.000{,}595{,}733$, ſed minor quàm $0.000{,}595{,}737$, et pro-inde verus valor quantitatis x, ſeu $42.083 + z$, erit major quàm $42.083{,}595{,}733$, ſed minor quàm $42.083{,}595{,}737$; hoc eſt, primæ decèm figuræ ejuſdem erunt $42.083{,}595{,}73$.
Q. E. I.

Numerus à Raphſono inventus pro valore radicis x in hâc æquatione $x^4 - 6808\,xx + 672{,}792\,x = 43{,}507{,}216$ eſt $42.083{,}595{,}734$, qui cum noſtro numero $42.083{,}595{,}737$ convenit in decèm primis figuris $42.083{,}595{,}73$.

Exemplum 6, à Raphſoni Problemate decimo octavo deſumptum.

Reſolvenda ſit æquatio biquadratica $17575.969,772,051\,x - 1221.125\,xx + 62.145,069,375\,x^3 - x^4 = 135,869.138,875,123,885$, quæ oritur ex diviſione terminorum omnium æquationis $141,873,228\,x - 9,856,921\,xx + 501,636\,x^3 - 8072\,x^4 = 1,096,735,689$ per numerum 8072. Hæc autèm æquatio derivatur ex ſolutione cujuſdam Problematis Geometrici, quod à Raphſono eſt propoſitum, et cujus ſolutio eſt ab eo, ſed breviſſimè, indicata. Poteſt autèm hoc Problema pleniùs enunciari et ad æquationem biquadraticam $141,873,228\,x - 9,856,921\,xx + 501,636\,x^3 - 8072\,x^4 = 1,096,735,689$ perduci hoc modo.

Problema Geometricum.



Sit triangulum planum ABI habens angulum B, ſeu ABI, obtuſum; et à puncto B ducatur recta linea BO ad rectos angulos lateri BA, et occurrens baſi AI in puncto O. Et ſint datæ lineæ ſequentes, ſcilicet, Latus BI, et recta linea BO, et ſumma lateris AB et baſis AI; nempe, ſit Latus BI = 32, et recta linea BO = 21, et ſumma lateris AB et baſis AI, ſeu quantitas AB + AI, = 51. Requiritur invenire longitudinem lateris AB.

Solutio.

A puncto B ducatur recta linea BS ad rectos angulos baſi AI trianguli ABI, et eidem occurrens in puncto S. Et dividetur triangulum obtuſangulum ABI in duo triangula rectangula ABS et BSI. Ergò AB^q erit = $AS^q + BS^q$, et BI^q erit = $BS^q + SI^q$, et pro-inde $AB^q - AS^q$ erit = BS^q, et $BI^q - SI^q$ erit etiàm = BS^q. Ergò $AB^q - AS^q$ erit = $BI^q - SI^q$. Sed SI

eſt

eſt = AI − AS, et pro-inde SI^q erit = $\overline{AI - AS}^2$ = AI^q − 2AI × AS + AS^q. Ergò BI^q − SI^q erit (= BI^q − $\overline{AI^q - 2AI \times AS + AS^q}$) = BI^q − AI^q + 2AI × AS − AS^q. Ergò AB^q − AS^q, (quòd oſtenſum eſt eſſe = BI^q − SI^q,) erit etiàm = BI^q − AI^q + 2AI × AS − AS^q, et (addendo utrinque AS^q,) erit AB^q = BI^q − AI^q + 2AI × AS, et (addendo utrinque AI^q,) erit AB^q + AI^q = BI^q + 2AI × AS, et (auferendo utrinque BI^q,) erit AB^q + AI^q − BI^q = 2AI × AS, et denique (diviſione factâ per 2AI) erit $\frac{AB^q + AI^q - BI^q}{2AI}$ = AS, ut Raphſonus affirmat. Sed, quoniàm angulus ABO eſt rectus, AO^q erit = AB^q + BO^q, et pro-inde AO erit = $\sqrt{AB^q + BO^q}$; et, per propoſitionem octavam Libri ſexti Elementorum Euclidis, AO eſt ad AB ut AB ad AS, et pro-inde AS erit = $\frac{AB^q}{AO}$, ſeu $\frac{AB^q}{\sqrt{AB^q + BO^q}}$, ut Raphſonus affirmat. Sed ante oſtenſum eſt quòd AS eſt = $\frac{AB^q + AI^q - BI^q}{2AI}$. Ergò $\frac{AB^q + AI^q - BI^q}{2AI}$ erit = $\frac{AB^q}{\sqrt{AB^q + BO^q}}$. Et hinc æquatio biquadratica ſuprà deſcripta, nempe, 141,873,228 x − 9,856,921 xx + 501,636 x^3 − 8072 x^4 = 1,096,735,689 derivari poterit hoc modo.

Ponatur x pro quæſitâ lineâ AB. Et, quoniam AB + AI eſt = 51, erit AI = 51 − AB = 51 − x, et pro-inde AI^q erit (= $\overline{51 - x}^2$) = 2601 − 102 x + xx, et AB^q + AI^q erit (= xx + 2601 − 102x + xx) = 2601 − 102 x + 2 xx.

Et, quoniàm BI, per conditiones Problematis, eſt = 32, BI^q erit (= $\overline{32}^2$) = 1024, et pro-inde $\frac{AB^q + AI^q - BI^q}{2AI}$ erit = $\frac{2601 - 102x + 2xx - 1024}{2 \times \overline{51 - x}}$ = $\frac{1577 - 102x + 2xx}{102 - 2x}$.

Porrò, quoniàm BO eſt = 21, BO^q erit = ($\overline{21}^2$) = 441, et AB^q + BO^q erit = xx + 441, et fractio $\frac{AB^q}{\sqrt{AB^q + BO^q}}$ erit = $\frac{xx}{\sqrt{xx + 441}}$.

Sed

Sed $\frac{AB^q + AI^q - BI^q}{2AI}$ eſt $= \frac{AB^q}{\sqrt{AB^q + BO^q}}$.

Ergò fractio $\frac{1577 - 102x + 2xx}{102 - 2x}$ erit $=$ fractioni $\frac{xx}{\sqrt{xx + 441}}$. Et pro-inde (quadrando has fractiones,) erit fractio $\frac{2,486,929 - 321,708x + 16,712xx - 408x^3 + 4x^4}{10,404 - 408x + 4xx}$ $= \frac{x^4}{xx + 441}$, et (multiplicando utraſque partes hujus æquationis per quantitatem trinomiam $10,404 - 408x + 4xx$,) quantitas quinquinomia $2,486,929 - 321,708x + 16,712xx - 408x^3 + 4x^4$ erit $= \frac{10,404x^4 - 408x^5 + 4x^6}{xx + 441}$, et (multiplicando utraſque partes æquationis per quantitatem binomiam $xx + 441$,) quantitas multinomia $2,486,929xx - 321,708x^3 + 16,712x^4 - 408x^5 + 4x^6 + 441 \times 2,486,929 - 441 \times 321,708x + 441 \times 16,712xx - 441 \times 408x^3 + 441 \times 4x^4$ erit $= 10,404x^4 - 408x^5 + 4x^6$, et (addendo utrinque $408x^5$,) quantitas multinomia $2,486,929xx - 321,708x^3 + 16,712x^4 + 4x^6 + 441 \times 2,486,929 - 441 \times 321,708x + 441 \times 16,712xx - 441 \times 408x^3 + 441 \times 4x^4$ erit $= 10,404x^4 + 4x^6$, et (auferendo utrinque $4x^6$,) quantitas multinomia $2,486,929xx - 321,708x^3 + 16,712x^4 + 441 \times 2,486,929 - 441 \times 321,708x + 441 \times 16,712xx - 441 \times 408x^3 + 441 \times 4x^4$ erit $= 10,404x^4$, hoc eſt, $2,486,929xx - 321,708x^3 + 16,712x^4 + 1,096,735,689 - 141,773,228x + 7,369,992xx - 179,928x^3 + 1764x^4$ erit $= 10,404x^4$,

ſeu $\left\{\begin{array}{l} 2,486,929xx - 321,708x^3 + 16,712x^4 - 141,773,228x \\ + 7,369,992xx - 179,928x^3 + 1,764x^4 + 1,096,735,689 \end{array}\right\}$

erit $= 10,404x^4$, ſeu $9,856,921xx - 501,636x^3 + 18,476x^4 - 141,773,228x + 1,096,735,689$ erit $= 10,404xx$, et (addendo utrinque $501,636x^3 + 141,773,228x$) quantitas trinomia $9,856,921xx + 18,476x^4 + 1,096,735,689$ erit $= 10,404xx + 501,636x^3 + 141,773,228x$, et (auferendo utrinque $10,404x^4$) quantitas trinomia $9,856,921xx + 8072x^4 + 1,096,735,689$ erit $= 501,636x^3 + 141,773,228x$, et (auferendo utrinque $9,856,921xx + 8072x^4$,) quantitas $1,096,735,689$ erit $= 141,773,228x - 9,856,921xx + 501,636x^3 - 8072x^4$, ſeu quantitas quadrinomia $141,773,228x - 9,856,921xx + 501,636x^3 - 8072x^4$ erit $= 1,096,735,689$, et (dividendo omnes terminos per 8072, co-efficientem quantitatis x^4,) erit quantitas quadrinomia $17,575.969,772,051x - 1221.125, \times xx + 62.145,069,375x^3 - x^4 = 135,869.138,875,123,885$; quæ eſt æquatio nunc reſolvenda.

Hæc

Hæc æquatio numeralis $17,575.969,772,051\,x - 1221.125\,xx + 62.145,069,375\,x^3 - x^4 = 135,869.138,875,123,885$ eſt ejuſdem formæ ac æquatio generalis $n^3x - mmxx + lx^3 - x^4 = p^4$ in Propoſitione primâ capitis ſexti tractatûs præcedentis; et co-efficiens litteralis l in termino lx^3 in iſtâ æquatione generali fit in hâc numerali $= 62.145,069,375$, ſeu (negligendo partem hujus numeri fractionalem,) $= 62$, et mm in iſtâ æquatione generali fit in hâc numerali $= 1221.125$, ſeu, propè, $= 1221$, et n^3 in iſtâ generali fit in hâc numerali $= 17,575.969,772,051$, ſeu, propè, $17,575$, et p^4 in iſtâ generali fit in hâc numerali $= 135,869.138,875,123,885$, ſeu, propè, $135,869$. Sed, per iſtam propoſitionem primam, æquatio generalis $n^3x - mmxx + lx^3 - x^4 = p^4$ poteſt habere quatuòr radices veras; et quælibet harum radicum erit major quàm $\frac{p^4}{\overline{l+n}^3}$, ſeu $\frac{p^4}{\overline{l^3+3lln+3lnn+n^3}}$, et quàm $\frac{p^4}{\overline{l^3+2lln+lnn+n^3}}$, et quàm minor horum duorum terminorum, n et $\frac{p^3}{lnn+n^3}$, ſed minor quàm $l+n$. Ergò æquatio numeralis $17,575.969$ &c $\times x - 1221.125 \times xx + 62.145$ &c $\times x^3 - x^4 = 135,869.138$, &c poterit, fortaſſe, habere quatuòr radices veras; et quælibet harum radicum erit major quàm $\frac{135,869}{\overline{62}^3+2\times\overline{62}^2\times\overline{17,575}^{\frac{1}{3}}+62\times\overline{17,575}^{\frac{2}{3}}+17,575}$, et quàm minor horum duorum terminorum $\overline{17,575}^{\frac{1}{3}}$ et $\frac{\overline{135,869}^{\frac{3}{4}}}{62\times\overline{17,575}^{\frac{2}{3}}+17,575}$. Hos igitùr limites nunc neceſſe eſt ut computemus. Hoc verò fieri poteſt hoc modo.

Quoniàm p^4 eſt $= 135,869$, pp erit $(= \sqrt{135,869}) = 368$, et p erit $(= \sqrt{368}) = 19$; et, quoniàm l eſt $= 62$, ll erit $= (\overline{62}^2) = 3744$, et l^3 erit $(= \overline{62}^3) = 232,128$; et, quoniàm n^3 eſt $= 17,575$, n erit $= 26$, et nn erit $= 676$. Ergò $2lln$ erit $(= 2 \times 3744 \times 26 = 7488 \times 26) = 194,688$, et lnn erit $(= 62 \times 676) = 41,912$, et $l^3 + 2lln + lnn + n^3$ erit $(= 232,128 + 194,688 + 41,912 + 17,575) = 286,303$, et $\frac{p^4}{l^3+2lln+lnn+n^3}$ erit $(= \frac{135,869}{286,303}) = 0.47$; et $\frac{p^3}{lnn+n^3}$ erit $(= \frac{pp\times p}{lnn+n^3} = \frac{368\times 19}{62\times 676+17,575} = \frac{6992}{41,912+17,575} = \frac{6992}{59,487}) = 0.117$. Ergò quælibet radicum æquationis $17,575.969$ &c $\times x - 1221.125 \times xx + 62.145$, &c $\times x^3 - x^4 =$

$135,869.138$,

135,869.138, &c erit major quàm 0.47 et quàm minor harum duarum quantitatum 26 et 0.117, hoc eſt, quàm 0.117, ſed minor quàm ($l + n$, ſeu 62 + 26, ſeu) 88. Ergò maxima harum radicum (ſi ſint plures,) erit major quàm 0.117 et quàm 0.47, ſed minor quàm 88, ideóque erit mediæ cujuſdam magnitudinis inter 0.47 et 88. Hi limites ſunt à ſe invicèm admodùm remoti, et non multum nos adjuvant in conjecturâ faciendâ de magnitudine radicis maximæ x. Ponamus tamèn quòd hæc radix maxima ſit = 40, et ſubſtituamus 40 pro x in quantitate quadrinomiâ $17,575x - 1221xx + 62x^3 - x^4$, ut videamus an valor ejuſdem quantitatis ex iſtâ ſubſtitutione oriundus erit major, vel erit minor, quàm numerus abſolutus, ſeu quantitas nota, 135,869, ſeu valor ejuſdem quantitatis in propoſitâ æquatione $17,575x - 1221xx + 62x^3 - x^4 = 135,869$, et quænam erit horum valorum differentia, ut inde fiamus aptiores ad ſecundam conjecturam faciendam de hujus radicis magnitudine.

Poſito autèm quòd x ſit = 40, xx erit ($= 40^2$) 1600, et x^3 erit ($= \overline{40}^3$) = 64,000, et x^4 erit ($= \overline{40}^4$) = 2,560,000, et $17,575x$ erit ($= 17,575 \times 40$) = 703,000, et $1221xx$ erit ($= 1221 \times 1600$) = 1,753,600, et $62x^3$ erit ($= 62 \times 64,000$) = 3,968,000; et pro-inde quantitas quadrinomia $17,575x - 1221xx + 62x^3 - x^4$ erit ($= 703,000 - 1,753,600 + 3,968,000 - 2,560,000 = 4,671,000 - 4,313,600$) = 357,400; qui numerus eſt multo major quàm 135,869, ſeu valor ejuſdem quantitatis quadrinomiæ in æquatione $17,575x - 1221xx + 62x^3 - x^4 = 135,869$. Ergò radix maxima iſtius æquationis non eſt æqualis numero 40. Non autèm à numero 40 multùm differt, quoniam, ſi ponamus x eſſe æqualem numero 41, (qui numerum 40 tantùm unitate major eſt,) quantitas quadrinomia $17,575x - 1221xx + 62x^3 - x^4$ erit = numero 115,415, ut ex ſubſtitutione numeri 41 pro x in iſtâ quantitate quadrinomiâ fiet manifeſtum. Hæc autem ſubſtitutio ita fiet. Si x ſit = 41, xx erit ($= \overline{41}^2$) = 1681, et x^3 erit ($= \overline{41}^3$) = 68,921, et x^4 erit ($= \overline{41}^4$) = 2,825,761, et $17,575x$ erit ($= 17,575 \times 41$) = 720,575, et $1221xx$ erit ($= 1221 \times 1681$) = 2,052,501, et $62x^3$ erit ($= 62 \times 68,921$) = 4,273,102, et pro-inde quantitas quadrinomia $17,575x - 1221xx + 62x^3 - x^4$ erit ($= 720,575 - 2,052,501 + 4,273,102 - 2,825,761 = 4,993,677 - 4,878,262$) = 115,415; qui numerus eſt minor quàm 135,869, ſeu valor ejuſdem quantitatis quadrinomiæ in æquatione $17,575x - 1221xx + 62x^3 - x^4 = 135,869$. Ergò nunc concludere debemus quòd radix maxima hujus æquationis eſt major quàm 40, ſed minor quàm 41: Et hinc poterimus commodè inſtituere pro-

ceſſum approximatorium ad verum hujus radicis valorem accuratiùs inveſtigandum, per Methodum à Raphſono traditam, ponendo z pro exceſſu numeri 41 ſuper verum iſtum valorem, et ſubſtituendo $41 - z$ pro x in æquatione propoſitâ $17,575.969 \&c \times x - 1221.125 \times xx + 62.145 \&c \times x^3 - x^4 = 135,869.138$, &c, modo ſequente.

Si x ſit æqualis $41 - z$, xx erit $= \overline{41 - z}^2$ $(= \overline{41}^2 - 2 \times 41 \times z + \&c) = 1681 - 82 \times z + \&c$, et x^3 erit $= \overline{41 - z}^3$ $(= \overline{41}^3 - 3 \times \overline{42}^2 \times z + \&c = \overline{41}^3 - 3 \times 1681 \times z + \&c) = 68,921 - 5043 \times z + \&c$, et x^4 erit $= \overline{41 - z}^4$ $(= \overline{41}^4 - 4 \times 41^3 \times z + \&c = \overline{41}^4 - 4 \times 68,921, \times z + \&c) = 2,825,761 - 275,684 \times z + \&c$, et $17,575.969,772,051\, x$ erit $= 17,575.969,772,051 \times 41 - z$ $(= 17,575.969,772,051 \times 41 - 17,575.969,772,051 \times z) = 720,614.760,654,091 - 17,575.969,772,051 \times z$, et $1221.125\, xx$ erit $(= 1221.125 \times 1681 - 82\, z + \&c = 1221.125 \times 1681 - 1221.125 \times 82\, z + \&c) = 2,052,711.125 - 100,132.250 \times z + \&c$, et $62.145,069,375\, x^3$ erit $(= 62.145,069,375 \times 68,921 - 5043 \times z + \&c = 62.145,069,375 \times 68,921 - 62.145,069,375 \times 5043 \times z + \&c) = 4,283,100.326,394,375 - 313,397.584,758,125 \times z + \&c$. Ergò quantitas quadrinomia $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4$ erit æqualis quantitati multinomiæ

$$\left\{\begin{array}{rrl} 720,614.760,654,091 & - & 17,575.969,772,051 \times z + \&c \\ - 2,052,711.125,000,000 & + & 100,132.250,000,000 \times z - \&c \\ + 4,283,100.326,394,375 & - & 313,397.584,758,125 \times z + \&c \\ - 2,825,761.000,000,000 & + & 275,684.000,000,000 \times z - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{rrl} 5,003,715.087,048,466 & + & 375,816.250,000,000 \times z + \&c \\ - 4,878,472.125,000,000 & - & 330,973.554,530,176 \times z - \&c \end{array}\right\}$$

$$= 125,242.962,048,466 + 44,842.695,469,824 \times z \&c.$$

Sed quantitas quadrinomia $17,575.969,772,051\, x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4$ eſt $= 135,869.138,875,123,885$.

Ergò quantitas $125,242.962,048,466 + 44,842.695,469,824 \times z$ &c erit etiàm $= 135,869.138,875,123,885$; et pro-inde $44,842.695,469,824 \times z$

$\times z$ erit ($= 135,869.138,875,123,885 - 125,242.962,048,466$) $=$ 10,626.176,826,657,885, et z erit $= \frac{10,626.176,826,657,885}{44,842.695,469,824} = 0.23$. Ergò x, ſeu $41 - z$, erit ($= 41 - 0.23 = 41.00 - 0.23$) $= 40.77$, hoc eſt, maxima radix æquationis propoſitæ $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4 = 135,869.138,875,123,885$ erit $= 40.77$ propè. Q. E. I.

Subſtituatur jam 40.77 pro x in quantitate quadrinomiâ $17,575.969,772.051x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4$, ut videamus an valor iſtius quantitatis ex hâc ſubſtitutione oriundus erit major, vel erit minor, quàm 135,869.138,875,123,885, ſeu valor ejuſdem quantitatis in æquatione propoſitâ. Hæc ſubſtitutio fiet hoc modo.

Si x ſit $= 40.77$, xx erit ($= \overline{40.77}^2$) $= 1662.1929$, et x^3 erit ($= \overline{40.77}^3$) $= 67,767.604,533$, et x^4 erit ($= \overline{40.77}^4$) $= 2,762,885.236,810,41$, et $17,575.969,772,051 \times x$ erit ($= 17,575.969,772,051 \times 40.77$) $=$ 716,572.287,606,519,27, et $1221.125 \times xx$ erit ($= 1221.125 \times 1662.1929$) $= 2,029,745.305,012,5$, et $62.145,069,375 \times x^3$ erit ($= 62.145,069,375 \times 67,767.604,533$) $= 4,211,422.485,080,849,476,875$; et pro-inde quantitas quadrinomia $17,575.969,772,051 \times x - 1221.125 \times x^2 + 62.145,069,375 \times x^3 - x^4$ erit $=$

$$\left\{\begin{array}{rr} 716,572.287,606,519,27 & - 2,029,745.305,012,5 \\ + 4,211,422.485,080,849,476,875 & - 2,762,885.236,810,41 \end{array}\right\}$$

$$= 4,927,994.772,687,368,746,875 - 4,792,630.541,822,91$$

$= 135,364.230,864,458,746,875$; qui numerus eſt paulò minor quàm 135,869.138,875,123,885, ſeu valor ejuſdem quantitatis quadrinomiæ in æquatione propoſitâ $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375, \times x^3 - x^4 = 135,869.138,875,123,885$.

Ex ſubſtitutionibus ſuprà deſcriptis numerorum 41 et 40.77 et 40 pro x in quantitate quadrinomiâ $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4$ manifeſtum eſt, quòd, quando x eſt $= 41$, iſta quantitas quadrinomia erit $= 115,415$, et, quando x eſt $= 40.77$, iſta quantitas quadrinomia eſt $= 135,364$, et, quando x eſt $= 40$, iſta quantitas eſt $=$

357,400. Ergò, quando x eſt paulò minor quàm 40.77, ſed major quàm 40, hæc quantitas erit = 135,869.138,875,123,885; et pro-inde x, ſeu radix maxima æquationis propoſitæ $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375\,x^3 - x^4 = 135,869,138,875,123,885$ erit paulò minor quàm 40.77, et, ſi z ponatur pro differentiâ inter ipſam et 40.77, erit = $40.77 - z$.

Poſito autèm quòd x ſit = $40.77 - z$, xx erit = $\overline{40.77 - z}^2$ (= $\overline{40.77}^2 - 2 \times 40.77 \times z +$ &c = $\overline{40.77}^2 - 81.54 \times z +$ &c) = 1662.1929 $- 81.54 \times z +$ &c, et x^3 erit = $\overline{40.77 - z}^3$ (= $\overline{40.77}^3 - 3 \times \overline{40.77}^2 \times z +$ &c = $\overline{40.77}^3 - 3 \times 1662.1929 \times z +$ &c) = 67,767.604,533 $-$ 4986.5787 $\times z +$ &c, et x^4 erit = $\overline{40.77 - z}^4$ (= $\overline{40.77}^4 - 4 \times \overline{40.77}^3 \times z +$ &c = $\overline{40.77}^4 - 4 \times 67,767.604,533 \times z +$ &c) = 2,762,885.236,810,41 $-$ 271,070.418,132 $\times z +$ &c; et $17,575.969,772,051 \times x$ erit = $17,575.969,772,051 \times \overline{40.77 - z}$ (= $17,575.969,772,051 \times 40.77 - 17,575.969,772,051 \times z$,) = 716,572.287,606.519,27 $-$ $17,575.969,772,051 \times z$, et $1221.125 \times xx$ erit (= $1221.125 \times 1662.1929 - 81.54 \times z +$ &c = $1221.125 \times 1662.1929 - 1221.125 \times 81.54 \times z +$ &c) = 2,029,745.305,012,5 $-$ 99,570.532,50 $\times z +$ &c, et $62.145,069,375\,x^3$ erit (= $62.145,069,375 \times 67,767.604,533 - 4,986.578,7 \times z +$ &c = $62.145,069,375 \times 67,767.604,533 - 62.145,069,375 \times 4,986.578,7 \times z +$ &c) = $4,211,422.485,080,849,476,875 - 309,891.279,255,397,312,5 \times z +$ &c.

Ergò quantitas quadrinomia $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4$ erit = quantitati multinomiæ

$$\left\{\begin{array}{ll} 716,572.287,606,519,27 & - 17,575.969,772,051 \times z \\ - 2,029,745.305,012,500,00 & + 99,570.532,500,000 \times z - \&c \\ + 4,211,422.485,080,849,476,875 & - 309,891.279,255,397,312,5 \times z \\ & \quad [+ \&c \\ - 2,762,885.236,810,410 & + 271,070.418,132 \times z - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{ll} 4,927,994.772,687,368,746,875 & + 370,640.950,632 \times z \\ - 4,792,630.541,822,910,000,000 & - 327,467.249,027,448,312,5 \times z \end{array}\right\}$$

$$= 135,364.230,864,458,746,875 + 43,173.701,604,551,687,5 \times z \text{ \&c.}$$

Sed

Sed quantitas quadrinomia 17,575.969,772,051 × x — 1221.125 × xx + 62.145,069,375 × x^3 — x^4 eſt = 135,869.138,875,123,885.

Ergò quantitas 135,364.230,864,458,746,875 + 43,173.701,604,551,687,5 × z erit etiàm = 135,869.138,875,123,885 ; et pro-inde 43,173.701,604,551,687,5 × z erit (= 135,869.138,875,123,885 — 135,364.230,864,458,746,875) = 504,908,010,665,138,125, et z erit = $\frac{504.908,010,665,138,125}{43,173.701,604.551,687,5}$ = 0.011,694. Ergò x, ſeu 40.77 — z, erit (= 40.770,000 — 0.011,694) = 40.758,306, hoc eſt, radix maxima æquationis propoſitæ 17,575.969,772,051 × x — 1221.125 × xx + 62.145,069,375 × x^3 — x^4 = 135,869.138,875,123,885 erit = 40.758,306. Q. E. I.

Hujus numeri, ſeu valoris radicis x, ſex primæ figuræ 40.7583, ſunt, ut opinor, veræ.

Hæc autèm radix maxima æquationis propoſitæ 17,575.969,772,051 × x — 1221.125 × xx + 62.145,069,375 x^3 — x^4 = 135,869.138,875,123,885 non eſt illa ex pluribus ejuſdem æquationis radicibus quæ ſolutioni Problematis Geometrici ſuperiùs deſcripti, (ex quo iſta æquatio derivata fuit,) accommodatur. Non enim fieri poteſt ut recta linea, ſeu Latus, AB in triangulo ABI ſit æqualis numero 40.7583. Nam, quoniam angulus ABI eſt obtuſus, Baſis AI, quæ iſti angulo eſt oppoſitus, erit major quàm recta linea BI, quæ ſubtendit angulum acutum BAI. Sed BI eſt = 32. Ergò AI eſt major quàm 32. Sed, ſi linea AB (pro quâ x eſt poſita,) ſit = 40.7583, AI erit minor quàm BI. Nam, per conditiones Problematis, AB + AI eſt = 51, ideóque AI eſt = 51 — AB, hoc eſt, ſi AB ſit = 40.7583, AI erit = 51.0000 — 40.7583 = 10.241, ideóque multo minor quàm 32, ſeu BI. Ergò linea AB in triangulo ABI non poteſt eſſe = 40.7583, ſed iſto numero erit multo minor.

Ergò, ut Problema iſtud ſolvatur, et longitudo lineæ AB inveniatur, recurrendum eſt ad aliam aliquam ex pluribus radicibus æquationis propoſitæ 17,575.969,772,051 × x — 1221.125 × xx + 62.145,069,375 × x^3 — x^4 = 135,869.138,875,123,885 quàm iſtam maximam 40.7583. Et, in hunc finem nunc inveſtigabimus magnitudinem minimæ ex iſtis radicibus, quæ, per limites harum radicum ſuprà inventos, erit major quàm 0.47.

Quoniam igitùr hæc minima radix prædictæ æquationis eſt major quàm 0.47, ponamus primò quòd ſit = 1, et ſubſtituamus 1 pro x in quantitate quadrinomiâ 17,575 x — 1221 xx + 62 x^3 — x^4. Et valor iſtius quantitatis

inde

inde oriundus erit (= 17,575 — 1221 + 62 — 1 = 17,637 — 1222) = 16,415; qui numerus eſt multo minor quàm 135,869, ſeu valor ejuſdem quantitatis quadrinomiæ in æquatione $17,575x - 1221xx + 62x^3 - x^4 = 135,869$. Ergò 1 erit multo minor quàm minima radix iſtius æquationis.

Ponamus igitùr, ſecundo loco, quòd hæc radix minima ſit = 10, et ſubſtituamus 10 pro x in dictâ quantitate quadrinomiâ $17,575x - 1221xx + 62x^3 - x^4$. Et valor iſtius quantitatis inde oriundus erit 175,750 — 122,100 + 62,000 — 10,000 = 237,750 — 132,100 = 105,650; qui numerus eſt minor quàm 135,869, ſeu valor ejuſdem quantitatis quadrinomiæ in æquatione $17,575x - 1221xx + 62x^3 - x^4 = 135,869$. Ergò 10 erit minor quàm iſta radix minima.

Ponamus igitùr, tertio loco, quòd hæc radix minima ſit = 12, et ſubſtituamus 12 pro x in quantitate quadrinomiâ $17,575x - 1221xx + 62x^3 - x^4$. Et valor iſtius quantitatis inde oriundus erit (= 17,575 × 12 — 1221 × 144 + 62 × 1728 — 20,736 = 210,900 — 175,824 + 107,136 — 20,736 = 318,036 — 196,560) = 121,476; qui numerus eſt minor quàm 135,869. Ergò 12 erit etiàm minor quàm radix minima æquationis $17,575x - 1221xx + 62x^3 - x^4 = 135,869$.

Ponamus igitùr, quarto loco, quòd hæc radix minima ſit = 13, et ſubſtituamus 13 pro x in quantitate quadrinomiâ $17,575x - 1221xx + 62x^3 - x^4$. Et valor iſtius quantitatis inde oriundus erit (= 17,575 × 13 — 1221 × 169 + 62 × 2,197 — 28,561 = 228,475 — 207,349 + 136,214 — 28,561 = 364,689 — 235,910) = 128,779; qui numerus eſt minor quàm 135,869. Ergò 13 erit etiàm minor quàm radix minima æquationis $17,575x - 1221xx + 62x^3 - x^4 = 135,869$.

Ponamus igitùr, quinto loco, quòd hæc radix minima ſit = 14, et ſubſtituamus 14 pro x in quantitate quadrinomiâ $17,575x - 1221xx + 62x^3 - x^4$. Et valor iſtius quantitatis inde oriundus erit (= 17,575 × 14 — 1221 × 196 + 62 × 2744 — 38,416 = 246,050 — 239,316 + 170,128 — 38,416 = 416,178 — 277,732) = 138,446; qui numerus eſt major quàm 135,869. Ergò 14 erit major quàm radix minima æquationis $17,575x - 1221xx + 62x^3 - x^4 = 135,869$.

Ponamus igitùr, ſexto loco, quòd 13.6 ſit = x, ſeu radici minimæ æquationis $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4 = 135,869.138,875,123,885$, et ſubſtituamus 13.6 pro x in quantitate quadrinomiâ $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4$. Et xx erit $(= \overline{13.6}^2) = 184.96$, et x^3 erit $(= \overline{13.6}^3) = 2,515$.

2,515.456, et x^4 erit $(=\overline{13.6}^4) = 34,210.2016$, et pro-inde 17,575.969,772,051 $\times x$ erit $(= 17,575.969,772,051 \times 13.6) = 239,033.188,899,893,6$, et $1221.125 \times xx$ erit $(= 1221.125 \times 184.96) = 225,859.280,00$, et $62.145,069,375 \times x^3$ erit $(= 62.145,069,375 \times 2,515.456) = 156,323.187,629,760,000$. Ergò quantitas quadrinomia $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4$ erit $(= 239,033.188,899,893,6 - 225,859.280,000 + 156,323.187,629,760,000 - 34,210.2016 = 395,356.376,529,653,600 - 260,069.481,60) = 135,286.894,929,653,600$; qui numerus eſt minor quàm 135,869.138,875,123,885. Ergò numerus 13.6 erit minor quàm radix minima æquationis propoſitæ $17,575.969,772.051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4 = 135,869.138,875,123,885$.

Ponatur jam z pro differentiâ inter hanc minimam radicem x et numerum 13.6; et x erit $= 13.6 + z$. Ergò xx erit $= \overline{13.6 + z}^2$ $(= \overline{13.6}^2 + 2 \times 13.6 \times z + zz) = 184.96 + 27.2 \times z + zz$, et x^3 erit $= \overline{13.6 + z}^3$ $(= \overline{13.6}^3 + 3 \times \overline{13.6}^2 \times z + 3 \times \overline{13.6} \times zz + \&c = \overline{13.6}^3 + 3 \times 184.96 \times z + 3 \times 13.6 \times zz + \&c) = 2,515.456 + 554\,88 \times z + 40.8 \times zz + \&c$, et x^4 erit $= \overline{13.6 + z}^4$ $(= \overline{13.6}^4 + 4 \times \overline{13.6}^3 \times z + 6 \times \overline{13.6}^2 \times zz + \&c = \overline{13.6}^4 + 4 \times 2,515.456 \times z + 6 \times 184.96 \times zz + \&c) = 34,210.2016 + 10,061.824 \times z + 1109.76 \times zz + \&c$. Ergò 17,575.969,772,051 $\times x$ erit $(= 17,575.969,772,051 \times \overline{13.6 + z} = 17,575.969,772,051 \times 13.6 + 17,575.969,772,051 \times z) = 239,033.188,899,893,6 + 17,575.969,772,051 \times z$, et $1221.125 \times xx$ erit $= 1221.125 \times \overline{13.6 + z}^2$ $(= 1221.125 \times \overline{184.96 + 27.2 \times z + zz} = 1221.125 \times 184.96 + 1221.125 \times 27.2 \times z + 1221.125 \times zz) = 225,859.280,00 + 33,214.6000 \times z + 1221.125 \times zz$, et $62.145,069,375 \times x^3$ erit $= 62.145,069,375 \times \overline{13.6 + z}^3$ $(= 62.145,069,375 \times 2,515.456 + 554.88 \times z + 40.8 \times zz + \&c = 62.145,069,375 \times 2,515.456 + 62.145,069,375 \times 554.88 \times z + 62.145,069,375 \times 40.8 \times zz + \&c) = 156,323.187,629,760,000 + 34,483.056,094,800,00 \times z + 2,535.518,830,500,0 \times zz + \&c$; et pro-inde quantitas quadrinomia $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4$ erit æqualis quantitati multinomiæ

$$\left\{\begin{array}{rrl} 239,033.188,899,893,6 & + & 17,575.969,772,051 \times z \\ - 225,859.280,000,000,0 & - & 33,214.600,000,000 \times z \\ + 156,323.187,629,760,0 & + & 34,483.056,094,800 \times z \\ - 34.210.201,600,000,0 & - & 10,061.824,000,000 \times z \end{array}\right. \left.\begin{array}{l} - 1221.125 \times zz \\ + 2,535.518,830,5 \times zz + \&c \\ - 1,109.760,000,0 \times zz - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{rrl} 395,356.376,529,653,6 & + & 52,059.025,866,851 \times z \\ - 260,069.481,600,000,0 & - & 43,276.424,000,000 \times z \end{array}\right. \left.\begin{array}{l} \\ + 204.633,830,5 \times zz \ \&c \end{array}\right\}$$

$$= 135,286.894,929,653,6 + 8,782.601,866,851 \times z + 204.633,830,5 \times zz \ \&c.$$

Sed quantitas quadrinomia $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4$ eſt $= 135,869.138,875,123,885$.

Ergò quantitas $135,286.894,929,653,6 + 8,782.601,866,851 \times z + 204.633,830,5 \times zz$ &c erit etiàm $= 135,869.138,875,123,885$; et pro-inde $8,782.601,866,851 \times z + 204.633,830,5 \times zz$ erit $(= 135,869.138,875,123,885 - 135,286.894,929,653,6) = 582.243,945,470,285$, et $8,782.601,866,851 \times z$ erit $= 582.243,945,470,285 - 204.633,830,5 \times zz$, et z erit $= \frac{582.243,945,470,285}{8,782.601,866,851} - \frac{204.633,830,5 \times zz}{8,782.601,866,851} = 0.066,295 -$ &c. Ergò x, ſeu $13.6 + z$, erit $= 13.6 + 0.066,295 = 13.666,295, -$ &c, hoc eſt, radix minima æquationis $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 x^3 - x^4 = 135,869.138,875,123,885$ erit $= 13.666,295 -$ &c, ſeu aliquantulo minor quàm 13.666,295. Q. E. I.

Hujus numeri 13.666,295, inventi pro valore radicis x, quinque primæ figuræ 13.666 ſunt, ut opinor, veræ. Sed, ut hoc omninò certum fiat, inquirendum reſtat in quo figurarum decimalium loco prima figura fractionis decimalis quæ eſt æqualis termino $\frac{204.633,830,5 \times zz}{8,782.601,866,851}$, (qui ſubtrahendus eſt à valore quantitatis z jam invento, ſcilicet, à fractione decimali 0.066,295, ut ad verum ejus valorem propiùs accedamus,) calculum intrabit. Hæc autèm inquiſitio ſatìs commodè inſtitui poteſt hoc modo.

Terminus

Terminus $\frac{204.633,830,5 \times zz}{8,782.601,866,851}$ eſt minor quàm fractio $\frac{204.7 \times zz}{8,782}$, ſeu quàm 0.023. Sed z eſt minor quàm 0.066,295, et à fortiori, minor quàm 0.067; et pro-inde zz erit minor quàm $\overline{0.067}^2$, ſeu quàm 0.004,489, et, à fortiori, minor quàm 0.0045. Ergò terminus $\frac{204.633,830,5 \times zz}{8,782.601,866,851}$ erit minor quàm 0.023 × 0.0045, ſeu quàm 0.000,103,5; et pro-inde ejus prima figura non intrabit calculum ante quartum locum figurarum locum. Ergò ſubtractio hujus termini à quantitate, ſeu fractione decimali, 0.066,295, hîc inventâ pro valore quantitatis z, non efficiet ullam mutationem in tribus primis locis fractionum decimalium, ſeu in figuris 0.066; et pro-inde concludere poſſumus quòd tres primæ figuræ, 0.066, fractionis 0.066,295 hîc inventæ pro valore z, erunt veræ, ideóque quòd 13.666,295, ſeu valor radicis *x*, ſeu 13.6 + 0.066,295, erit vera in quinque primis figuris 13.666. Q. E D.

Per Raphſoni calculum hæc radix minima *x* eſt = 13.6792, qui valor differt à valore 13.666 hîc invento in figuris quartâ et quintâ; neſcio, an ejus errore, an meo. Sed cum meum calculum, poſt hanc diſcrepantiam inter valores radicis *x*, à nobis inventos, obſervatam, diligentèr repetiveram et ad examen revocaveram, nullum potui errorem in eo deprehendere. Adde quòd amicus meus *Gulielmus Frend*, vir doctiſſimus et in æquationum numeralium reſolutione peritiſſimus, cui hanc differentiam inter valores hujus radicis *x* à Raphſono et à me inventos indicavi, hanc æquationem $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 x^3 - x^4 = 135,869.138,875,123,885$ de novo reſolvendam in ſe ſuſcepit, et, computo diligentèr perfecto, invenit ejus radicem minimam eſſe = 13.666,19, qui numerus convenit cum numero 13.666,295 ſuprà invento in quinque primis figuris 13.666; et, ſi ponamus duas primas figuras fractionis decimalis quæ eſt æqualis termino $\frac{204\,633,830,5 \times zz}{8782.601,866,851}$ (qui à fractione decimali 0.066,295 eſt ſubtrahendus, ut valor quantitas z accuratiùs obtineatur,) et quæ oſtenſa eſt eſſe minor quam 0.000,1035, eſſe 0.000,10, et pro-inde valorem magìs accuratum quantitatis z eſſe = 0.066,295 — 0.000,10, ſeu 0.066,195, ſeu (negligendo figuram 5 in ſexto decimalium fractionum loco poſitam,) 0.066,19, numerus in hoc calculo inventus pro valore radicis *x* erit = 13.666,19, qui in omnibus ſeptèm figuris convenit cum numero per Frendi calculum invento. Et hinc concludo, ſine

ullâ de hâc re dubitatione, quòd quinque primæ figuræ numeri 13.666,19 inventi pro valore radicis x, nempe, 13.666, ſunt veræ, et quòd idem maximè eſt veriſimile, licèt non adhùc omninò certum, de duabus ultimis figuris 0.000,19 ejuſdem numeri 13.666,19.

Invento igitùr hoc numero 13.666 pro valore radicis minimæ æquationis $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375\, x^3 - x^4 = 135,869.138,875,123,885$, ſupereſt ut eum nunc adhibeamus ad ſolutionem Problematis Geometrici ſuprà deſcripti, ex quo hæc æquatio eſt derivata. Ponamus igitùr quod recta linea AB, in triangulo obtuſangulo BAI, cujus longitudo quæſita erat, et per reſolutionem hujus æquationis biquadraticæ determinanda, ſit æqualis 13.666; et inquiramus an hæc longitudo lineæ AB conditionibus Problematis ſatisfacere poterit, an (ut fieri ſuprà deprehendimus de numero 40.7583, quæ eſt maxima radix hujuſmet æquationis,) ſit iiſdem conditionibus, ſeu earum alicui, contraria. Hæc autèm inquiſitio ita poterit inſtitui.

Si AB ſit $=$ 13.666, AB^q erit $(= \overline{13.666}^2) = 186.759,556$.

Sed BO, per conditiones Problematis, eſt $= 21$; et pro-inde BO^q erit $(= \overline{21}^2) = 441$.

Ergò $AB^q + BO^q$ erit $(= 186.759,556 + 441) = 627.759,556$, hoc eſt, AO^q erit $= 627.759,556$; et pro-inde AO erit $(= \surd\, 627.759,556) = 25.055,130,3$.

Eſt verò AO ad AB ut AB ad AS. Ergò AS erit $= \frac{AB^q}{AO}$ $(= \frac{186.759,556}{25,055,130,3}) = 7.453,944$, et pro-inde AS^q erit $(= \overline{7.453,944}^2) = 55.561,281,155,136$.

Eſt autèm, ob angulum rectum ASB, $AB^q = AS^q + BS^q$; et pro-inde BS^q erit $= AB^q - AS^q$. Ergò BS^q erit $(= 186.759,556 - 55.561,281,155,136) = 131.198,274,844,864$.

Porrò, in triangulo rectangulo BSI, BI^q eſt $= BS^q + SI^q$, et pro-inde SI^q erit $= BI^q - BS^q$. Eſt autèm, per conditiones Problematis, BI $=$ 32.

Ergò

Ergò BI^q erit ($= \overline{32}^2$) $=$ 1024, et BI^q $-$ BS^q erit ($=$ 1024 $-$ 131.198,274,844,864) $=$ 892.801,725,155,136, ideóque SI^q erit etiàm $=$ 892.801,725,155,136, et SI erit ($= \surd$ 892.801,725,155,136) $=$ 29.879,787.

Ergò AS $+$ SI erit ($=$ 7.453,944 $+$ 29.879,787) $=$ 37.333,731; hoc eſt, AI erit $=$ 37.333,731. Et pro-inde AB $+$ AI erit $=$ 13.666 $+$ 37.333,731 $=$ 50.999,731; qui numerus eſt quàm proximè æqualis numero 51, cui ſumma rectarum AB et AI debet eſſe æqualis per conditiones Problematis. Ergò quantitas 13.666, ſeu valor radicis minimæ æquationis 17,575.969,772,051 $\times x -$ 1221.125 $\times xx +$ 62.145,069,375 $\times x^3 - x^4 =$ 135,869.138,875,123,885 hîc inventus, ſatisfacit conditionibus Problematis, et verè exprimit longitudinem rectæ lineæ AB, quam propoſitum erat invenire; et pro-inde Problema jam tandèm eſt ſolutum.

Sed, quanquàm Problema Geometricum, à quo æquatio prædicta derivata fuit, ſit jam ſolutum, reſolutio iſtius æquationis non eſt omninò perfecta. Quæri enim poteſt an non aliæ duæ ſint ejuſdem radices, et, ſi ſint, quænam ſint. Has igitùr quæſtiones reſolvere nunc conabor.

Ponamus *a* pro hujus æquationis radice minimâ, 13.666, jàm inventâ; et *x* pro quâlibet aliarum radicum hujus æquationis, ſive ſit una (ſcilicet, quantitas 40.7583 ſuperiùs inveſtigata,) ſive plures. Et, brevitatis gratiâ, ponamus *p* pro numero 62.145,069,375, ſeu co-efficiente cubi radicis *x* in æquatione 17,575.969,772,051 $\times$ *x* $-$ 1221.125 $\times$ *xx* $+$ 62.145,069,375 $\times x^3 - x^4$ $=$ 135,869.138,875,123,885, et *q* pro numero 1221.125, ſeu co-efficiente quadrati radicis *x*, et *r* pro numero 17,575.969,772,051, ſeu co-efficiente ipſius radicis *x*. Et erit quantitas quadrinomia $r - qa^2 + pa^3 - a^4 =$ 135,869.138,875,123,885, et quantitas quadrinomia $rx - qx^2 + px^3 - x^4$ erit etiàm $=$ 135,869.138,875,123,885. Ergò quantitas quadrinomia $rx - qx^2 + px^3 - x^4$ erit $=$ quantitati quadrinomiæ $ra - qa^2 + pa^3 - a^4$; et pro-inde (addendo utrinque $qx^2 + x^4$,) quantitas $rx + px^3$ erit $=$ quantitati $ra + qx^2 - qa^2 + pa^3 + x^4 - a^4$, et (auferendo utrinque $ra + pa^3$,) quantitas $rx - ra + px^3 - pa^3$ erit $=$ quantitati $qx^2 - qa^2 + x^4 - a^4$, hoc eſt, quantitas $r \times \overline{x - a} + p \times \overline{x^3 - a^3}$ erit $=$ quantitati $q \times \overline{x^2 - a^2} + \overline{x^4 - a^4}$. Ergò (dividendo omnes terminos per $x - a$,) quantitas $r + p \times xx + xa$

$\overline{\overline{xx + xa} + \overline{aa}}$ erit = quantitati $q \times \overline{x + a} + x^3 + x^2a + xa^2 + a^3$, hoc eſt, quantitas quadrinomia $r + pxx + pxa + paa$ erit = quantitati ſextinomiæ $qx + qa + x^3 + x^2a + xa^2 + a^3$, ſeu (ordinando terminos ſecundùm poteſtates ignotæ radicis x,) quantitas ſextinomia

$$\left\{\begin{array}{rrrr} x^3 + ax^2 + a^2x & + a^3 \\ & + qx & + qa \end{array}\right\}$$ erit = quantitati quadrinomiæ $pxx + pax + paa + r$, et (auferendo utrinque $pxx + pax$,) quantitas octinomia

$$\left\{\begin{array}{rrrr} x^3 + ax^2 + a^2x & + a^3 \\ - px^2 + qx & + qa \\ - pax \end{array}\right\}$$ erit = quantitati binomiæ $paa + r$.

Sed a eſt = 13.666, et pro-inde a^2 eſt $(= \overline{13.666}^2) = 186.759,556$, et a^3 eſt $(= \overline{13.666}^3) = 2552.256,092,296$, et pa erit $(= p \times 13.666 = 62.145,069,375 \times 13.666) = 849.274,518,078,750$, et qa erit $(= 1221.125 \times 13.666) = 16,687.894,250$, et paa erit $(= 62.145,069,375 \times 186.759,556) = 11,606.185,564,064,197,500$. Ergò, ſi in æquatione litterali $\left\{\begin{array}{rrrr} x^3 + ax^2 + a^2x & + a^3 \\ - px^2 + qx & + qa \\ - pax \end{array}\right\} = paa + r$ ſubſtituamus prò a, p, a^2, a^3, q, pa, paa, et r, eorum valores numerales, hæc æquatio convertetur in hanc æquationem numeralem, ſcilicet,

$$\left\{\begin{array}{rrrr} x^3 + 13.666 \times x^2 & & + 186.759,556 \times x + 2552.256,092,296 \\ - 62.145,069,375 \times x^2 & + 1221.125, & \times x + 16,687.894,250, \\ & & - 849.274,518,078,750 \times x \end{array}\right\}$$

$$\left\{\begin{array}{l} = 11,606.185,564,064,197,500 \\ + 17,575.969,772,051, \end{array}\right\}$$, ſeu $x^3 - 48.479,069,375 \times xx + 558.610,037,921,250 \times x + 19,240.150,342,296 = 29,182.155,336,115,197,500$; quæ (ſi utrinque auferatur 19,240.150,342,296) fiet $x^3 - 48.479,069,375 \times xx + 558.610,037,921,250 \times x = 9942.004,993,819,197,500$; quæ eſt æquatio cubica ejuſdem formæ ac æquatio generalis $x^3 - px^2 + qx = r$. Ergò, ſi hæc æquatio cubica habet duas, vel tres, radices, æquatio biquadratica $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4 = 135,869.138,875,123,885$, à quà eſt derivata, habebit etiàm duas, vel tres, radices, præter radicem 13.666, ultimò inventam;

inventam; ſed, ſi hæc æquatio cubica habet tantùm unam radicem, æquatio biquadratica habebit etiàm tantùm unam aliam radicem præter radicem 13.666 jàm inventam. Inquirendum eſt igitùr, nùm æquatio cubica $x^3 - 48.479,069,375 \times xx + 558.610,037,921,250 \times x = 9942.004,993,819,197,500$ habeat duas aut tres radices, an tantùm unam radicem. Hoc autèm determinari poteſt per articulùm XIX meæ Diſſertationis ſuprà impreſſæ in hoc tomo Scriptorum Logarithmicorum *de Methodo inveniendi maximos et minimos valores quarundam quantitatum compoſitarum, ſeu multinomialium.*

Nam in hoc articulo XIX, (in paginis 184,185,186, &c, - - - - 193 contento,) demonſtratur (in Corollario nono, paginâ 192,) "quòd, ſi in æquatione cubicâ hujus formæ, $qx - px^2 + x^3 = r$, ſeu $x^3 - px^2 + qx = r$, quantitas q ſit minor quantitate $\frac{pp}{4}$, et quantitas cognita r ſit major quantitate $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\overline{2pp - 6q} \times \sqrt{pp - 3q}}{27}$, (quæ oritur ex ſubſtitutione quantitatis $\frac{p - \sqrt{pp - 3q}}{3}$ pro x in quantitate trinomiâ $qx - px^2 + x^3$,) æquatio habebit tantùm unam radicem; et hæc radix unica erit major quàm $\frac{p + \sqrt{pp - 4q}}{2}$." Hoc autèm accidit in æquatione cubicâ $x^3 - 48.479,069,375 \times xx + 558.610,037,921,250 \times x = 9942.004,993,819,197,500$, quæ eſt ejuſdem formæ ac æquatio $x^3 - px^2 + qx = r$, et in quâ p eſt $= 48.479,069,375$, et q eſt $= 558.610,037,921,250$, et r eſt $= 9942.004,993,819,197,500$. Nam, ſi, brevitatis gratiâ, omittamus omnes partes fractionales horum numerorum, exceptis earum duabus figuris primis, et ponamus p eſſe $= 48.47$, et q eſſe $= 558.61$, et r eſſe $= 9942.00$, ſeu 9942, habebimus pp $(= \overline{48.47}^2) = 2,349.3409$, et p^3 $(= \overline{48.47}^3) = 113,872.553,423$, et $2p^3$ $(= 2 \times 113,872.553,423) = 227,745.106,846$, et $\frac{2p^3}{27}$ $\left(= \frac{227,745.106,846}{27}\right)$ $= 8431.300,253$, et pq $(= 48.47 \times 558.61) = 27,075.8267$, et $\frac{pq}{3}$ $\left(= \frac{27.075.8267}{3}\right) = 9025.2755$, et pro-inde $\frac{pq}{3} - \frac{2p^3}{27}$ $(= 9025.2755 - 8431.300,253) = 593.975,247$. Et $3q$ erit $(= 3 \times 558.61) = 1675.83$, et $pp - 3q$ erit $(= 2349.3409 - 1675.83) = 673.5109$, et $\sqrt{pp - 3q}$ erit $(= \sqrt{673.5109}) = 25.95$, et $\overline{2pp - 6q}$ erit $(= 2 \times \overline{pp - 3q} = 2 \times 673$.

$\times$ 673.5109) = 1347.0218, et $\sqrt{2pp - 6q} \times \sqrt{pp - 3q}$ erit (= 1347.0218 $\times$ 25.95) = 34,955.215,710, et $\frac{\sqrt{2pp - 6q} \times \sqrt{pp - 3q}}{27}$ erit (= $\frac{34,955.215,710}{27}$) = 1220.563,544. Ergò quantitas $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\sqrt{2pp - 6q} \times \sqrt{pp - 3q}}{27}$ erit (= 593.975,247 + 1220.563,544) = 1814.538,791.

Et $\frac{pp}{4}$ erit (= $\frac{2349.3409}{4}$) = 587.3352; et $4q$ erit (= 4 $\times$ 558.61) = 2,234.44, et $pp - 4q$ erit (= 2349.3409 — 2234.44) = 114.9009, et $\sqrt{pp - 4q}$ erit (= $\sqrt{114.9009}$) = 10.71, et $p + \sqrt{pp - 4q}$ erit = 48.47 + 10.71 = 59.18, et $\frac{p + \sqrt{pp - 4q}}{2}$ erit (= $\frac{59.18}{2}$) = 29.59. Ergò in hâc æquatione cubicâ $x^3 - 48.47 \times xx - 558.61 \times x = 9942.00$ quantitas q, ſeu 558.61, eſt minor quàm $\frac{pp}{4}$, quæ eſt = 587.3352, et quantitas r, ſeu 9942.00, eſt major quàm quantitas $\frac{pq}{3} - \frac{2p^3}{27} + \frac{\sqrt{2pp - 6q} \times \sqrt{pp - 3q}}{27}$, quæ eſt = 1814.538,791; et pro-inde, per dictum Corollarium nonum, hæc æquatio habebit tantùm unam radicem, et hæc radix unica erit major quàm $\frac{p + \sqrt{pp - 4q}}{2}$, ſeu quàm 29.59. Ergò et æquatio cubica $x^3 - 48.479,069,375 \times xx + 558.610,037,921,250 \times x = 9942.004,993,819,197,500$, (cujus termini ſunt quàm proximè æquales terminis æquationis $x^3 - 48.47 \times xx + 558.61 \times x = 9942.00$) habebit tantùm unam radicem, et hæc radix unica erit major quàm 29.59. Erit verò hæc radix unica æqualis numero 40.7583, quem anteà inveneramus eſſe radicem maximam æquationis biquadraticæ $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4 = 135,869.138,875,123,885$, ut demonſtrari poterit aut per reſolutionem dictæ æquationis cubicæ $x^3 - 48.479,069,375 \times xx + 558.610,037,921,250 \times x = 9942.004,993,819,197,500$, aut per methodum breviorem et minùs operoſam, nempe, ſubſtituendo dictum numerum 40.7583 pro x in quantitate trinomiâ $x^3 - 48.479,069,375 \times xx + 558.610,037,921,250 \times x$, et conferendo valorem hujus quantitatis trinomiæ ex hâc ſubſtitutione oriundum cum

termino

termino abſolutò, 9942.004,993,819,197,500, prædictæ æquationis, cui invenietur eſſe propemodùm æqualis. Hæc ſubſtitutio fieri poterit hoc modo.

Si x fit = 40.7583, xx erit $(= \overline{40.7583}^{2}) = 1661.239,018,89$, et x^3 erit $(= \overline{40.7583}^{3}) = 67,707.278,303,624,287$, et $48.479,069,375 \times xx$ erit $(= 48.479,069,375 \times 1661.239,018,89) = 80,535.321,645,225,245,493,75$, et $558.610,037,921,250 \times x$ erit $(= 558.610,037,921,250 \times 40.7583) = 22,767.995,508,605,683,875,0$, et $x^3 + 558.610,037,921,250 \times x$ erit $(= 67,707.278,303,624,287 + 22,767.995,508,605,683,875,0) = 90,475.273,812,229,970,875,0$, et quantitas trinomia $x^3 - 48.479,069,375 \times xx + 558.610,037,921,250 \times x$ erit $(= 90,475.273,812,229,970,875,0 - 80,535.321,645,225,245,493,75) = 9939.952,167,004,725,381,25$; qui numerus eſt paulò minor quàm 9942.004,993,819,197,500, ſeu terminus abſolutus æquationis cubicæ $x^3 - 48.479,069,375 \times xx + 558.610,037,921,250 \times x = 9942.004,993,819,197,500$, et pro-inde numerus 40.7583 erit paulò minor quàm radix unica hujus æquationis, ſeu illi propemodùm æqualis.

Q. E. D.

Ergò nunc demùm concludere certò poſſumus, quòd æquatio biquadratica $17,575.969,772,051 \times x - 1221.125 \times xx + 62.145,069,375 \times x^3 - x^4 = 135,869.138,875,123,885$ non habet plùres quàm duas radices quas ſuprà inveſtigavimus et invenimus eſſe æquales numeris 40.7583 et 13.666, adeò ut hæc æquatio ſit jàm plenè reſoluta.

Exemplum 7, à Raphſoni Problemate 22 deſumptum.

Reſolvenda ſit æquatio biquadratica $4,228,931.085,087,852 \times x + 323,609.663,689 \times xx - x^4 = 22,540,483,202.613,561,987,516$, in quâ cubus ignotæ quantitatis x de-eſt.

Hæc æquatio eſt ejuſdem formæ ac æquatio $n^3x + mmxx - x^4 = p^4$ in Propoſitione ſeptimâ Capitis ſeptimi tractatûs ſuperioris; et co-efficiens mm quadrati xx in iſtâ æquatione generali fit in hâc æquatione numerali = 323,609.663,689, et co-efficiens n^3 ipſius quantitatis ignotæ x in iſtâ æquatione

tione fit in hâc = 4,228,931.085,087,852, et terminus abſolutus, ſeu quantitas nota, p^4, in iſtâ æquatione fit in hâc æquatione numerali = 22,540,483,202.613,561,987,516. Sed, per iſtam Propoſitionem ſeptimam, æquatio $n^3x + mmxx - x^4 = p^4$ poteſt habere duas radices veras; et harum radicum utraque erit major quàm fractio $\frac{p^4}{mm \times \sqrt{mm + nn} + n^3}$ et quàm minor duarum quantitatum n et $\frac{p^4}{mmm + n^3}$, ſed erit minor quàm $\sqrt{mm + nn}$. Ergò æquatio numeralis 4,228,931.085,087,852 × x + 323,609.663,689 × xx − x^4 = 22,540,483,202.613,561,987,516 poterit habere duas radices, et harum radicum utraque erit major quàm fractio quæ eſt æqualis fractioni $\frac{p^4}{mm \times \sqrt{mm + nn} + n^3}$, et quàm minor duarum quantitatum quæ ſunt æquales quantitatibus n et $\frac{p^4}{mmn + n^3}$, ſed minor quàm quantitas quæ eſt æqualis quantitati $\sqrt{mm + nn}$. Ut igitùr hos harum radicum limites obtinere poſſimus, quærendi ſunt valores quantitatum quæ ſunt æquales quantitatibus $\frac{p^4}{mm \times \sqrt{mm + nn} + n^3}$ et n et $\frac{p^4}{mmn + n^3}$, et $\sqrt{mm + nn}$. Hoc verò fieri poterit hoc modo.

Inſpectâ Tabulâ cuborum qui oriuntur ex centùm primis numeris naturalibus 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, &c, apparebit quòd cubus proximè minor quàm numerus 4,228.931,085,087,852, ſeu $\frac{4,228,931.085,087,852}{1000}$, ſeu $\frac{n^3}{1000}$, eſt 4096, ſeu cubus numeri 16. Ergò cubus proximè minor quàm n^3, ſeu 4,228,931.085,087,852, erit = 1000 × 4096, ſeu 4,096,000, ſeu cubus numeri 160; et pro-inde n in æquatione numerali propoſitâ erit propemodùm æqualis numero 160, ſed eodem paulò major, et nn erit (= $\overline{160}^2$) = 25600. Ergò $mm + nn$ erit (= 323,609.663,689 + 25,600) = 349,209.663,689, et $\sqrt{mm + nn}$ erit (= $\sqrt{349,209.663,689}$) = 591, et $mm \times \sqrt{mm + nn}$ erit (= 323,609.663,689 × 591) = 191,253,311.240,199, et $mm \times \sqrt{mm + nn} + n^3$ erit (= 191,253,311.240,199 + 4,228,931.085,087,852) = 195,482,242.325,286,852, et fractio $\frac{p^4}{mm \times \sqrt{mm + nn} + n^3}$ erit

erit $(= \frac{22,540,483,202.613,561,987,516}{195,482,242.325,286,852}) = 115$. Ergò utraque radix hujus æquationis erit major quàm numerus 115.

Porrò, quoniàm n eſt $= 160$, mmn erit $(= 323,609.663,689 \times 160) = 51,777,546.190,240$, et $mmn + n^3$ erit $(= 51,777,546.190,240 + 4,228,931.085,087,852) = 56,006,477.275,327,852$, et $\frac{p^4}{mmn + n^3}$ erit $(= \frac{22,540,483,202.613,\&c}{56,006,477.275,\&c}) = 402.4\&c$, qui numerus eſt major quàm 160, ſeu n. Ergò utraque radix erit major quàm 115, et major etiàm quàm 160, ſed minor quàm $(\sqrt{mm + nn}$, ſeu) 591.

Inventis his Limitibus 160 et 591 duarum radicum æquationis propoſitæ, quæramus jàm, eorum ope, valorem minoris harum radicum, quæ erit propior quem radix major minori Limiti 160. Et, ut de hujus minoris radicis magnitudine conjecturam facere poſſimus ſatìs commodam et veritati propinquam, ſubſtituamus, primò, ipſum limitem 160 (quem ſcimus eſſe minorem hâc radice,) pro x in quantitate trinomiâ $4,228,931.085,\&c \times x + 323,609.663,\&c \times xx - x^4$, (negligendo figuras in co-efficientium partibus fractionalibus quæ ſequuntur tres primas figuras, ut labor calculi minuatur,) et conferamus valorem hujus quantitatis trinomiæ ex hâc ſubſtitutione oriundum cum termino abſoluto, ſeu noto, 22,540,483,202.613, &c, ut appareat quanto minor hic valor erit quàm dictus terminus abſolutus, ſeu valor ejuſdem quantitatis trinomiæ in propoſitâ æquatione, et pro-inde ut felicitèr conjecturam facere poſſimus de magnitudine quantitatis addendæ ipſi Limiti 160 ut fiat multo propinquior quàm anteà iſtius radicis veræ magnitudini. Hæc ſubſtitutio erit hujuſmodi.

Si x ſit $= 160$, xx erit $(= \overline{160}^2) = 25,600$, et x^4 erit $(= xx \times xx = 25,600 \times 25,600) = 655,360,000$, et $4,228,931.085 \times x$ erit $(= 4,228,931.085 \times 160) = 676,628,973.600$, et $323,609.663 \times xx$ erit $(= 323,609.663 \times 25,600) = 8,284,407,372.800$; et pro-inde quantitas trinomia $4,228,931.085 \times x + 323,609.663 \times xx - x^4$ erit $(= 676,628,973.600 + 8,284,407,372.800 - 655,360,000 = 8,961,036,346.400 - 655,360,000) = 8,305,676,346.400$; qui numerus eſt non multo major quàm tertia pars numeri 22,540,483,202.613, &c, qui eſt terminus abſolutus æquationis pro-

positæ. Ergò numerus 160 erit multo minor quàm vera magnitudo quantitatis x seu ejusdem æquationis radicis minoris.

Ponamus igitùr, secundo loco, quòd x sit = 200, et substituamus hunc numerum pro x in quantitate trinomiâ $4{,}228{,}931.085 \times x + 323{,}609.663 \times xx - x^4$; quod fieri potest hoc modo.

Si x sit = 200, xx erit = 40,000, et x^4 erit = 1,600,000,000, et $4{,}228{,}931.085 \times x$ erit (= $4{,}228{,}931.085 \times 200$) = 845,786,217.000, et $323{,}609.663 \times xx$ erit (= $323{,}609.663 \times 40{,}000$) = 12,944,386,520.000; et pro-inde quantitas trinomia $4{,}228{,}931.085 \times x + 323{,}609.663 \times xx - x^4$ erit = 845,786,217.000 + 12,944,386,520.000 — 1,600,000,000 = 13,790,172,737.000 — 1,600,000,000) = 12,190,172,737.000; qui numerus est minor quàm 22,540,483,202.613, &c, seu terminus absolutus æquationis propositæ, et non multo major quàm ejusdem termini dimidium. Ergò numerus 200 est minor, et multo minor, quàm x, seu radix minor ejusdem æquationis.

Ponamus igitùr, tertio loco, quòd x sit = 300, et substituamus hunc numerum pro x in quantitate trinomiâ $4{,}228{,}931.085 \times x + 323{,}609.663 \times xx - x^4$; quod ita fieri potest.

Si x sit = 300, xx erit = 90,000, et x^4 erit = 8,100,000,000, et $4{,}228{,}931.085 \times x$ erit (= $4{,}228{,}931.085 \times 300$) = 1,268,679,325.500, et $323{,}609.663 \times xx$ erit (= $323{,}609.663 \times 90{,}000$) = 29,124,869,670.000; et pro-inde quantitas trinomia $4{,}228{,}931.085 \times x + 323{,}609.663 \times xx - x^4$ erit (= 1,268,679,325.500 + 29,124,869,670.000 — 8,100,000,000 = 30,393,548,995.500 — 8,100,000,000) = 22,293,548,995.500; qui numerus est minor, sed paulo tantùm minor, quàm 22,540,483,202.613, &c, seu terminus absolutus æquationis propositæ. Ergò numerus 300 est minor quàm x, seu radix ejusdem æquationis, sed est illi satìs propinquus, ut commodè possit assumi pro fundamento processûs approximatorii ad ejus verum valorem secundùm Methodum à Raphsono traditam.

Ponatur igitur z pro differentiâ inter numerum 300 et verum valorem quantitatis x, seu radicis minoris æquationis propositæ $4{,}228{,}931.085{,}087{,}852 \times$

$x +$

$x + 323{,}609.663{,}689 \times xx - x^4 = 22{,}540{,}483{,}202.613{,}561{,}987{,}516$. Et erit $x = 300 + z$.

Ergò xx erit $= \overline{300 + z}^2$ $(= \overline{300}^2 + 2 \times 300 \times z +$ &c$) = 90{,}000 + 600z +$ &c, et x^4 erit $= \overline{300 + z}^4$ $(= \overline{300}^4 + 4 \times \overline{300}^3 \times z +$ &c $= 8{,}100{,}000{,}000 + 4 \times 27{,}000{,}000 \times z +$ &c$) = 8{,}100{,}000{,}000 + 108{,}000{,}000 \times z +$ &c, et $4{,}228{,}931.085{,}087{,}852 \times x$ erit $= 4{,}228{,}931.085{,}087{,}852 \times \overline{300 + z}$ $(= 4{,}228{,}931.085{,}087{,}352 \times 300 + 4{,}228{,}931.085{,}087{,}852 \times z) = 1{,}268{,}679{,}325.526{,}355{,}600 + 4{,}228{,}931.085{,}087{,}852 \times z$, et $323{,}609.663{,}689 \times xx$ erit $(= 323{,}609.663{,}689 \times 90{,}000 + 600z +$ &c $= 323{,}609.663{,}689 \times 90{,}000 + 323{,}609.663{,}689 \times 600z +$ &c$) = 29{,}124{,}869{,}732.010{,}000 + 194{,}165{,}798.213{,}400 \times z +$ &c; et pro-inde quantitas trinomia $4{,}228{,}931.085{,}087{,}852 \times x + 323{,}609.663{,}689\ xx - x^4$ erit æqualis quantitati multinomiæ

$$\left\{\begin{array}{rrr} 1{,}268{,}679{,}325.526{,}355{,}600 & + 4{,}228{,}931.085{,}087{,}852 & \times z \\ + 29{,}124{,}869{,}732.010{,}000{,}000 & + 194{,}165{,}798.213{,}400 & \times z + \&c \\ - 8{,}100{,}000{,}000.000{,}000{,}000 & - 108{,}000{,}000.000{,}000{,}000 & \times z - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{rr} - 30{,}393{,}549{,}057.536{,}355{,}600 & + 198{,}394{,}729.298{,}487{,}852 \times z \\ - 8{,}100{,}000{,}000.000{,}000{,}000 & - 108{,}000{,}000.000{,}000{,}000 \times z - \&c \end{array}\right\}$$

$$22{,}293{,}549{,}057.536{,}355{,}600 + 90{,}394{,}729.298{,}487{,}852 \times z - \&c.$$

Sed quantitas trinomia $4{,}228{,}931.085{,}087{,}852 \times x + 323{,}609.663{,}689 \times xx - x^4$ eſt $= 22{,}540{,}483{,}202.613{,}561{,}987{,}516$.

Ergò et quantitas multinomia $22{,}293{,}549{,}057.536{,}355{,}600 + 90{,}394{,}729.298{,}487{,}852 \times z -$ &c erit $= 22{,}540{,}483{,}202.613{,}561{,}987{,}516$. Et pro-inde (auferendo utrinque $22{,}293{,}549{,}057.536{,}355{,}600$,) quantitas $90{,}394{,}729.298{,}487{,}852 \times z -$ &c erit $= 246{,}934{,}145.077{,}206{,}387{,}516$, et $+$ &c, et z erit $= \frac{246{,}934{,}145.077{,}206{,}387{,}516}{90{,}394{,}729.298{,}487{,}852} +$ &c $= 2.7 +$ &c. Ergò x, ſeu $300 + z$, erit $(= 300 + 2.7 +$ &c$) = 302.7 +$ &c, hoc eſt, radix minor æquationis propoſitæ $4{,}228{,}931.085{,}087{,}852 \times x + 323{,}609.663{,}689 \times xx - x^4 = 22{,}540{,}483{,}202.613{,}561{,}987{,}516$ erit propemodùm æqualis numero 302.7, ſed eodem paulò major. Q. E. I.

Subſtituamus jam numerum 302.7 pro x in quantitate trinomiâ 4,228,931.085,087,852 × x + 323,609.663,689 × xx — x^4, ut videamus quantum valor iſtius quantitatis inde oriundus diſtabit à numero 22,540,483,202.613,561,987,516, ſeu ejuſdem quantitatis valore in propoſitâ æquatione. Hæc ſubſtitutio erit hujuſmodi.

Si x ſit = 302.7, xx erit $(= \overline{302.7}^2) = 91,627.29$, et x^4 erit $(= xx \times xx = 91,627.29 \times 91,627.29) = 8,395,560,272.7441$, et 4,228,931.085,087,852 × x erit $(= 4,228,931.085,087,852 \times 302.7) = 1,280,097,439.456,092,800,4$, et 323,609.663,689 × xx erit $(= 323,609.663,689 \times 91,627.29) = 29,651,476,501.634,472,81$; et pro-inde quantitas trinomia 4,228,931.085,087,852 × x + 323,609.663,689 × xx — x^4 erit $(= 1,280,097,439.456,092,800,4 + 29,651,476,501.634,472,81 - 8,395,560,272.7441 = 30,931,573,941.090,565,610,4 - 8,395,560,272.7441) = 22,536,013,668.346,465,610,4$; qui numerus eſt paulò minor quàm numerus 22,540,483,202.613,561,987,516, ſeu terminus abſolutus propoſitæ æquationis, ſed in tribus primis figuris 225 cum eodem omninò convenit. Ergò numerus 302.7 erit paulò minor quàm x, ſeu radix minor æquationis propoſitæ 4,228,931.085,087,852 × x + 323,609.663,689 × xx — x^4 = 22,540,483,202.613,561,987,516.

Ponamus jam z pro differentiâ inter x et 302.7. Et erit $x = 302.7 + z$, et pro-inde xx erit $= \overline{302.7 + z}^2 (= \overline{302.7}^2 + 2 \times 302.7 \times z + zz) = 91,627.29 + 605.4 \times z + zz$, et x^4 erit $= \overline{302.7 + z}^4 (= \overline{302.7}^4 + 4 \times \overline{302.7}^3 \times z + 6 \times \overline{302.7}^2 \times zz + \&c = \overline{302.7}^4 + 4 \times 27,735.580.683 \times z + 6 \times 91,627.29 \times zz + \&c) = 8,395,560,272.7441 + 110,942,322.732 \times z + 549,763.74 \times zz + \&c$, et 4,228,931.085,087,852 × x erit $= 4,228,931.085,087,852 \times \overline{302.7 + z} (= 4,228,931.085,087,852 \times 302.7 + 4,228,931.085,087,852 \times z) = 1,280,097,439.456,092,800,4 + 4,228,931.085,087,852 \times z$, et 323,609.663,689 × xx erit $(= 323,609.663,689 \times \overline{91,627.29 + 605.4 \times z + zz} = 323,609.663,689 \times 91,627.29 + 323,609.663,689 \times 605.4 \times z + 323,609.663,689 \times zz) = 29,651,476,501.634,472,81 + 195,913,290.397,320,6 \times z + 323,609.663,689 \times zz$. Ergò quantitas trinomia 4,228,931.085,087,852 × x + 323,609.663,689 × xx — x^4 erit æqualis quantitati multinomiæ

1,280,097,

$$\left\{\begin{array}{rrl} 1,280,097,439.456,092,800.4 & + & 4,228,931.085,087,852 \times z \\ + 29,651,476,501.634,472,810,0 & + & 195,913,290.397,320,600 \times z \\ & + & 323,609.663,689 \times zz \\ - 8,395,560,272.744,100,000,0 & - & 110,942,322.732,000,000 \times z \\ & - & 549,763.74 \times zz - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{rrl} 30,931,573,941.090,563,610,4 & + & 200,142,221.482,408,452 \times z \\ - 8,395,560,272.744,100,000,0 & - & 110,942,322.732,000,000 \times z \\ & - & 226,154.076,311 \times zz - \&c \end{array}\right\}$$

$= 22,536,013,668.346,465,610,4 + 89,199,898.750,408,452 \times z$
$- 226,154.076,311 \times zz - \&c.$

Sed quantitas trinomia $4,228,931.085,087,852 \times x + 323,609.663,689 \times xx - x^4$ eſt $= 22,540,483,202.613,561,987,516.$

Ergò quantitas $22,536,013,668.346,465,610,4 + 89,199,898.750,408,452 \times z - 226,154.076,311 \times zz - \&c$ erit etiàm $= 22,540,483,202.613,561,987,516$, et pro-inde (auferendo utrinque $22,536,013,668.346,465,610,4$) quantitas $89,199,898.750,408,452 \times z - 226,154.076,311 \times zz - \&c$ erit $= 4,469,534.267,096,377,116$, et (addendo utrinque $226,154.076,311 \times zz + \&c$,) quantitas $89,199,898.750,408,452 \times z$ erit $= 4,469,534.267,096,377,116 + 226,154.076,311 \times zz + \&c$, et z erit $= \frac{4,469,534.267,096,377,116}{89,199,898.750,408,452} + \frac{226,154.076,311 \times zz}{89,199,898.750,408,452} + \&c = 0.050,106 + \&c.$

Ergò x, ſeu $302.7 + z$, erit $(= 302.7 + 0.050,106 + \&c) = 302.750,106 + \&c$, hoc eſt, radix minor æquationis propoſitæ erit propemodùm æqualis numero 302.750,106, ſed eodem aliquantulum major. Q. E. I.

Hic valor radicis x, ſcilicet, 302.750,106, cum ejus vero, ſeu magìs accurato, valore conveniet, ut opinor, in primis octò figuris 302.750,10. Sed, ut hoc certò ſciamus, computemus valorem termini $\frac{226,154.076,311 \times zz}{89,199,898.750,408,452}$, (qui addendus eſt fractioni decimali 0.050,106, ſeu valori quantitatis z jàm invento, ut ad ejus verum valorem propiùs accedamus,) ad unam vel duas figuras, vel, ad hunc laborem minuendum, computemus valorem quantitatis cujuſdam quæ ſit hoc termino paulò major, ut inde dignoſcere poſſimus in quo

fractionum

fractionum decimalium loco prima figura valoris hujuſmet quantitatis calculum intrabit. Hoc autèm fieri poterit hoc modo.

Terminus $\frac{226,154.076,311 \times zz}{89,199,898.750,408,452}$ eſt minor quàm quantitas $\frac{226,155 \times zz}{89,199,898}$, ſeu quantitas $\frac{226,155 \times \overline{0.050,106}^{2}}{89,199,898}$, et à fortiori, eſt minor quàm quantitas $\frac{226,155 \times \overline{0.051}^{2}}{89,199,898}$, ſeu quàm quantitas $\frac{226,155 \times 0.002,601}{89,199,898}$, ſeu quàm quantitas $\frac{135.919,155}{89,199,898}$, ſeu quàm fractio decimalis 0.000,001,5, cujus prima figura, 1, occupat ſextum locum fractionum decimalium. Ergò prima figura valoris termini $\frac{226,154.076,311 \times zz}{89,199,898.750,408,452}$ non poterit intrare calculum ante ſextum locum fractionum decimalium; et, ſi intrat calculum in hoc loco, non poterit eſſe major quàm 1; ideóque nullam mutationem efficiet in quinque primis locis fractionis decimalis 0.050,106, ſeu valoris quantitatis z hîc jàm inventi, ſed tantùm in ejus ultimâ figurâ, 6, in ſexto loco poſitâ, quam fortaſſe convertet in figuram 7. Ergò numerus 302.750,106, hic inventus pro valore radicis x, ſeu radicis minoris æquationis propoſitæ $4,228,931.085,087,852 \times x + 323,609.663,689 \times xx - x^4 = 22,540,483,202.613,561,987,516$, erit accuratus in primis octò figuris 302.750,10, ſeu cum ejus vero, ſeu magìs accurato, valore in iſtis octò figuris omninò co-incidet. Q. E. D.

Inveſtigatio radicis majoris æquationis propoſitæ $4,228,931.085,087,852 \times x + 323,609.663,689 \times xx - x^4 = 22,540,483,202.613,561,987,516$.

Pergo nunc ad inveſtigationem radicis majoris æquationis propoſitæ $4,228,931.085,087,852 \times x + 323,609.663,689 \times xx - x^4 = 22,540,483,202.613,561,987,516$. Hanc autem ſcimus eſſe minorem quàm 591. Ponamus ergò quòd ſit $=$ 500, et ſubſtituamus 500 pro x in quantitate trinomiâ $4,228,931.085,087,852 \times x + 323,609.663,689 \times xx - x^4$, et conferamus valorem hujus quantitatis ex hâc ſubſtitutione oriundum cum numero 22,540,483,202.613,561,987,516, ſeu termino abſoluto propoſitæ æquationis. Hæc ſubſtitutio fieri poterit hoc modo.

Si

Si x fit $=$ 500, xx erit $=$ 250,000, et x^4 erit $=$ 62,500,000,000; et pro-inde 4,228,913.085 $\times$ x erit ($=$ 4,228,913.085 $\times$ 500) $=$ 2,114,456,542.500, et 323,609.663 $\times$ xx erit ($=$ 323,609.663 $\times$ 250,000) $=$ 16,180,483,150.000, et quantitas trinomia 4,228,931.085 $\times$ x + 323,609.663 $\times$ xx — x^4 erit ($=$ 2,114,456,542.500 + 16,180,483,150.000 — 62,500,000,000) $=$ 18,294,939,692.500 — 62,500,000,000. Hæc autèm quantitas eſt impoſſibilis, quoniam 62,500,000,000 eſt major quàm 18,294,939,692.500, et pro-inde ab eo non poteſt ſubtrahi. Ergò 500 eſt major quàm x, ſeu radix major æquationis propoſitæ.

Ponamus igitùr, ſecundo loco, quòd x ſit $=$ 400, et ſubſtituamus 400 pro x in quantitate trinomiâ 4,228,931.085 $\times$ x + 323,609.663 $\times$ xx — x^4. Hoc verò fieri poteſt hoc modo.

Si x ſit $=$ 400, xx erit $=$ 160,000, et x^4 erit $=$ 25,600,000,000, et 4,228,931.085 $\times$ x erit ($=$ 4,228,931.085 $\times$ 400) $=$ 1,691,572,434.000, et 323,609.663 $\times$ xx erit ($=$ 323,609.663 $\times$ 160,000) $=$ 54,777,546,080.000; et pro-inde quantitas trinomia 4,228,931.085 $\times$ x + 323,609.663 $\times$ xx — x^4 erit ($=$ 1,691,572,434.000 + 54,777,546,080.000 — 25,600,000,000 $=$ 56,469,118,514 — 25,600,000,000) $=$ 30,869,118,514; qui numerus major eſt quàm 22,540,483,202.613, &c, ſeu terminus abſolutus æquationis propoſitæ. Ergò numerus 400 erit minor quàm x, ſeu ejuſdem æquationis radix major.

Ponamus igitùr, tertio loco, quòd x ſit $=$ 450, et ſubſtituamus 450 pro x in quantitate trinomiâ 4,228,931.085 $\times$ x + 323,609.663 $\times$ xx — x^4; quod fieri poterit hoc modo.

Si x ſit $=$ 450, xx erit ($=$ $\overline{450}^2$) $=$ 202,500, et x^4 erit ($=$ $\overline{450}^4$) $=$ 41,006,250,000, et 4,228,931.085 $\times$ x erit ($=$ 4,228,931.085 $\times$ 450) $=$ 1,903,018,988.250, et 323,609.663 $\times$ xx erit ($=$ 323,609.663 $\times$ 202,500) $=$ 65,530,956,757.500; et pro-inde quantitas trinomia 4,228,931.085 $\times$ x + 323,609.663 $\times$ xx — x^4 erit ($=$ 1,903,018,988.250 + 65,530,956,757.500 — 41,006,250,000 $=$ 67,433,975,745.750 — 41,006,250,000) $=$ 26,427,725,745.750; qui numerus eſt major quàm 22,540,483,202.613,&c, ſeu terminus abſolutus æquationis propoſitæ. Ergò numerus 450 erit minor quàm x, ſeu radix major ejuſdem æquationis.

Ponamus

Ponamus igitùr, quarto loco, quòd x fit $=$ 470, et ſubſtituamus 470 pro x in quantitate trinomiâ $4{,}228{,}931.085 \times x + 323{,}609.663 \times xx - x^4$; quod fieri poterit hoc modo.

Si x fit $=$ 470, xx erit $(= \overline{470}^2) =$ 220,900, et x^4 erit $(= \overline{470}^4) =$ 48,786,810,000, et $4{,}228{,}931.085 \times x$ erit $(= 4{,}228{,}931.085 \times 470) =$ 1,987,597,609.950, et $323{,}609.663 \times xx$ erit $(= 323{,}609.663 \times 220{,}900) =$ 71,485,374,556.700, et pro-inde quantitas trinomia $4{,}228{,}931.085 \times x + 323{,}609.663 \times xx - x^4$ erit $(= 1{,}987{,}597{,}609.950 + 71{,}485{,}374{,}556.700 - 48{,}786{,}810{,}000 = 73{,}472{,}972{,}166.650 - 48{,}786{,}810{,}000) =$ 24,686,162,166.650; qui numerus eſt major quàm 22,540,483,202.613,&c, ſeu terminus abſolutus æquationis propoſitæ. Ergò numerus 470 erit minor quàm x, ſeu radix major æquationis propoſitæ.

Ponamus igitur, quinto loco, x eſſe $=$ 480, et ſubſtituamus 480 pro x in quantitate trinomiâ $4{,}228{,}931.085 \times x + 323{,}609.663 \times xx - x^4$; quod fieri poteſt hoc modo.

Si x fit $=$ 480, xx erit $(= \overline{480}^2) =$ 230,400, et x^4 erit $=$ 53,084,160,000, et $4{,}228{,}931.085 \times x$ erit $(= 4{,}228{,}931.085 \times 480) =$ 2,029,886,920.800 et $323{,}609.663 \times xx$ erit $(= 323{,}609.663 \times 230{,}400) =$ 74,559,666,355.200; et pro-inde quantitas trinomia $4{,}228{,}931.085 \times x + 323{,}609.663 \times xx - x^4$ erit $(= 2{,}029{,}886{,}920.800 + 74{,}559{,}666{,}355.200 - 53{,}084{,}160{,}000 = 76{,}589{,}553{,}276.000 - 53{,}084{,}160{,}000) =$ 23,505,393,276; qui numerus eſt major quàm 22,540,483,202.613,&c, ſeu terminus abſolutus æquationis propoſitæ. Ergò numerus 480 erit minor quàm x, ſeu radix major æquationis propoſitæ.

Ponamus igitùr, ſexto loco, quòd x fit $=$ 490, et ſubſtituamus 490 pro x in quantitate trinomiâ $4{,}228{,}931.085 \times x + 323{,}609.663 \times xx - x^4$; quod fieri poteſt hoc modo.

Si x fit $=$ 490, xx erit $(= \overline{490}^2) =$ 240,100, et x^4 erit $(= \overline{490}^4) =$ 57,648,010,000, et $4{,}228{,}931.085 \times x$ erit $(= 4{,}228{,}931.085 \times 490) =$ 2,072,176,231.650, et $323{,}609.663 \times xx$ erit $(= 323{,}609.663 \times 240{,}100) =$ 77,698,680,086.300; et pro-inde quantitas trinomia $4{,}228{,}931.085 \times x$

$+ 323,609.663 \times xx - x^4$ erit (= 2,072,176,231.650 + 77,698,680,086.300 − 57,648,010,000 = 79,770,856,317.950 − 57,648,010,000) = 22,122,846,317.950; qui numerus eſt minor quàm 22,540,483,202.613,&c, ſeu terminus abſolutus æquationis propoſitæ. Ergò numerus 490 eſt major quàm x, ſeu radix major iſtius æquationis, et pro-inde iſta radix erit magnitudinis cujuſdem intermediæ inter 480 et 490.

Ad inveniendum valorem hujus radicis qui ſit ejus vero valori magìs propinquus quàm aut 480 aut 490, adhibere nunc commodè poterimus proceſſum Methodi Differentialis; quod fieri poterit hoc modo.

Numerus 480, et numerus x, ſeu radix major æquationis propoſitæ, et numerus 490, ſunt tres quantitates quæ non multùm à ſe invicèm differunt, quoniam differentia inter maximam, 490, et minimam, 480, eſt tantùm 10, ſeu pars quadrageſima octava earum minimæ, 480; et tres valores quantitatis trinomiæ $4,228,931.085 \times x + 323,609.663 \times xx - x^4$ his tribus numeris correſpondentes, ſeu qui oriuntur ex ſubſtitutione horum trium numerorum pro x, ſucceſſivè, in iſtâ quantitate trinomiâ, ſunt 23,505,393,276.000, et 22,540,483,202.613,&c, et 22,122,846,317.950, ſeu (negligendo omnes figuras in his magnis numeris præter tres primas) 23,500,000,000 et 22,500,000,000 et 22,100,000,000. Ergò, per methodum Differentialem, Differentia inter primam et tertiam priorum trium quantitatum 480, x, et 490, erit ad Differentiam inter mediam et primam earum, propè, in eâdem ratione ac Differentia inter primam et tertiam poſteriorum trium quantitatum ad Differentiam inter mediam et primam, hoc eſt, 490 − 480 erit ad x − 480, propè, in eâdem ratione ac 23,500,000,000 − 22,100,000,000 eſt ad 23,500,000,000 − 22,500,000,000, ideóque in eâdem ratione ac 235 − 221 ad 235 − 225, ſeu ut 14 ad 10, ſeu ut 7 ad 5; hoc eſt, 10 erit ad x − 480 propè in eâdem ratione ac 7 ad 5. Ergò $\overline{x - 480} \times 7$ erit, propemodùm, = 5 × 10 = 50, ſeu $7x - 7 \times 480$ erit = 50, et pro-inde $7x$ erit (= 50 + $7 \times 480 = 50 + 3360$) = 3410, et x erit = $\frac{3410}{7}$ = 487, hoc eſt, 487 erit valor radicis majoris æquationis propoſitæ ejus vero valori magìs propinquus quàm aut 480 aut 490.

Ponamus ergò, ſeptimo loco, quòd x ſit = 487, et ſubſtituamus 487 pro x

in quantitate trinomiâ 4,228,931.085,087,852 × x + 323,609.663,689 × $xx - x^4$; quod fieri poterit hoc modo.

Si x fit = 487, xx erit $(= \overline{487}^2) = 237,169$, et x^4 erit $(= \overline{487}^4) = 56,249,134,561$, et 4,228,931.085,087,852 × x erit (=4,228,931.085,087,852 × 487) = 2,059,489,438.437,783,924, et 323,609.663,689 × xx erit (= 323,609.663,689 × 237,169) = 76,750,180,327.456,441 ; et pro-inde quantitas trinomia 4,228,931.085,087,852 × x + 323,609.663,689 × $xx - x^4$ erit (= 2,059,489,438.437,783,924 + 76,750,180,327.456,441 − 56,249,134,561 = 78,809,669,765.894,224,924 − 56,249,104,561) = 22,560,535,204.894,224,924 ; qui numerus eſt paulò major quàm 22,540,483,202.613,561,987,516, ſeu terminus abſolutus æquationis propoſitæ. Ergò numerus 487 erit paulò minor quàm x, ſeu radix major iſtius æquationis. Erit autèm eidem radici tàm propinquus ut commodiſſimè aſſumi poſſit pro fundamento proceſſûs approximatorii ad alium ejuſdem radicis valorem, ipſo 487 multo magìs accuratum, obtinendum, ſecundùm Methodum à Raphſono traditam. Hoc verò fieri poterit hoc modo.

Ponatur z pro differentiâ inter 487 et x ; et erit $x = 487 + z$.

Ergò xx erit $= \overline{487 + z}^2 (= \overline{487}^2 + 2 \times 487 \times z + \&c) = 237,169 + 974 \times z + \&c$, et x^4 erit $= \overline{487 + z}^4 (= \overline{487}^4 + 4 \times \overline{487}^3 \times z + \&c = \overline{487}^4 + 4 \times 115,501,303 \times z + \&c) = 56,249,134,561 + 462,005,212 \times z + \&c$, et 4,228,931.085,087,852 × x erit = 4,228,931.085,087,852 × $\overline{487 + z}$ (= 4,228,931.085,087,852 × 487 + 4,228,931.085,087,852 × z) = 2,059,489,438.437,783,924 + 4,228,931.085,087,852 × z, et 323,609.663,689 × xx erit = 323,609.663,689 × $\overline{237,169 + 974 \times z + \&c}$ (= 323,609.663,689 × 237,169 + 323,609.663,689 × 974 × z + &c) = 76,750,180,327.456,441 + 315,195,812.433,086 × z + &c ; et pro-inde quantitas trinomia 4,228,931.085,087,852 × x + 323,609.663,689 × $xx - x^4$ erit = quantitati multinomiæ

$$\left\{\begin{array}{rrr} 2,059,489,438.437,783,924 & + & 4,228,931.085,087,852 \times z \\ + 76,750,180,327.456,441,000 & + & 315,195,812.433,086,000 \times z + \&c \\ - 56,249,134,561.000,000,000 & - & 462,005,212,000.000,000 \times z - \&c \end{array}\right\}$$

$$= \left\{ \begin{array}{l} 78,809,669,765.894,224,924 + 319,424,743.518,173,852 \times z + \&c \\ -56,249,134,561.000,000,000 - 462,005,212.000,000,000 \times z - \&c \end{array} \right\}$$

$$= 22,560,535,204.894,224,924 - 142,580,468.481,826,148 \times z \ \&c.$$

Sed quantitas trinomia $4,228,931.085,087,852 \times x + 323,609.663,689 \times xx - x^4$ eſt $= 22,540,483,202.613,561,987,516$.

Ergò quantitas $22,560,535,204.894,224,924 - 142,580,468.481,826,148 \times z$ &c erit etiàm $= 22,540,483,202.613,561,987,516$; et pro-inde (addendo utrinque $142,580,468.481,826,148 \times z$,) quantitas $22,560,535,204.894,224,924$ erit $= 22,540,483,202.613,561,987,516 + 142,580,468.481,826,148 \times z$, et (auferendo utrinque $22,540,483,202.613,561,987,516$,) quantitas $142,580,468.481,826,148 \times z$ erit $= 20,052,002.280,662,936,484$, et z erit $= \frac{20,052,002.280,662,936,484}{142,580,468.481,826,148} = 0.1406$. Ergò x, ſeu $487 + z$, erit $(= 487 + 0.1406) = 487.1406$, hoc eſt, radix major æquationis propoſitæ $4,228,931.085,087,852 \times x + 323,609.663,689 \times xx - x^4 = 22,540,483,202.613,561,987,516$ erit $= 487.1406$. Q. E. I.

Hujus numeri 487.1406 ſex primæ figuræ 487.140 veræ ſunt, ſeu cum ſex primis figuris valoris magìs accurati radicis x omninò conveniunt, ut manifeſtum erit ex alio proceſſu approximatorio ſecundùm Raphſoni Methodum ad ejus veram magnitudinem, ſumendo numerum 487.140, ſeu 487.14, pro dicti proceſſûs fundamento. Hic autèm proceſſus erit hujuſmodi.

Subſtituatur numerus 487.14 pro x in quantitate trinomiâ $4,228,931.085,087,852 \times x + 323,609.663,689 \times xx - x^4$; quod fieri poterit hoc modo.

Si x ſit $= 487.14$, xx erit $(= \overline{487.14}^2) = 237,305.3796$, et x^4 erit $(= \overline{487.14}^4) = 56,313,843,187,100,096,16$, et $4,228,931.085,087,852 \times x$ erit $(= 4,228,931.085,087,852 \times 487.14) = 2,060,081,488.789,696,223,28$, et $323,609.663,689 \times xx$ erit $(= 323,609.663,689 \times 237,305.3796) = 76,794,314,083.946,481,344,4$; et pro-inde quantitas trinomia $4,228,931.085,087,$

085,087,852 × x + 323,609.663,689 × xx — x^4 erit = 2,060,081,488.789,696,223,28 + 76,794,314,083.946,481,344,4 — 56,313,843,187.100,096,16 = 78,854,395,572.736,177,567,68 — 56,313,843,187.100,096,16) = 22,540,552,385.636,081,407,68 ; qui numerus eſt paulò major quàm numerus 22,540,483,202.613,561,987,516, ſeu terminus abſolutus æquationis propoſitæ 4,228,931.085,087,852 × x + 323,609.663,689 × xx — x^4 = 22,540,483,202.613,561,987,516. Ergò numerus 487.14 erit paulò minor quàm x, ſeu radix major iſtius æquationis.

Ponatur jàm z pro differentiâ inter 487.14 et x; et x erit = 487.14 + z. Ergò 4,228,931.085,087,852 × x erit = 4,228,931.085,087,852 × $\overline{487.14 + z}$ (= 4,228,931.085,087,852 × 487.14 + 4,228,931.085,087,852 × z) = 2,060,081,488.789,696,223,28 + 4,228,931.085,087,852 × z, et xx erit = $\overline{487.14 + z}^2$ (= $\overline{487.14}^2$ + 2 × 487.14 × z + zz = $\overline{487.14}^2$ + 974.28 × z + zz) = 237,305.3796 + 974.28 × z + zz, et pro-inde 323,609.663,689 × xx erit = 323,609.663,689 × $\overline{237,305.3796 + 974.28 \times z + zz}$ = 323,609.663,689 × 237,305.3796 + 323,609.663,689 × 974.28 × z + 323,609.663,689 × zz) = 76,794,314,083.946,481,344,4 + 315,286,423.238,918,92 × z + 323,609.663,689 × zz; et x^4 erit = $\overline{487.14 + z}^4$ (= $\overline{487.14}^4$ + 4 × $\overline{487.14}^3$ × z + 6 × $\overline{487.14}^2$ × zz — &c = $\overline{487.14}^4$ + 4 × 115,600,942.618,344 × z + 6 × 237,305.3796 × zz + &c) = 56,313,843,187.100,096,16 + 462,403,770.473,376 × z + 1,423,832.2776 × zz + &c. Ergò quantitas trinomia 4,228,931.085,087,852 × z + 323,609.663,689 × xx — x^4 erit æqualis quantitati multinomiæ

$$\left\{\begin{array}{rrr} 2,060,081,488.789,696,223,28 & + & 4,228,931.085,087.852 \times z \\ + 76,794,314,083.946,481,344,4 & + & 315,286,423.238,918,92 \times z \\ & + & 323,609.663,689 \times zz \\ - 56,313,843,187.100,096,16 & - & 462,403,770.473,376 \times z \\ & - & 1,423,832.277,600 \times zz - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{rrr} 78,854,395,572.736,177,567,68 & + & 319,515,354.324,006,772 \times z \\ - 56,313,843,187.100,096,160,00 & - & 462,403,770.473,376,000 \times z \\ & - & 1,100,222.613,911 \times zz - \&c \end{array}\right\}$$

$$= 22,540,552,385.636,081,407,68 - 142,888,416.149,369,228 \times z - 1,100,222.613,911 \times zz - \&c.$$

Sed

Sed quantitas trinomia 4,228,931.085,087,852 × x + 323,609.663,689 × xx — x^4 eſt = 22,540,483,202.613,561,987,516.

Ergò quantitas 22, 540,552,385.636,081,407,68 — 142,888,416.149,369,228 × z — 1,100,222.613,911 × zz — &c erit etiàm = 22,540,483,202.613,561,987,516; et pro-inde (addendo utrinque 142,888,416.149,369,228 × z,) quantitas 22,540,552,385.636,081,407,68 — 1,100,222.613,911 × zz — &c erit = 22,540,483,202.613,561,987,516 + 142,888,416.149,369,228 × z, et (auferendo utrinque 22,540,483,202.613,561,987,516) quantitas 142, 888,416.149, 369,228 × z erit = 69, 183. 022, 519, 420,164 — 1,100,222.613,911 × zz — &c, et z erit = $\frac{69,183.022,519,420,164}{142,888,416.149,369,228}$ — $\frac{1,100,222.613,911 \times zz}{142,888,416.149,369,228}$ — &c = 0.000,484,175 — &c. Ergò x, ſeu 487.14 + z, erit (= 487.14 + 0.000,484,175 — &c) = 487.140,484,175, — &c, hoc eſt, radix major æquationis propoſitæ erit propemodùm æqualis numero 487.140,484,175, ſed eodem aliquantulum minor. Q. E. I.

Hic numerus 487.140,484,175 pro valore radicis x inventus, erit, ut opinor, accuratus in decèm primis figuris 487.140,484,1, niſi forte aliquis error in præcedentem calculum ſit illapſus. Sed, ut hoc certò ſciatur, inquiremus quis-nam erit locus fractionum decimalium in quo prima figura termini proximi, ſcilicet, $\frac{1,100,222.613,911 \times zz}{142,888,416.149,369,228}$, (qui eſt ſubtrahendus à numero 0. 000,484,175, ſeu valore quantitatis z hîc invento, ut ad ejus verum valorem propiùs accedamus,) calculum intrabit. Hoc autèm inveniri poterit hoc modo.

Terminus $\frac{1,100,222.613,911 \times zz}{142,888,416.149,369,228}$ eſt minor quàm fractio $\frac{1,100,222.613,911 \times zz}{142,888,416}$, ſeu quàm fractio decimalis 0.007,6 × zz, ſeu quàm 0.007,6 × $\overline{0.000,484,175}^2$) et pro-inde, à fortiori, erit minor quàm 0.0076 × $\overline{0.000,5}^2$, ſeu quàm 0.0076 × 0.000,000,25, hoc eſt, quàm fractio decimalis 0.000,000,001,900, cujus prima figura 1, occupat nonum ab unitate locum. Ergò prima figura valoris iſtius termini non poterit occupare ullum locum ante iſtum nonum; et, ſi in illo

illo nono loco calculum intrat, non poterit esse major quàm 1. Ergò subtractio istius termini à numero 0.000,484,175 non poterit aliquam mutationem efficere in octò primis ab unitate figuris numeri 0.000,484,175, sed tantùm in ultimâ ejusdem figurâ 5 in nonoloco positâ, quam fortasse converti faciet in figuram 4. Ergò omnes figuræ in octò primis ab unitate locis numeri 0.000,484,175 immutatæ manebunt; et, pro-inde undecìm primæ figuræ numeri 487.140,484,175, hîc pro valore radicis x invento, scilicet, figuræ 487.140,484,17, erunt veræ, seu cum undecìm primis figuris veri valoris radicis x, seu valoris ejusdem ad plures quàm undecìm figuras investigati, omninò convenient.

Numerus à Raphsono inventus pro valore x, seu radicis majoris hujus æquationis $4,228,931.085,087,852 \times x + 323,609.663,689 \times xx - x^4 = 22,540,483,202.613,561,987,516$, est 487.140,483,51, qui in primis octò figuris 487.140,48 cum nostro numero hîc invento, scilicet, 487.140,484,175, convenit. Ergò hæ saltèm octò figuræ 487.140,48 videntur tutò posse pro veris haberi. De minore hujus æquationis radice, (quam in primâ parte hujus resolutionis invenimus esse = 302.750,106,) Raphsonus nullam omninò facit mentionem.

Exemplum 8, à Raphsoni Problemate 23 desumptum.

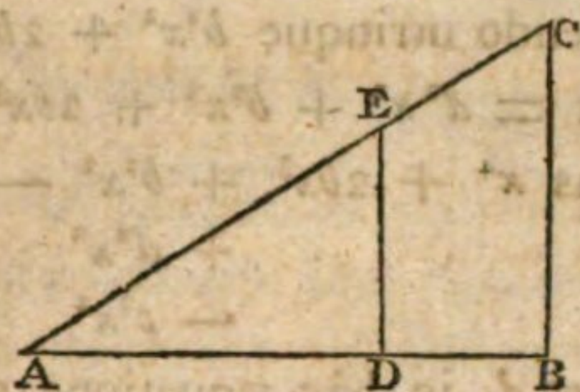
Resolvenda sit æquatio biquadratica $x^4 + 4x^3 + 751\ xx - 9000\ x = 90,000$.

Hæc æquatio derivatur à Problemate Geometrico sequente.

In triangulo rectangulo ABC, cujus basis est AB, altitudo BC, et hypotenusa AC, à puncto quodam D in basi AB ducatur recta linea DE parallela rectæ BC et occurrens hypotenusæ AC in puncto E. Et erit triangulum ADE rectangulum et simile priori triangulo ABC. In his triangulis datæ sint longitudines linearum AD, CB, et EC; nempe, sit AD = 20, CB = 24, et EC = 15; et ex his datis requiratur invenire longitudinem lineæ DB.

Solutio.

Solutio.



Ponatur b = AD, ſeu 20, et c = EC = 15, et d = CB = 24, et x = quæſitæ lineæ DB.

Ob ſimilia triangula ABC et ADE, erit AB ad AD ut BC ad DE.

Sed AD eſt = b, et DB eſt = x, et pro-inde AB, ſeu AD + DB, erit = $b + x$. Et BC eſt = d.

Ergò $b + x$ erit ad b ut d ad DE, et pro-inde DE erit = $\frac{bd}{b+x}$, et DE^q erit = $\frac{b^2d^2}{b^2+2bx+x^2}$.

Sed, ob rectum angulum ADE, erit AE^q = AD^q + DE^q, et pro-inde DE^q erit = AE^q — AD^q = AE^q — bb.

Eſt verò, ob ſimilia triangula ABC, ADE, AE ad EC ut AD ad DB, hoc eſt, AE ad c ut b ad x. Ergò AE erit = $\frac{bc}{x}$, et AE^q erit = $\frac{b^2c^2}{xx}$, et AE^q — bb erit = $\frac{b^2c^2}{xx} - bb = \frac{b^2c^2 - b^2x^2}{xx}$. Ergò DE^q erit etiàm $\frac{b^2c^2 - b^2x^2}{xx}$.

Sed ſuprà oſtenſum eſt quòd DE^q eſt = $\frac{b^2d^2}{b^2+2bx+x^2}$.

Ergò $\frac{b^2c^2 - b^2x^2}{xx}$ erit = $\frac{b^2d^2}{b^2+2bx+x^2}$, et pro-inde (multiplicando utrinque per xx) $b^2c^2 - b^2x^2$ erit = $\frac{b^2d^2x^2}{b^2+2bx+x^2}$, et (dividendo omnes terminos

per

per b^2,) $c^2 - x^2$ erit $= \frac{d^2 x^2}{b^2 + 2bx + x^2}$, et (multiplicando utrinque per quantitatem trinomiam $b^2 + 2bx + x^2$) quantitas ſextinomia $b^2c^2 - b^2x^2 + 2bc^2x - 2bx^3 + c^2x^2 - x^4$ erit $= d^2x^2$, et (addendo utrinque $b^2x^2 + 2bx^3 + x^4$,) quantitas trinomia $b^2c^2 + 2bc^2x + c^2x^2$ erit $= d^2x^2 + b^2x^2 + 2bx^3 + x^4$, et (auferendo utrinque $2bc^2x + c^2x^2$,) quantitas $x^4 + 2bx^3 + b^2x^2 - 2bc^2x$ erit

$+ d^2x^2$

$- c^2x^2$

$= b^2c^2$; et, ſubſtituendo pro Litteris b, c, et d, in hâc æquatione earum valores, 20, 15, et 24, erit quantitas $x^4 + 40x^3 + 400x^2 - 40 \times 225 \times x$

$+ 576x^2$

$- 225x^2$

$= 400 \times 225$, ſeu quantitas $x^4 + 40x^3 + 976x^2 - 9000x = 90{,}000$, ſeu

$- 225x^2$

quantitas $x^4 + 40x^3 + 751x^2 - 9000x = 90{,}000$; quæ eſt æquatio à Raphſono propoſita et reſoluta.

Hæc æquatio eſt ejuſdem formæ ac æquatio generalis $x^4 + lx^3 + mmx^2 - n^3x = p^4$ in Propoſitione duodecimâ Capitis Decimi tractatûs ſuperioris; et co-efficiens l cubi x^3 in iſtâ æquatione generali fit in hâc æquatione numerali $= 40$, et co-efficiens mm quadrati xx in iſtâ æquatione fit in hâc $= 751$, et co-efficiens ipſius quantitatis x in iſtâ æquatione fit in hâc $= 9000$, et terminus abſolutus, ſeu quantitas nota, p^4, in iſtâ æquatione fit in hâc æquatione numerali $= 90{,}000$. Sed, per iſtam Propoſitionem duodecimam, æquatio $x^4 + lx^3 + mmxx - n^3x = p^4$ habet tantùm unam radicem veram; et, cum p eſt $= \frac{n^3}{lp + mm}$, ſeu $lpp + mmp$ eſt $= n^3$, hæc radix unica erit æqualis p, vel $\frac{n^3}{lp + mm}$; et, cum p non eſt æqualis $\frac{n^3}{lp + mm}$, ſeu $lpp + mmp$ non eſt æqualis n^3, hæc radix unica erit mediæ cujuſdam magnitudinis inter maximam et minimam harum trium quantitatum, p, $\sqrt{\frac{n^3}{l + m}}$, et $\frac{n^3}{lm + mm}$. Vide ſuprà, paginam 285. Ergò æquatio numeralis $x^4 + 40x^3 + 751x^2 - 9000x = 90{,}000$ habebit tantùm unam radicem veram; et, ſi p, (ſeu $\sqrt[4]{p^4}$,) ſeu $\sqrt[4]{90{,}000}$, fit $= \frac{n^3}{lp + mm}$, ſeu $\frac{9000}{40 \times \sqrt[4]{90{,}000} + 751}$, ſeu ſi $lpp + mmp$, ſeu

40 ×

$40 \times \sqrt[2]{90,000} + 751 \times \sqrt[4]{90,000}$ ſit $= n^3$, ſeu 9000, hæc radix unica erit $= p$, ſeu $\sqrt[4]{90,000}$, vel $\frac{n^3}{lp + mm}$, ſeu $\frac{9000}{40 \times \sqrt[4]{90,000} + 751}$; et, ſi p, ſeu $\sqrt[4]{90,000}$, non ſit $= \frac{n^3}{lp + mm}$, vel $\frac{9000}{40 \times \sqrt[4]{90,000} + 751}$, ſeu ſi $lpp + mmp$, ſeu $40 \times \sqrt[2]{90,000} + 751 \times \sqrt[4]{90,000}$, non ſit $= n^3$ ſeu 9000, hæc radix unica non erit $= p$, ſeu $\sqrt[4]{90,000}$, ſed erit mediæ cujuſdam magnitudinis inter maximam et minimam harum trium quantitatum, ſcilicet, p, ſeu $\sqrt[4]{90,000}$, $\sqrt{\frac{n^3}{l + m}}$, ſeu $\sqrt{\frac{9000}{40 + \sqrt{751}}}$, et $\frac{n^3}{lm + mm}$, ſeu $\frac{9000}{40 \times \sqrt{751} + 751}$. Igitùr, ut hos hujus radicis limites obtinere poſſimus, quærendi ſunt valores quantitatum p, $\frac{n^3}{lp + mm}$, $\sqrt{\frac{n^3}{l + m}}$, et $\frac{n^3}{lm + mm}$, ſeu $\sqrt[4]{90,000}$, $\frac{9000}{40 \times \sqrt[4]{90,000} + 751}$, $\sqrt{\frac{9000}{40 + \sqrt{751}}}$, et $\frac{9000}{40 \times \sqrt{751} + 751}$. Hoc verò fieri poterit hoc modo.

Quantitas p^4, ſeu 90,000, eſt $(= 9 \times 10,000) = 3 \times 3 \times 100 \times 100$. Ergò pp erit $(= 3 \times 100) = 300$, et p erit $(= \sqrt{3} \times 10 = 1.732 \times 10) = 17.32$, et lp erit $(= 40 \times 17.32) = 692.80$, et $lp + mm$ erit $(= 692.80 + 751) = 1443.80$, et $\frac{n^3}{lp + mm}$ erit $(= \frac{9000}{1443.80}) = 6.2$, et m erit $(= \sqrt{751}) = 27.4$, et $l + m$ erit $(= 40 + 27.4) = 67.4$, et $\frac{n^3}{l + m}$ erit $(= \frac{9000}{67.4}) = 133$, et $\sqrt{\frac{n^3}{l + m}}$ erit $(= \sqrt{133}) = 11.5$, et lm erit $(= 40 \times 27.4) = 1096$, et $lm + mm$ erit $(= 1096 + 751) = 1847$, et $\frac{n^3}{lm + mm}$ erit $(= \frac{9000}{1847}) = 4.8$.

Ergò in hâc æquatione $x^4 + 40x^3 + 751x^2 - 9000x = 90,000$ quantitas p, ſeu $\sqrt[4]{90,000}$, non eſt æqualis $\frac{n^3}{lp + mm}$, ſeu $\frac{9000}{40 \times \sqrt[4]{90,000} + 751}$, ſed eâdem major; cum prior ſit $= 17.32$, et poſterior ſit $= 6.2$. Ergò radix unica hujus æquationis non erit æqualis p, ſeu 17.32, ſed erit mediæ cujuſdam magnitudinis inter maximam et minimam harum trium quantitatum,

ſcilicet, p, $\sqrt{\frac{n^3}{l+m}}$, et $\frac{n^3}{lm+mm}$, ſeu 17.32, 11.5, et 4.8, hoc eſt, intèr quantitates 17.32 et 4.8.

Harum igitùr trium quantitatum, 17.32, 11.5, et 4.8, mediam, ſcilicet, 11.5, (quæ à duabus extremis, 17.32 et 4.8, ferè æqualitèr diſtat,) aſſumemus per conjecturam pro primo valore radicis x æquationis propoſitæ $x^4 + 40x^3 + 751xx - 9000x = 90{,}000$, et eam ſubſtituemus pro x in quantitate quadrinomiâ $x^4 + 40x^3 + 751x^2 - 9000x$; quod fieri poterit hoc modo.

Si x ſit $= 11.5$, xx erit $(= \overline{11.5}^2) = 132.25$, et x^3 erit $(= \overline{11.5}^3 = 132.25 \times 11.5) = 1520.875$, et x^4 erit $(= \overline{11.5}^4 = 1520.875 \times 11.5) = 17{,}490.0625$, et $40x^3$ erit $(= 40 \times 1520.875) = 60{,}835.000$, et $751xx$ erit $(= 751 \times 132.25) = 99{,}319.75$, et $9000x$ erit $(= 9000 \times 11.5) = 103{,}500.0$; et pro-inde quantitas quadrinomia $x^4 + 40x^3 + 751xx - 9000x$ erit $= 17{,}490.0625 + 60{,}835.000 + 99{,}319.75 - 103{,}500.0 = 177{,}644.8125 - 103{,}500.0 = 74{,}144.8125$; qui numerus eſt minor quàm 90,000, ſeu terminus abſolutus æquationis propoſitæ. Ergò 11.5 erit minor quàm x, ſeu radix iſtius æquationis. Erit tamen eidem radici ſatìs propinquus ut commodè aſſumi poſſit pro fundamento proceſſûs approximatorii ad dictam radicem, ſecundùm Methodum à Raphſono traditam.

Ponatur jàm z pro differentiâ inter 11.5 et x; et x erit $= 11.5 + z$. Ergò xx erit $= \overline{11.5 + z}^2$ $(= \overline{11.5}^2 + 2 \times 11.5 \times z + \&c) = 132.25 + 23.0 \times z + \&c$, et x^3 erit $= \overline{11.5 + z}^3$ $(= \overline{11.5}^3 + 3 \times \overline{11.5}^2 \times z + \&c = \overline{11.5}^3 + 3 \times 132.25 \times z + \&c) = 1520.875 + 396.75 \times z + \&c$, et x^4 erit $= \overline{11.5 + z}^4$ $(= \overline{11.5}^4 + 4 \times \overline{11.5}^3 \times z + \&c \overline{11.5}^4 + 4 \times 1520.875 \times z + \&c) = 17{,}490.0625 + 6083.500 \times z + \&c$; et pro-inde $40x^3$ erit $(= 40 \times \overline{1520.875 + 396.75 \times z + \&c} = 40 \times 1520.875 + 40 \times 396.75 \times z + \&c) = 60{,}835.000 + 15{,}870.00 \times z + \&c$, et $751 \times xx$ erit $= 751 \times \overline{132.25 + 23.0 \times z + \&c}$ $(= 751 \times 132.25 + 751 \times 23.0 \times z + \&c) = 99{,}319.7500 + 17{,}273.0 \times z + \&c$, et $9000x$ erit $= 9000 \times \overline{11.5 + z}$ $(= 9000 \times 11.5 + 9000 \times z) = 103{,}500.0 + 9000 \times z$. Ergò quantitas quadrinomia $x^4 + 40x^3 + 751x^2 - 9000x$ erit æqualis quantitati multinomiæ

17,490.

$$\left\{\begin{array}{rrrr} 17,490.0625 & + & 6,083.500 \times z & + \&c \\ + \ 60,835.0000 & + & 15,870.000 \times z & + \&c \\ + \ 99,319.7500 & + & 17,273.000 \times z & + \&c \\ - \ 103,500.0000 & - & 9,000.000 \times z & \end{array}\right\} =$$

$$\left\{\begin{array}{rrrr} 177,644.8125 & + & 39,226.5 \times z & + \&c \\ - \ 103,500,0000 & - & 9000.0 \times z & \end{array}\right\}$$

$$= 74,144.8125 + 30,226.5 \times z + \&c.$$

Sed quantitas quadrinomia $x^4 + 40x^3 + 751x^2 - 9000x$ eſt $= 90,000$.

Ergò quantitas $74,144.8125 + 30,226.5 \times z + \&c$ erit etiàm $= 90,000$. Et pro-inde (auferendo utrinque $74,144.8125$) quantitas $30,226.5 \times z + \&c$ erit $= 15,855.1875$, et $z + \&c$ erit $(= \frac{15,855.1875}{30,226.5}) = 0.524$; et z erit $0.524 - \&c$. Ergò x, ſeu $11.5 + z$, erit $(= 11.5 + 0.524 - \&c) = 12.024 - \&c$, ſeu paulò minor quàm 12.024; hoc eſt, radix unica æquationis propoſitæ $x^4 + 40x^3 + 721x^2 - 9000x = 90,000$, erit paulò minor quàm 12.024. Q. E. I.

Hîc verò ſuſpicari licet quòd hæc radix (quæ minor eſt quàm 12.024,) poſſit eſſe accuratè æqualis numero integro 12. Et ita reverâ ſe res habet. Nam, ſi x ſit $= 12$, xx erit $= 144$, et x^3 erit $= 1728$, et x^4 erit $= 20,736$, et $40x^3$ erit $(= 40 \times 1728) = 69,120$, et $751xx$ erit $(= 751 \times 144) = 108,144$, et $9000x$ erit $(= 9000 \times 12) = 108,000$; et pro-inde quantitas quadrinomia $x^4 + 40x^3 + 751xx - 9000x$ erit $(= 20,736 + 69,120 + 108,144 - 108,000 = 198,000 - 108,000) = 90,000$; quæ eſt æquatio propoſita. Ergò numerus integer 12 eſt ejuſdem æquationis radix.

Hæc exempla æquationum Cubicarum et Biquadraticarum per Methodum approximationis à Raphſono traditam reſolutarum, ſufficere poſſunt ad illuſtrationem utilitatis Limitum (inter quos jacent harum æquationum radices,) à Florimondo De Beaune deſcriptorum, ad inveniendum, per commodas conjecturas ex his Limitibus derivatas, primos propinquos valores radicum quæſitarum qui poſſint felicitèr aſſumi pro fundamentis proceſſuum per quos ad

earum veros valores multo propiùs, per Raphſoni aut aliorum Methodos, accedere poſſimus. Itaque huic tractatui hîc finem impono.

FRANCISCUS MASERES.

Londini, in Templo Interiore,
Die 7 Junii, A. D. 1804.

A METHOD

OF

ASCERTAINING THE NUMBER OF FIGURES THAT ARE EXACT IN THE VALUE OF A ROOT OF AN ALGEBRÄICK EQUATION,

THAT HAS BEEN OBTAINED BY

MR. RAPHSON'S METHOD OF APPROXIMATION.

BY MR. JAMES IVORY,

ONE OF THE MATHEMATICAL INSTRUCTORS IN THE NEW MILITARY ACADEMY AT MARLOW, IN BUCKINGHAMSHIRE.

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BY MR. JAMES THORP.

A METHOD OF ASCERTAINING THE NUMBER OF FIGURES THAT ARE EXACT IN THE VALUE OF A ROOT OF AN ALGEBRÄICK EQUATION, THAT HAS BEEN OBTAINED BY MR. RAPHSON'S METHOD OF APPROXIMATION.

BY MR. JAMES IVORY,
ONE OF THE MATHEMATICAL INSTRUCTORS IN THE NEW MILITARY ACADEMY AT MARLOW, IN BUCKINGHAMSHIRE.

WHEN the root of an Algebräick equation has been obtained to eight or ten, or more, places of figures by Mr. Raphſon's Method of Approximation, it is neceſſary, in order to make the reſolution of ſuch equation compleatly ſatisfactory, that we ſhould determine with certainty how many of the figures of the value of the root ſo obtained are true, or would be the ſame with the like number of figures in a ſtill nearer value of the ſame root, if we were to carry the approximation a ſtep further, ſo as to obtain ſuch ſtill nearer value of it. Now the plaineſt, or moſt obvious way of doing this is, firſt, to ſubſtitute ſo many of the figures of the number obtained for the value of the ſaid root as we conceive to be exact, (as, for example, the firſt ten figures, if we conceive that ten figures of the ſaid number are exact,) inſtead of x, or the unknown quantity, in the terms of the compound, or multinomial, quantity that forms the firſt, or left-hand, ſide of the equation, and to compare the value of the ſaid multinomial quantity reſulting from ſuch ſubſtitution with the abſolute term that forms the ſecond, or right-hand, ſide of the equation, in order to diſcover whether it is leſs, or is greater, than the ſaid abſolute

abſolute term; and then, if it is leſs than the ſaid abſolute term, to increaſe the ſaid number that had been ſo ſubſtituted, by adding to it an unit in the laſt, or loweſt, place of figures, and to ſubſtitute ſuch increaſed number, inſtead of x, or the unknown quantity of the equation, in the ſaid compound, or multinomial, quantity which forms the firſt ſide of the equation, and to compare the value of the ſaid compound quantity reſulting from ſuch ſecond ſubſtitution with the abſolute term of the equation. And, if it ſhall appear that this ſecond value of the ſaid compound quantity is greater than the abſolute term of the equation, we may conclude with certainty that the former, or leſſer, number that was ſubſtituted for x, in the ſaid compound quantity, will be exact in all it's figures. But the labour of making theſe ſubſtitutions, by raiſing the number of eight or ten figures that has been found for a near value of the root x, to it's ſquare, cube, fourth power, and other higher powers, and multiplying theſe powers into the ſeveral co-efficients of the terms of the equation, is very great and inconvenient; and therefore it is very much to be wiſhed that ſome eaſier expedients may be found-out for determining the number of figures that may be depended-upon as exact in the values of the roots that have been obtained by this method of approximation, without having recourſe to theſe ſubſtitutions. And, for this purpoſe we may obſerve, in the firſt place, that, if no miſtakes have been made in the arithmetical operations that have been employed in obtaining, by Mr. Raphſon's Method of Approximation, the number we have found for the value of the root ſought, we may almoſt always conclude that, if the former near value of the root x, which was made the baſis, or ground, of the laſt proceſs of approximation, according to Mr. Raphſon's Method, was true to n places of figures, the new near value of it, which will have been obtained by ſuch proceſs of approximation, will be true to $n + n - 1$, or to $2n - 1$ places of figures; and it will often be true even to $2n$ places. But there are alſo other methods of determining this point without having recourſe to the above-mentioned laborious ſubſtitutions. And one of theſe methods has been deſcribed and exemplified in the reſolutions of ſome of the cubick and biquadratick equations, taken from Mr. Raphſon's *Analyſis Æquationum Univerſalis*, that have been given in the laſt preceeding pages of this volume. This method conſiſts in retaining in the final equation, with which we mean to cloſe our proceſſes of approximation to the value of the root ſought, the term involving zz, or the ſquare of the new difference z, (by the addition, or ſubtraction, of which, to, or from, the former known near value of the root x, we mean to approach nearer to it's true value,) and in computing the value of the ſaid term involving zz, or, rather, (in order to avoid unneceſſary labour,) in computing the value of a quantity a little greater than the ſaid term, to only one or two places of decimal figures, whereby we ſhall diſcover in what place of decimal figures the firſt, or higheſt, figure of the ſaid term would appear. And this method, (from the trials I have made of it in ſome of the examples of the reſolution of Cubick and Biquadratick Equations, given in the foregoing Tract,) ſeems

to

to anſwer this purpoſe very well. But the learned and ſagacious Mr. James Ivory, (who is now ſettled at Marlow, in Buckinghamſhire, as one of the Mathematical Inſtructors at the new Military Academy lately erected there,) has alſo diſcovered another very ſatisfactory method of doing the ſame thing; which he has deſcribed in a very full and clear manner, in a Letter which he ſent me in October, 1801, after having peruſed with care and attention my Volume of *Tracts on the Reſolution of Algebraick Equations by Dr. Halley's, Mr. Raphſon's, and Sir Iſaac Newton's, Methods of Approximation*, which was publiſhed in the year 1800. This Method of Mr. Ivory I ſhall now proceed to ſtate in his own words from the above-mentioned Letter with which he honoured me, and which is dated from Douglaſtown, near Dundee, on the 17th of October, 1801.

An Extract from a Letter of Mr. James Ivory, *to* Francis Maseres, *Eſq.*

dated from Douglaſtown, near Dundee.

Douglaſtown, near Dundee, October 17, 1801.

—— I perfectly ſubſcribe to your opinion, "That, all things conſidered, "Mr. Raphſon's Method of approximating to the roots of Affected Equations "is, in general, preferable to every other." The peruſal of theſe tracts has drawn my attention a good deal to the different Methods of Approximation that are conſidered and compared in them. And I remark that they are all defective in one reſpect: which is this, "That, after a near value of the root is found, that is true in the firſt figure, or the two firſt figures, there is no ſure and certain rule by which to determine how near the ſeveral corrections in the different ſteps of the inveſtigation approach to the truth, or how many figures of them are exact." It is true, indeed, that, after the inveſtigation is advanced a certain length, we may reckon that there will be as many new figures obtained exactly in each ſtep,—or, more ſafely, as many figures, wanting one,—as there are figures already aſcertained in the root; ſo that every new proceſs will double, or nearly double, the number of the known figures of the root: but in the firſt ſtages of the inveſtigation this rule is often of no uſe, and we are left intirely to conjectures, or to the neceſſity of making laborious ſubſtitutions, in order to aſcertain the progreſs of the approximation; or we are obliged to have recourſe to auxiliary methods, (ſuch as the

Differential Method,) to advance the operation to the requiſite degree of exactneſs.

In reflecting on this matter I hit upon a method of remedying the imperfection, I have juſt mentioned, in a way that ſeems to me to be ſatisfactory. My Method is to be conſidered only as a more general view of Mr. Raphſon's Method; and, like his, requires the reſolution of ſimple equations only: And, as I apprehend that it will be found greatly to abridge the labour of the inveſtigation, as well as to render it more ſatisfactory, eſpecially in the firſt ſtages, I think you will not be diſpleaſed to have it communicated to you.

Subjects of this kind are, I believe, better taught by examples than by general precepts. I ſhall therefore immediately proceed to apply my Method to ſome example: and I have purpoſely choſen the difficult Biquadratick Equation that has been ſo much diſcuſſed in the Tracts; becauſe, having already applied to that equation all the other methods, you will be the better able to judge of the comparative merit of the new method.

Taking, then, the equation $14{,}937x - 1998x^2 + 80x^3 - x^4 = 5000$, I ſhall, firſt, conſider *that* root of it which is determined, in Art. 28 of the Tracts, to be greater than 12, but leſs than 13.

Let a be $= 12$, and let e be the ſupplemental part of the root, ſo that x ſhall be $= a + e$. And, becauſe x is greater than 12, or a, but leſs than 13, the correction e muſt be a fraction leſs than 1; or, in other words, the correction e muſt lie between the limits 0 and 1.

I now ſubſtitute $a + e$ for x in the given equation, and (referring to the Tracts, pages 83 and 84, for the work,) I get this equation

$$5000 = \left\{\begin{array}{rrrrrr} & 14{,}937\,a & + & 14{,}937\,e & & \\ - & 1{,}998\,a^2 & - & 3{,}996\,ae & - & 1998\,e^2 \\ + & 80\,a^3 & + & 240\,a^2e & + & 240\,ae^2 + 80\,e^3 \\ - & a^4 & - & 4\,a^3e & - & 6\,a^2e^2 - 4ae^3 - e^4. \end{array}\right\}$$

But a is $= 12$. Therefore $14{,}937\,a - 1998\,a^2 + 80\,a^3 - a^4$ is $(= 179{,}244 - 287{,}712 + 138{,}240 - 20{,}736) = 9{,}036$, by Art. 8;

and $14{,}937\,e - 3{,}996\,ae + 240\,a^2e - 4\,a^3e$ is $(= 14{,}937\,e - 47{,}952\,e + 34{,}560\,e - 6912\,e) = 49{,}497\,e - 54{,}864\,e$;

and $-1998\,e^2 + 240\,ae^2 - 6\,a^2e^2$ is $(= -1998\,e^2 + 2880\,e^2 - 864\,e^2) = -2862\,e^2 + 2880\,e^2$;

and $80\,e^3 - 4\,ae^3$ is $(= 80\,e^3 - 48\,e^3) = 32\,e^3$.

Therefore

Therefore the equation becomes

$$5000 = \left\{ \begin{array}{l} 9036 + 49{,}497\,e - 2862\,e^2 \\ \quad - 54{,}864\,e + 2880\,e^2 + 32\,e^3 - e^4 \end{array} \right\},$$

or $5000 = 9036 - 5367\,e + 18\,e^2 + 32\,e^3 - e^4$;

And (adding the compound quantity $5367\,e - 18\,e^2 - 32\,e^3 + e^4$ to both ſides,) $5000 + 5367\,e - 18\,e^2 - 32\,e^3 + e^4 = 9036$; and, (ſubtracting 5000 from both ſides,) $5367\,e - 18\,e^2 - 32\,e^3 + e^4 = 4036$; and, finally, (dividing both ſides by the compound quantity $5367 - 18\,e - 32\,e^2 + e^3$,) $e = \frac{4036}{5367 - 18\,e - 32\,e^2 + e^3}$.

Now, if we reject the three terms of the compound denominator $5367 - 18\,e - 32\,e^2 + e^3$ that involve in them the ſmall quantity e, and it's ſquare and cube, as inconſiderable in compariſon of the firſt term 5367, or, in other words, if we take the value of the denominator correſponding to the ſmaller Limit of e, or to the ſuppoſition that e is $= 0$, we ſhall have ſimply $e = \frac{4036}{5367}$; which is the very reſult that is obtained by Mr. Raphſon's Method.

But there appears to be no reaſon for ſuppoſing e to co-incide with it's ſmaller Limit rather than with it's greater Limit, or for ſuppoſing $e = 0$ rather than $e = 1$. Let us then ſuppoſe $e = 1$. And the correſponding value of the denominator is $5367 - 18 - 32 + 1 = 5318$; which gives $e = \frac{4036}{5318}$.

Now we know that the true value of e is between the Limits 0 and 1. And conſequently the true value of the denominator muſt be between the numbers 5367 and 5318, which are it's values correſponding to the ſuppoſitions of $e = 0$ and $e = 1$. Therefore the exact value of e, or of the correction of the root ſought, muſt be between the two fractions $\frac{4036}{5367}$ and $\frac{4036}{5318}$.

I have detailed the preceeding reaſoning at the greater length, becauſe on it the Method I am explaining intirely depends. And I remark further, that the preceeding reaſoning is juſt in all caſes, provided that no more than one root of the given equation lies between the two Limits with which the Inveſ-

tigation ſets out; that is, in the preſent example, provided that no more than one root of the given equation lies between the Limits 12 and 13. This remark, inſtead of reſtricting the extenſiveneſs of the Method, only tends to introduce greater clearneſs and preciſion into ſuch inveſtigations.

But, to reſume: the fraction $\frac{4036}{5367}$ is $= 0.750$, and the fraction $\frac{4036}{5318}$ is $= 0.758$. Therefore, omitting the doubtful figures, e will be $= 0.75$, and $x = a + e$, will be $(= 12 + 0.75) = 12.75$. And we are certain that all the four figures are exact.

To approximate ſtill nearer to the root ſought, we have only to repeat the ſame reaſonings and operations that have already been gone through, building upon the concluſions derived from the firſt proceſs. We now know that the root ſought is greater than 12.75, but leſs than $12 + \frac{4036}{5318}$, or 12.758. Therefore, if we now put $a = 12.75$, and ſuppoſe $x = a + e$, the correction e will be leſs than 0.008; that is, e will be between the Limits 0 and 0.008. But before I proceed to the neceſſary ſubſtitutions, it will be proper to notice an obſervation that will enable us to ſhorten our labour.

In the proceſs which we have already gone through, the greater Limit of e was 1; and this made it neceſſary to compute all the terms of the transformed equation that involved any powers of e. But, if the greater Limit of e, inſtead of being 1, had been a ſmall fraction, as $\frac{1}{10}$ or $\frac{1}{100}$, we might have omitted all the terms of the transformed equation that involved in them the cube and other higher powers of e without hurting the accuracy of the reſult. Thus, therefore, the Method we are conſidering will, in moſt caſes, only require that one more term of the transformed equation ſhould be computed than in Mr. Raphſon's Method.

Subſtituting now $a + e$ for x in the given equation, when a is $= 12.75$, and omitting the terms that involve the cube and fourth power of e, according to the obſervation juſt now made, we ſhall have, very nearly,

$$5000 = \left\{\begin{array}{rrrr} & 14,937\,a & + & 14,937\,e \\ - & 1998\,a^2 & - & 3,996\,ae & - & 1998\,e^2 \\ + & 80\,a^3 & + & 240\,a^2e & + & 240\,ae^2 \\ - & a^4 & - & 4\,a^3e & - & 6\,a^2e^2 \end{array}\right\}$$

But

But a is $=$ 12.75. Therefore $14{,}937\,a - 1998\,a^2 + 80\,a^3 - a^4$ will be ($=$ 190,446.75 — 324,799.875 + 165,813.75 — 26,426.566,406,25 $=$ 356,260.5 — 351,226.441,406,25) $=$ 5034.058,593,75; and $14{,}937\,e - 3{,}996\,ae + 240\,a^2e - 4\,a^3e$ will be ($=$ $14{,}937\,e - 50{,}949\,e + 39{,}015\,e - 8{,}290.6875\,e$) $= 53{,}952\,e - 59{,}239.6875\,e$; and $-1998\,e^2 + 240\,ae^2 - 6\,a^2e^2$ will be ($=$ $-1998\,e^2 + 3060\,e^2 - 975.275\,e^2$) $= 3060\,e^2 - 2973.275\,e^2$.

Therefore 5000 will be $=$ (5034.058,593,75
$+ 53{,}952\,e$ $+ 3060\,e^2$
$- 59{,}239.6875\,e$ $- 2973.275\,e$ $\Big\} = 5034.058{,}593{,}75$
$- 5287.6875\,e + 86.725\,e^2$. And (adding the compound quantity $5287.6875e - 86.725\,e^2$ to both sides,) $5000 + 5287.6875\,e - 86.725\,e^2$ will be $=$ 5034.058,593,75; and (subtracting 5000 from both sides,) $5287.6875\,e - 86.725\,e^2$ will be $=$ 34.058,593,75; and, finally, (dividing both sides by the compound quantity $5287.6875 - 86.725\,e$) we shall have $e = \frac{34.058{,}593{,}75}{5287.6875 - 86.725\,e}$.

Let us now take that value of the denominator $5287.6875 - 86.725\,e$ which corresponds to the smaller Limit of e, or to the supposition that e is $= 0$; and we shall have one value of $e = \frac{34.058{,}593{,}75}{5288.6875} = 0.006{,}441{,}7$. And again, taking the other value of the denominator $5287.6875 - 86.725\,e$, which corresponds to the greater Limit of e, or to the supposition that e is $=$ 0.008, we shall have e ($= \frac{34.058{,}593{,}75}{5287.6875 - 86.725 \times 0.008} = \frac{34.058{,}593{,}75}{5287.6875 - 0.6938}$) $= \frac{34.058{,}593{,}75}{5286.9937} = 0.006{,}441{,}9$.

But the exact value of e lies between the two values now found. Therefore, omitting the doubtful figures, e will be $=$ 0.006,441. Consequently x, or $a + e$, or $12.75 + e$, will be $= 12.75 + 0.006{,}441 = 12.756{,}441$, and we are certain that all the eight figures are exact.

In order further to illustrate this Method, I shall now consider the least root of the same equation. And, for the sake of brevity, I shall assume (what it

is

is not difficult to diſcover,) that the root ſought is greater than $\frac{1}{3}$, but leſs than $\frac{1}{2}$. Put now $a = \frac{1}{3}$, and ſuppoſe x to be $= a + e$. And it is obvious that e will be of an intermediate magnitude between o and $(\frac{1}{2} - \frac{1}{3}$, or) $\frac{1}{6}$.

Now, if we ſubſtitute $a + e$ for x in the given equation, and neglect the terms that involve the cube and fourth power of e, we ſhall get, as before, the equation

$$5000, \text{ very nearly, } = \left\{\begin{array}{rrrr} & 14{,}937\,a & + & 14{,}937\,e \\ - & 1{,}998\,a^2 & - & 3{,}996\,ae - 1998\,e^2 \\ + & 80\,a^3 & + & 240\,a^2e + 240\,ae^2 \\ - & a^4 & - & 4a^3e - 6\,a^2e^2 \end{array}\right\}$$

But a is $= \frac{1}{3}$. Therefore $14{,}937\,a - 1998\,a^2 + 80\,a^3 - a^4$ will be $(= 4979 - 222 + 2.96 - 0.01) = 4759.95$; and $14{,}937\,e - 3{,}996\,ae + 240\,a^2e - 4\,a^3e$ will be $(= 14{,}937\,e - 1332\,e + 26.66\,e - 0.14\,e) = 13{,}631.52 \times e$; and $-1998\,e^2 + 240\,ae^2 - 6\,a^2e^2$ will be $(= -1998\,e^2 + 80\,e^2 - 0.66\,e^2) = -1998.66\,e^2 + 80\,e^2$. And conſequently 5000 will be $= 4759.95 + 13{,}631.52 \times e - 1918.66\,e^2$; and (ſubtracting 4759.95 from both ſides,) we ſhall have $240.05 = 13{,}631.52 \times e - 1{,}918.66 \times e^2$; and (dividing both ſides by the compound quantity $13{,}631.52 - 1{,}918.66e$) we ſhall have $e = \frac{240.05}{13{,}631.52 - 1{,}918.66\,e}$.

Let us now make e in the compound denominator $13{,}631.52 - 1{,}918.66\,e$ be $= 0$; and we ſhall have $e\,(= \frac{240.05}{13{,}631.52 - 0} = \frac{240.05}{13{,}631.52}) = 0.0176$. And, making $e = \frac{1}{6}$, it's higheſt Limit, we ſhall have $e = \frac{240.05}{13{,}631.52 - 1{,}918.66 \times \frac{1}{6}}$ $(= \frac{240.05}{13{,}631.52 - 319.77} = \frac{240.05}{13{,}311.75}) = 0.0180$.

Now the exact value of e muſt be between the two values of it here found.

Therefore

Therefore it will be greater than the leſſer of the two, that is, than 0.0176. Therefore x, or $a + e$, will be greater than $a + 0.0176$, (or $\frac{1}{3} + 0.0176$, or $0.3333 + 0.0176$,) or than 0.3509.

And the exact value of e will be leſs than the greater of the two values that have been here found, that is, than 0.0180. Therefore x, or $a + e$, will be leſs than $a + 0.0180$, (or than $\frac{1}{3} + 0.0180$, or than $0.3333 + 0.0180$,) or than 0.3513.

Therefore, omitting the doubtful figures, we ſhall have $x = 0.35$; and we are ſure that theſe two figures are exact.

But, if any one ſhould be inclined to approximate nearer to this root by means of a ſecond proceſs of Mr. Raphſon's Method of Approximation grounded on the numbers already found, it will be better to aſſume $a = 0.3509$ in the next proceſs than to aſſume it $=$ only to 0.35; becauſe, although we are not ſure that all theſe four figures are exact, yet the Limit of the correction e will be in that ſuppoſition only $(= 0.3513 - 0.3509) = 0.0004$, or $\frac{4}{10,000}$, which is much ſmaller than it would be by aſſuming x $=$ only to 0.35.

End of the Extract from Mr. IVORY's *Letter.*

This Method of aſcertaining the number of figures that are exact in the near value of the root of an equation obtained by Mr. Raphſon's Method of Approximation, invented by Mr. Ivory, is both very ingenious and very ſatisfactory; ſince nothing can be more evident than that ſo many of the figures of the numeral values of the two Limits of the magnitude of the correction e as are found to be the ſame with each other, muſt likewiſe be found in the value of the ſaid correction e itſelf, which lies between the ſaid Limits.

A NEW

A NEW SOLUTION OF COLONEL TITUS'S ARITHMETICAL PROBLEM.

BY MR. JAMES IVORY,

ONE OF THE MATHEMATICAL INSTRUCTORS AT THE NEW MILITARY ACADEMY AT MARLOW, IN BUCKINGHAMSHIRE.

IN the month of October, 1801, Mr. Ivory ſent me the following very elegant Solution of the difficult Arithmetical Problem, which had been propoſed to Dr. John Wallis, the celebrated Savilian Profeſſor of Geometry at Oxford, by Colonel Silas Titus, in the year 1662, and of which Dr. Wallis has given a Solution in the 62nd Chapter of his Algebra. This Solution of this Problem by Dr. Wallis I had introduced into the octavo volume of *Tracts on the Reſolution of Affected Algebräick Equations, by Dr. Halley's, Mr. Raphſon's, and Sir Iſaac Newton's Methods of Approximation*, publiſhed in the year 1800, of which I had ſent a copy to Mr. Ivory. And the peruſal of this volume of Tracts, and particularly of thoſe parts of it which related to this Problem, and the Solutions that had been given of it by Dr. Wallis and Mr. William Frend, excited the attention and curioſity of Mr. Ivory ſo much as to induce him to ſeek for a new Solution of it; in which he has been ſingularly ſucceſsful. In his Letter to me of the 17th of October, 1801, (in which he communicated to me the ingenious Method he had found-out for aſcertaining the number of figures that are exact in any near value of the Root of an Affected Algebräick Equation, that has been obtained by Mr. Raphſon's Method of Approximation, which I have here publiſhed in the foregoing pages of this volume,) he ſpeaks of this new Solution which he had found of Colonel Titus's Problem, in theſe words. "In peruſing the *Tracts on the Reſolution of* "*Equations*, my attention was ſo much attracted by Colonel Titus's Arithme- "tical Problem, that I ſought a Solution of it, although Mr. Frend's is a very "good one. The Solution which I found is very different both from Dr. "Wallis's

" Wallis's Solution and Mr. Frend's; and it is particularly remarkable on ac-
" count of the ſimplicity of the reſult. For, purſuing the notation at page 189 of
" the Tracts, my Solution conducts me, in the firſt place, to the Biquadratick
" Equation $8 = 35z + 5z^2 - 34z^3 - z^4$; from which having obtained
" the value of z, the three numbers a, b, and c, ſought are immediately found
" by means of the expreſſions, $b = \sqrt{\frac{2}{z - z^3}}$, $a = \frac{2 - z - z^3}{2} \times b$, and c
" $= \frac{2 + z - z^3}{2} \times b$. Thinking that I might contribute to your entertain-
" ment by ſending this Solution, I have written it out at conſiderable length,
" ſo as to omit none of the ſteps of the reaſoning; and I hope you will find
" no difficulty in peruſing it."

The Solution itſelf (which was ſent me at the ſame time as the foregoing Letter which announced it,) was as follows.

The Solution of Colonel Titus's Problem.

Article 1. The Arithmetical Problem propoſed by Colonel Titus to Dr. Wallis (which is mentioned in Mr. Maſeres's *Tracts on the Reſolution of Algebräick Equations by Approximation, page* 188,) leads to the three equations following, viz.

$$\left.\begin{aligned} a^2 + bc &= 16, \\ b^2 + ac &= 17, \\ \text{and } c^2 + ab &= 18, \end{aligned}\right\}$$

in which the Letters a, b, and c, denote the three numbers ſought. This ſet of equations I ſhall refer to by the name of Number 1.

Subtract the firſt of theſe three equations from the ſecond, and the ſecond of them from the third. And we ſhall thereby get the two equations following, to wit,

$$b^2 - a^2 + ac - bc = 1,$$
$$\text{and } c^2 - b^2 + ab - ac = 1.$$

Now it is very eaſy to be diſcovered that the left-hand ſide of the equation $b^2 - a^2 + ac - bc = 1$ is no other than the product of the two factors $b - a$ and $b + a - c$ multiplied together; and, in like manner, that the left-hand ſide of the equation $c^2 - b^2 + ab - ac$ is the product of the two fac-

tors $c - b$ and $c + b - a$ multiplied together. Therefore these two equations may be thus written,

$$\overline{b - a} \times \overline{b + a - c} = 1,$$

$$\text{and } \overline{c - b} \times \overline{c + b - a} = 1.$$

And, if we divide both sides of the first of these equations by $b - a$, and both sides of the second equation by $c - b$, we shall obtain the equations

$$b + a - c = \frac{1}{b-a},$$

$$\text{and } c + b - a = \frac{1}{c-b}.$$

Let us now put $b - a = m$, and $c - b = n$. And the two last equations will become, by substitution,

$$b + a - c = \frac{1}{m},$$

and $c + b - a = \frac{1}{n}$; and we shall have the four following equations between the five unknown quantities a, b, c, m, and n, to wit,

$$b - a = m,$$

$$c - b = n,$$

$$b + a - c = \frac{1}{m},$$

and $c + b - a = \frac{1}{n}$. And this set of equations I shall refer to by the name of Number 2.

Article 2. I now seek a value of each of the three unknown numbers a, b, and c, which shall contain only the two unknown numbers m and n. And I readily find, by adding together the second and third equations of Number 2, that a will be $= n + \frac{1}{m}$; and, by adding together the third and fourth equations of the same Number 2, that $2b$ will be $= \frac{1}{n} + \frac{1}{m}$, and consequently that b will be $= \frac{1}{2} \times \overline{\frac{1}{n} + \frac{1}{m}}$; and, lastly, by subtracting the first equation from the fourth, that c will be $= \frac{1}{n} - m$.

We have only now to ſubſtitute theſe values of a, b, and c, ſo found, in any one of the original equations of Number 1, in order to have an equation that ſhall contain only m and n and known numbers. Thus, by ſubſtituting theſe values in the firſt of the equations of Number 1, to wit, the equation $a^2 + bc = 16$, we ſhall get the equation $\overline{n + \frac{1}{m}}^2 + \frac{1}{2} \times \overline{\frac{1}{n} + \frac{1}{m}} \times \overline{\frac{1}{n} - m} = 16$.

In order to determine the values of m and n we have ſtill to ſeek another equation that ſhall contain only m and n and known numbers. But ſuch an equation may be eaſily deduced from the equations of Number 2. For, by adding together the firſt and ſecond of the equations of Number 2, we get $c - a = m + n$; and, by doubling both ſides, we get $2c - 2a = 2m + 2n$. Again, by ſubtracting the third of the equations of Number 2 from the fourth, we likewiſe get $2c - 2a = \frac{1}{n} - \frac{1}{m}$. Therefore, by equating the two values of $2c - 2a$, there reſults the equation $2m + 2n = \frac{1}{n} - \frac{1}{m}$.

We have now obtained two equations involving only the unknown numbers m and n and known numbers, to wit,

$$\overline{n + \frac{1}{m}}^2 + \frac{1}{2} \times \overline{\frac{1}{n} + \frac{1}{m}} \times \overline{\frac{1}{n} - m} = 16,$$

and $2m + 2n = \frac{1}{n} - \frac{1}{m}$; which two equations I ſhall refer to by the name of Number 3. And theſe two equations are ſufficient to determine the unknown numbers m and n. And, when the values of m and n are determined, the values of the three numbers a, b, c, (which are the quantities ſought by the Problem,) will be directly obtained by means of the three equations $a = n + \frac{1}{m}$, $b = \frac{1}{2} \times \overline{\frac{1}{n} + \frac{1}{m}}$, and $c = \frac{1}{n} - m$, which have been already inveſtigated, and which I ſhall refer to by the name of Number 4.

Article 3. It ſtill remains that we deduce from the two equations of Number 3, which contain the two unknown numbers m and n, a final equation that ſhall contain only one of thoſe unknown numbers. Now, if we proceed to

operate directly upon these two equations, (which are $\overline{n + \frac{1}{m}}^2 + \frac{1}{2} \times \sqrt{\frac{1}{n} + \frac{1}{m}} \times \sqrt{\frac{1}{n} - m} = 16$, and $2m + 2n = \frac{1}{n} - \frac{1}{m}$,) with the view of exterminating one of the two unknown numbers m and n, we shall find a good many pretty tedious and intricate Algebraick processes to be necessary. But we may attain our object more easily and shortly in the manner following.

By multiplying n into $\frac{m}{m}$ in order to bring it to a fraction, having the same denominator as the fraction $\frac{1}{m}$ (by which multiplication it's magnitude will not be altered,) we shall have $n + \frac{1}{m} (= \frac{mn}{m} + \frac{1}{m}) = \frac{mn + 1}{m}$. Therefore the first term of the left-hand side of the first of the two equations of Number 3, to wit, the term $\overline{n + \frac{1}{m}}^2$, will be $(= \overline{\frac{mn + 1}{m}}^2) = \frac{\overline{mn + 1}^2}{m^2}$.

In like manner the compound quantity $\frac{1}{m} + \frac{1}{n}$ will be $= \frac{n + m}{mn}$, and the compound quantity $\frac{1}{n} - m$ will be $= \frac{1 - mn}{n}$. Therefore the second term of the left-hand side of the first of the two equations of Number 3, or the term $\frac{1}{2} \times \sqrt{\frac{1}{n} + \frac{1}{m}} \times \sqrt{\frac{1}{n} - m}$, will be $= \frac{1}{2} \times \frac{m + n}{mn} \times \frac{1 - mn}{n} = \frac{1}{2} \times \frac{\sqrt{m + n} \times \sqrt{1 - mn}}{mn \times n} =$ (by multiplying both numerator and denominator by m, which does not alter the value of the fraction,) $\frac{1}{2} \times \frac{\sqrt{m \times m + n} \times \sqrt{1 - mn}}{m \times mn \times n} = \frac{1}{2} \times \frac{\sqrt{mm + mn} \times \sqrt{1 - mn}}{m^2 n^2}$.

Now let $\frac{\overline{mn + 1}^2}{m^2}$ be substituted for, it's equal, $\overline{n + \frac{1}{m}}^2$ in the first of the equations of Number 3, and likewise the quantity $\frac{1}{2} \times \frac{\sqrt{m^2 + mn} \times \sqrt{1 - mn}}{m^2 n^2}$ be substituted for, it's equal, $\frac{1}{2} \times \sqrt{\frac{1}{n} + \frac{1}{m}} \times \sqrt{\frac{1}{n} - m}$ in the same equation.

tion. And that equation will become $\frac{\overline{mn+1}^2}{m^2} + \frac{1}{2} \times \frac{\overline{m^2+mn} \times \overline{1-mn}}{m^2 n^2} = 16$.

In this equation I put $x = mn$, and $y = m^2$; and, by ſubſtitution, I get the following equation, which involves only x and y, to wit, $\frac{\overline{x+1}^2}{y} + \frac{1}{2} \times \frac{\overline{y+x} \times \overline{1-x}}{x^2} = 16$.

Further, take the ſecond of the equations of Number 3, to wit, the equation $2m + 2n = \frac{1}{n} - \frac{1}{m}$, and multiply all it's terms by m^2n. And it will become $2m^3n + 2m^2n^2 = m^2 - mn$. And, by writing x for mn and y for m^2 in this equation, I get another equation involving only x and y, to wit, the equation $2xy + 2x^2 = y - x$. And thus we have obtained two equations containing only x and y, to wit,

$$\frac{\overline{x+1}^2}{y} + \frac{1}{2} \times \frac{\overline{y+x} \times \overline{1-x}}{x^2} = 16,$$

and $2xy + 2x^2 = y - x$, in which equations x is $= mn$, and y is $= m^2$. Theſe two equations I ſhall refer to by the name of Number 5.

Article 4. It is now eaſy to deduce a final equation from the two equations of Number 5, becauſe the ſecond of them (beſides that both of them are ſimpler than the equations of Number 3,) involves one of the two unknown quantities, to wit, y, ſimply, or without any of it's powers.

Multiply all the terms of the firſt of the two equations of Number 5 by $2x^2y$; and it will become $2x^2 \times \overline{x+1}^2 + y \times \overline{y+x} \times \overline{1-x} = 32x^2y$.

Again, take the ſecond of the equations of Number 5, to wit, $2yx + 2x^2 = y - x$, and add x to both ſides of it; and we ſhall have $2yx + 2x^2 + x = y$. And, by ſubtracting $2yx$ from both ſides, we ſhall have $2x^2 + x = y - 2yx$; and, by dividing both ſides by the compound quantity $1 - 2x$, we ſhall have $y = \frac{2x^2+x}{1-2x}$; and, by adding x to both ſides, we ſhall have

have $y + x = \frac{2x^2+x}{1-2x} + x \left(= \frac{2x^2+x}{1-2x} + \frac{x \times \overline{1-2x}}{1-2x} = \frac{2x^2+x}{1-2x} + \frac{x-2x^2}{1-2x} = \frac{2x^2+2x-2x^2}{1-2x}\right) = \frac{2x}{1-2x}$.

In the equation $2x^2 \times \overline{x+1}^2 + y \times \overline{y+x} \times \overline{1-x} = 32x^2y$, obtained above, I now write $\frac{2x^2+x}{1-2x}$ for, it's equal, y, and $\frac{2x}{1-2x}$ for, it's equal, $y + x$; and I thereby obtain the equation $2x^2 \times \overline{x+1}^2 + \frac{2x^2+x}{1-2x} \times \frac{2x}{1-2x} \times \overline{1-x} = 32x^2 \times \frac{2x^2+x}{1-2x}$, or $2x^2 \times \overline{x+1}^2 + \frac{2x+1}{1-2x} \times \frac{2x^2}{1-2x} \times \overline{1-x} = 32x^2 \times \frac{2x^2+x}{1-2x}$; which is an equation that contains only one unknown number, to wit, x, and only requires to be further reduced in a proper manner, according to the ordinary rules given by writers on Algebra for that purpose, to prepare it for being resolved.

Article 5. Divide all the terms of the last equation by $2x^2$; and it will thereby become $\overline{x+1}^2 + \frac{2x+1}{1-2x} \times \frac{1}{1-2x} \times \overline{1-x} = 16 \times \frac{2x^2+x}{1-2x}$. Therefore, by multiplying all the terms by $\overline{1-2x}^2$, we shall have $\overline{x+1}^2 \times \overline{1-2x}^2 + \overline{2x+1} \times \overline{1-x} = 16 \times \overline{2x^2+x} \times \overline{1-2x}$, that is, $\overline{x^2+2x+1} \times \overline{1-4x+4x^2} + 1 + 2x - x - 2x^2$ will be $\left(= 16 \times \overline{2x^2+x-4x^3-2x^2} = 16 \times \overline{x-4x^3}\right) = 16x - 64x^3$, or $\overline{x^2+2x+1} \times \overline{1-4x+4x^2} + 1 + x - 2x^2$ will be $= 16x - 64x^3$; and (subtracting $x - 2x^2$ from both sides,) we shall have $\overline{x^2+2x+1} \times \overline{1-4x+4x^2} + 1 = 15x - 2x^2 - 64x^3$, that is, $4x^4 + 4x^3 - 3x^2 - 2x + 1 + 1$ will be $= 15x - 2x^2 - 64x^3$, or $4x^4 + 4x^3 - 3x^2 - 2x + 2$ will be $= 15x - 2x^2 - 64x^3$; and (subtracting the compound quantity $4x^4 + 4x^3 - 3x^2 - 2x$ from both sides,) we shall have $2 = 17x + 5x^2 - 68x^3 - 4x^4$, and (multiplying all the terms by 4,) we shall have $8 = 68x + 20x^2 - 272x^3 - 16x^4$, or (placing the absolute term 8 on the right-hand side of the mark of equality,) $68x + 20x^2 - 272x^3 - 16x^4 = 8$.

Lastly, to free x^4 from it's co-efficient 16, and to reduce the equation to the simplest form possible, let z be put $= 2x$. Then will zz be $= 4xx$, and z^3 be

be $= 8x^3$, and z^4 be $= 16x^4$, and consequently $34z$ will be $(= 34 \times 2x)$ $= 68x$, and $5zz$ will be $(= 5 \times 4xx) = 20xx$, and $34z^3$ will be $= 34 \times 8x^3 = 272x^3$, and the compound quantity $34z + 5zz - 34z^3 - z^4$ will be $= 68x + 20x^2 - 272x^3 - 16x^4$. Therefore $34z + 5z^2 - 34z^3 - z^4$ will alſo be $= 8$. Q. E. I.

Article 6. This equation is ſufficient to determine the value, or values, of z. And, when theſe are obtained, we may diſcover the values of the three numbers a, b, and c, that were originally ſought, by tracing back the ſteps of the Analyſis. For, ſince z is $= 2x$, we ſhall have $x = \frac{z}{2}$. Therefore y (which is $= \frac{2x^2 + x}{1-2x}$, or $= \frac{2x+1}{1-2x} \times x$) will be $= \frac{z+1}{1-z} \times \frac{z}{2}$, or $\frac{1+z}{1-z} \times \frac{z}{2}$. Again, x is $= mn$, and y is $= m^2$. Therefore mn is $\frac{z}{2}$, and m^2 is $= \frac{1+z}{1-z} \times \frac{z}{2}$; and conſequently m is $= \sqrt{\frac{1+z}{1-z} \times \frac{z}{2}}$, and n is $(= \frac{mn}{m}$

$$= \frac{\frac{z}{2}}{\sqrt{\frac{1+z}{1-z} \times \frac{z}{2}}} = \frac{\sqrt{\frac{z}{2}}}{\sqrt{\frac{1+z}{1-z}}} = \frac{\sqrt{\frac{z}{2}}}{\frac{\sqrt{1+z}}{\sqrt{1-z}}} = \sqrt{\frac{z}{2}} \times \frac{\sqrt{1-z}}{\sqrt{1+z}}) =$$

$\sqrt{\frac{z}{2} \times \frac{1-z}{1+z}}$, or $\sqrt{\frac{1-z}{1+z} \times \frac{z}{2}}$.

The equations of Number 4 now give us the values of a, b, and c expreſſed by the Letters m and n. For a is $= n + \frac{1}{m}$, and b is $= \frac{1}{2} \times \sqrt{\frac{1}{n} + \frac{1}{m}}$, and c is $= \frac{1}{n} - m$. We muſt therefore now endeavour to expreſs theſe values by means of the Letter z and known numbers.

Article 7. Now we have already ſeen that m is $= \sqrt{\frac{1+z}{1-z} \times \frac{z}{2}}$, and that n is $= \sqrt{\frac{1-z}{1+z} \times \frac{z}{2}}$. Therefore $\frac{1}{m}$ will be $= \frac{1}{\sqrt{\frac{1+z}{1-z} \times \frac{z}{2}}}$ $(=$

$$\frac{1}{\frac{\sqrt{1+z}}{\sqrt{1-z}}}$$

$$\frac{1}{\frac{\sqrt{1+z}}{\sqrt{1-z}} \times \frac{\sqrt{z}}{\sqrt{2}}} = \frac{1}{\frac{\sqrt{1+z} \times \sqrt{z}}{\sqrt{1-z} \times \sqrt{2}}} = 1 \times \frac{\sqrt{1-z} \times \sqrt{2}}{\sqrt{1+z} \times \sqrt{z}} = \frac{\sqrt{1-z} \times \sqrt{2}}{\sqrt{1+z} \times \sqrt{z}}$$

$$= \frac{\sqrt{1-z}}{\sqrt{1+z}} \times \frac{\sqrt{2}}{\sqrt{z}}) = \sqrt{\frac{1-z}{1+z}} \times \sqrt{\frac{2}{z}}$$; and in like manner $\frac{1}{n}$ will be

$$= \frac{1}{\sqrt{\frac{1-z}{1+z}} \times \sqrt{\frac{z}{2}}} (= \frac{1}{\frac{\sqrt{1-z}}{\sqrt{1+z}} \times \frac{\sqrt{z}}{\sqrt{2}}} = \frac{1}{\frac{\sqrt{1-z} \times \sqrt{z}}{\sqrt{1+z} \times \sqrt{2}}} = 1 \times \frac{\sqrt{1+z} \times \sqrt{2}}{\sqrt{1-z} \times \sqrt{z}}$$

$$= \frac{\sqrt{1+z} \times \sqrt{2}}{\sqrt{1-z} \times \sqrt{z}} = \frac{\sqrt{1+z}}{\sqrt{1-z}} \times \frac{\sqrt{2}}{\sqrt{z}}) = \sqrt{\frac{1+z}{1-z}} \times \sqrt{\frac{2}{z}}.$$

Therefore a, or $n + \frac{1}{m}$, will be $= \sqrt{\frac{1-z}{1+z}} \times \sqrt{\frac{z}{2}} + \sqrt{\frac{1-z}{1+z}} \times \sqrt{\frac{2}{z}}$

$$(= \sqrt{\frac{1-z}{1+z}} \times \sqrt{\frac{z}{2} + \sqrt{\frac{2}{z}}} = \sqrt{\frac{1-z}{1+z}} \times \sqrt{\frac{\sqrt{z}}{\sqrt{2}} + \frac{\sqrt{2}}{\sqrt{z}}} = \sqrt{\frac{1-z}{1+z}} \times$$

$$\frac{z+2}{\sqrt{2z}} = \frac{\sqrt{1-z}}{\sqrt{1+z}} \times \frac{z+2}{\sqrt{2z}} = \frac{\sqrt{1-z} \times \sqrt{1-z}}{\sqrt{1+z} \times \sqrt{1-z}} \times \frac{z+2}{\sqrt{2z}} = \frac{1-z}{\sqrt{1+z} \times 1-z}$$

$$+ \frac{z+2}{\sqrt{2z}} = \frac{1-z}{\sqrt{1-zz}} \times \frac{z+2}{\sqrt{2z}} = \frac{\sqrt{1-z} \times \sqrt{z+2}}{\sqrt{1-zz} \times \sqrt{2z}} = \frac{2-z-zz}{\sqrt{1-zz \times 2z}} =$$

$$\frac{2-z-zz}{\sqrt{2z-2z^3}} = \frac{2-z-zz}{\sqrt{2} \times \sqrt{z-z^3}} = \frac{2-z-zz}{\sqrt{2} \times \sqrt{z-z^3}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2-z-zz}{2} \times \frac{\sqrt{2}}{\sqrt{z-z^3}})$$

$$= \frac{2-z-zz}{2} \times \sqrt{\frac{2}{z-z^3}};$$

and b, or $\frac{1}{2} \times \sqrt{\frac{1}{n} + \frac{1}{m}}$, will be $= \frac{1}{2}\sqrt{\sqrt{\frac{1+z}{1-z}} \times \sqrt{\frac{2}{z}} + \sqrt{\frac{1-z}{1+z}} \times \sqrt{\frac{2}{z}}}$

$$(= \frac{1}{2} \times \frac{\sqrt{1+z}}{\sqrt{1-z}} \times \sqrt{\frac{2}{z} + \frac{\sqrt{1-z}}{\sqrt{1+z}}} \times \sqrt{\frac{2}{z}} = \frac{1}{2} \times \sqrt{\frac{2}{z}} \times$$

$$\sqrt{\frac{\sqrt{1+z}}{\sqrt{1-z}} + \frac{\sqrt{1-z}}{\sqrt{1+z}}} = \frac{1}{2} \times \sqrt{\frac{2}{z}} \times \frac{1+z+1-z}{\sqrt{1-z} \times \sqrt{1+z}} = \frac{1}{2} \times \sqrt{\frac{2}{z}} \times$$

$$\frac{2}{\sqrt{1-z \times 1+z}} = \frac{1}{2} \times \sqrt{\frac{2}{z}} \times \frac{2}{\sqrt{1-zz}} = \sqrt{\frac{2}{z}} \times \frac{1}{\sqrt{1-zz}} = \frac{\sqrt{2}}{\sqrt{z}} \times$$

$$\frac{1}{\sqrt{1-zz}} = \frac{\sqrt{2}}{\sqrt{z-z^3}}) = \sqrt{\frac{2}{z-z^3}};$$

and

and c, or $\frac{1}{n} - m$, will be $= \sqrt{\frac{1+z}{1-z}} \times \sqrt{\frac{2}{z}} - \sqrt{\frac{1+z}{1-z}} \times \sqrt{\frac{z}{2}}$

$(= \sqrt{\frac{1+z}{1-z}} \times \sqrt{\frac{2}{z}} - \sqrt{\frac{z}{2}} = \sqrt{\frac{1+z}{1-z}} \times \frac{\sqrt{2}}{\sqrt{z}} - \frac{\sqrt{z}}{\sqrt{2}} = \sqrt{\frac{1+z}{1-z}} \times$

$\frac{2-z}{\sqrt{2z}} = \frac{\sqrt{1+z}}{\sqrt{1-z}} \times \frac{2-z}{\sqrt{2z}} = \frac{\sqrt{1+z} \times \sqrt{1+z}}{\sqrt{1-z} \times \sqrt{1+z}} \times \frac{2-z}{\sqrt{2z}} = \frac{1+z}{\sqrt{1-z \times 1+z}}$

$\times \frac{2-z}{\sqrt{2z}} = \frac{1+z}{\sqrt{1-zz}} \times \frac{2-z}{\sqrt{2z}} = \frac{2+z-zz}{\sqrt{1-zz} \times \sqrt{2z}} = \frac{2+z-zz}{\sqrt{2} \times \sqrt{z} \times \sqrt{1-zz}} =$

$\frac{2+z-zz}{\sqrt{2} \times \sqrt{z-z^3}} = \frac{2+z-zz}{\sqrt{2} \times \sqrt{z-z^3}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{2+z-zz}{2} \times \frac{\sqrt{2}}{\sqrt{z-z^3}}) =$

$\frac{2+z-zz}{2} \times \sqrt{\frac{2}{z-z^3}}.$

Article 8. The result, then, of the whole Analysis is this; That we must, first, resolve the equation $34z + 5z^2 - 34z^3 - z^4 = 8$, in order to obtain the value, or values, of z; and, this being done, we shall directly obtain the values of the three numbers a, b, and c, that are sought, by means of the expressions $a = \frac{2-z-zz}{2} \times \sqrt{\frac{2}{z-z^3}}$,

and $b = \sqrt{\frac{2}{z-z^3}}$,

and $c = \frac{2+z-zz}{2} \times \sqrt{\frac{2}{z-z^3}}$, or by means of these, to wit,

$b = \sqrt{\frac{2}{z-z^3}}$,

$a = \frac{2-z-zz}{2} \times b$,

and $c = \frac{2+z-zz}{2} \times b$.

Article 9. The equation $34z + 5z^2 - 34z^3 - z^4 = 8$, which it is necessary to resolve, (being a biquadratick equation in which the two terms which involve the two highest powers of z are marked with the sign $-$, or are subtracted from the other two terms on the same side of the equation,) may have more than one real and affirmative root: and, in order to solve the Problem compleatly, it is necessary to find all it's roots, or at least so many of them as are fitted to answer the

conditions of the Problem from which the equation was derived. Now a very little attention to the nature of the expreſſions for a, b, and c, will ſhew us that z muſt be a fraction leſs than unity. And therefore, if the equation has any roots (that is, real and poſitive roots,) greater than unity, it will be uſeleſs to inveſtigate ſuch roots, as they cannot be applied to the Problem under conſideration. For the values of a, b, and c all depend upon the radical quantity $\sqrt{\frac{z}{z - z^3}}$, which is poſitive only when z is leſs than 1, and which, in the language of Algebra, becomes impoſſible, or imaginary, when z is greater than 1.

In reality the equation $34z + 5z^2 - 34z^3 - z^4 = 8$ will be found to have two roots that are leſs than 1, one of which will be nearly $= \frac{8}{34}$, or $\frac{4}{17}$, which may be found to a great number of figures by two, or three, proceſſes, of Mr. Raphſon's Method of Approximation, and, when ſubſtituted for z in the foregoing expreſſions of the values of a, b, and c, will produce a ſet of numbers that will anſwer the conditions of the Problem, and indeed will be exactly the ſame with thoſe that are found in Article 19 of Dr. Wallis's Solution, in pages 232, 233, and 234 of the Tracts: and the other root will be ſomething greater than $\frac{9}{10}$, and, when ſubſtituted for z in the ſame expreſſions found for the values of a, b, and c, will give us another ſet of numbers that will likewiſe anſwer the conditions of the Problem, and will be exactly the ſame with thoſe found in Article 16 of Dr. Wallis's Solution, in pages 228, 229, and 230 of the Tracts.

The End of Mr. Ivory's Solution of Colonel Titus's Problem.

Mr. Ivory did not ſend me a reſolution of the equation $34z + 5z^2 - 34z^3 - z^4 = 8$. But this may be performed by Mr. Raphſon's Method of Approximation, in the following manner.

THE RESOLUTION OF THE BIQUADRATICK EQUATION $34z + 5z^2 - 34z^3 - z^4 = 8$.

Article 10. This equation is of that form of biquadratick equations which admits of two roots, (that is, real and positive roots,) and no more. Of these I will, first, endeavour to find the leſſer.

Conjectural Approaches to the Value of the leſſer Root of this Equation.

Now the leſſer root of this equation is equal to the leſſer root of the equation $34z + 5zz = 8 + 34z^3 + z^4$, and therefore is greater than the root of the quadratick equation $34z + 5zz = 8$. We will therefore, in the firſt place, reſolve the quadratick equation $34z + 5zz = 8$. Now, ſince $34z + 5zz$ is $= 8$, we ſhall have $zz + \frac{34}{5} \times z = \frac{8}{5}$, that is, $zz + 6.8 \times z = 1.6$, and $zz + 6.8 \times z + \overline{3.4}^2 = 1.6 + \overline{3.4}^2 = 1.6 + 11.56$, $= 13.16$, and $z + 3.4 = \sqrt{13.16} = 3.6$, and $z\ (= 3.6 - 3.4) = 0.2$. Therefore the leſſer root of the equation $34z + 5zz - 34z^3 - z^4 = 8$ will be greater than 0.2. We will therefore ſuppoſe it to be $= 0.3$, and try the effect of this conjecture.

Now, if z is $= 0.3$, we ſhall have $zz = 0.09$, and $z^3 = 0.027$, and $z^4 = 0.0081$, and $34z\ (= 34 \times 0.3) = 10.2$, and $5zz\ (= 5 \times 0.09) = 0.45$, and $34z^3\ (= 34 \times 0.027) = 0.918$; and conſequently $34z + 5zz - 34z^3 - z^4\ (= 10.2 + 0.45 - 0.918 - 0.0081 = 10.65 - 0.9261) = 9.7239$; which is conſiderably greater than 8, or the abſolute term of the equation $34z + 5zz - 34z^3 - z^4 = 8$. Therefore 0.3 is greater than the true value of the leſſer root of that equation. And conſequently the ſaid leſſer root muſt be greater than 0.2, but leſs than 0.3. We will, therefore, in the next place, ſuppoſe it to be $= 0.25$, and try the effect of this conjecture.

Now, if z is $= 0.25$, we ſhall have $zz\ (= \overline{0.25}^2) = 0.0625$, and $z^3\ (= \overline{0.25}^3) = 0.015,625$, and $z^4\ (= \overline{0.25}^4) = 0.003,906,25$, and $34z$ $(= 34$

$(= 34 \times 0.25) = 8.50$, and $5zz (= 5 \times 0.0625) = 0.3125$, and $34z^3$ $(= 34 \times 0.015,625) = 0.531,250$, and conſequently $34z + 5zz - 34z^3$ $- z^4 (= 8.50 + 0.3125 - 0.531,250 - 0.003,906,25 = 8.8125 -$ $0.535,156,25) = 8.277,343,75$; which is greater than 8, or the abſolute term of the propoſed equation. And conſequently 0.25 is greater than the leſſer root of that equation. We will, therefore, in the third place, ſuppoſe z to be $= 0.24$, and try the effect of that conjecture.

Now, if z is $= 0.24$, we ſhall have $zz (= \overline{0.24}^2) = 0.0576$, and z^3 $(= \overline{0.24}^3) = 0.013,824$, and $z^4 (= \overline{0.24}^4) = 0.003,317,76$, and $34z (=$ $34 \times 0.24) = 8.16$, and $5zz (= 5 \times 0.0576) = 0.2880$, and $34z^3 (=$ $34 \times 0.013,824) = 0.470,016$, and conſequently $34z + 5zz - 34z^3 -$ $z^4 (= 8.16 + 0.2880 - 0.470,016 - 0.003,317,76 = 8.4480 - 0.473,$ $333,76) = 7.974,666,24$; which is a very little leſs than 8, or the abſolute term of the propoſed equation. Therefore 0.24 is a very little leſs than the leſſer root of the ſaid equation, and conſequently will be a very good foundation for a proceſs of Mr. Raphſon's Method of Approximation.

A further Approach to the Value of the ſaid leſſer Root, by a Proceſs of Mr. Raphſon's Method of Approximation.

Article 11. We will therefore put e for the unknown exceſs of z, or the leſſer root of the equation $34z + 5zz + 34z^3 - z^4 = 8$, above 0.24; and we ſhall then have $z = 0.24 + e$.

Therefore zz will be $= \overline{0.24 + e}^2 (= \overline{0.24}^2 + 2 \times 0.24 \times e +$ &c) $= 0.0576 + 0.48 \times e +$ &c, and z^3 will be $= \overline{0.24 + e}^3 (= \overline{0.24}^3 +$ $3 \times \overline{0.24}^2 \times e +$ &c $= \overline{0.24}^3 + 3 \times 0.0576 \times e +$ &c) $= 0.013,824$ $+ 0.1728 \times e +$ &c, and z^4 will be $= \overline{0.24 + e}^4 (= \overline{0.24}^4 + 4 \times$ $\overline{0.24}^3 \times e +$ &c $= \overline{0.24}^4 + 4 \times 0.013,824 \times e +$ &c) $= 0.003,317,76$ $+ 0.055,296 \times e +$ &c, and $34z$ will be $= 34 \times \overline{0.24 + e} (= 34 \times 0.24$ $+ 34e) = 8.16 + 34e$, and $5zz$ will be $= 5 \times \overline{0.0576 + 0.48 \times e +}$ &c $(= 5 \times 0.0576 + 5 \times 0.48e +$ &c $) = 0.2880 + 2.40 \times e +$ &c,

and

and $34z^3$ will be $= 34 \times \overline{0.013,824 + 0.1728 \times e} +$ &c $(= 34 \times 0.013,824 + 34 \times 0.1728 \times e +$ &c$) = 0.470,016 + 5.8752 \times e +$ &c. Therefore the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ will be equal to the following multinomial quantity, to wit,

$$\left\{\begin{array}{rrl} & 8.160,000 + & 34 \times e \\ + & 0.288,000 + & 2.40 \times e + \&c \\ - & 0.470,016 - & 5.8752 \times e - \&c \\ - & 0.003,317,76 - & 0.055,296 \times e - \&c \end{array}\right\} =$$

$$\left\{\begin{array}{rrl} & 8.448,000,00 + & 36.400,000 \times e + \&c \\ - & 0.473,333,76 - & 5.930,496 \times e - \&c \end{array}\right\} =$$

$$7.974,666,24 + 30.469,504 \times e \quad \&c.$$

But the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ is $= 8$.

Therefore the quantity $7.974,666,24 + 30.469,504 \times e$ &c will alſo be $= 8$; and conſequently $30.469,504 \times e$ will be $(= 8 - 7.974,666,24) = 0.025,333,76$, and e will be $= \frac{0.025,333,76}{30.469,504} = 0.000,831$. Therefore z, or $0.24 + e$, will be $(= 0.24 + 0.000,831) = 0.240,831$, that is, the leſſer root of the propoſed equation $34z + 5zz - 34z^3 - z^4 = 8$ will be $= 0.240,831$.

Q. E. I.

Now let $0.240,831$ be ſubſtituted inſtead of z in the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$, and let the value of the ſaid quantity reſulting from ſuch ſubſtitution be compared with the abſolute term, 8, of the propoſed equation. This may be done in the manner following.

If z is $= 0.240,831$, zz will be $(= \overline{0.240,831}^2) = 0.057,999,770,561$, and z^3 will be $(= \overline{0.240,831}^3) = 0.013,968,141,843,976,191$, and z^4 will be $= 0.003,363,961,568,426,630,054,721$, and $34z$ will be $(= 34 \times 0.240,831,) = 8.188,254$, and $5zz$ will be $(= 5 \times 0.057,999,770,561) = 0.289,998,852,805$, and $34z^3$ will be $(= 34 \times 0.013,968,141,843,976,191) = 0.474,916,822,695,190,494$. And conſequently the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ will be $=$

8.188,

$$\left\{\begin{array}{r} 8.188,254,000,000 - 0.474,916,822,695,190,494,000,000 \\ +\ 0.289,998,852,805 - 0.003,363,961,568,426,630,054,721 \end{array}\right\}$$

$= 8.478,252,852,805 - 0.478,280,784,263,617,124,054,721 = 7.999,972,068,541,382,875,945,279$; which is leſs than 8, or the abſolute term of the propoſed equation, by an exceeding ſmall quantity. Therefore 0.240,831 will be leſs than the true value of the leſſer root of the ſaid equation, but will be very nearly equal to it, inſomuch that it is probably exact in all it's figures.

But, in order to find the value of this root to a ſtill greater degree of exactneſs, we will now employ a ſecond proceſs of Mr. Raphſon's Method of Approximation; which may be done as follows.

A further Approach to the Value of the ſaid leſſer Root, by a ſecond Proceſs of Mr. Raphſon's Method of Approximation.

Article 12. Let e be put for the exceſs of z, or the leſſer root of the propoſed equation $34z + 5zz - 34z^3 - z^4 = 8$, above the laſt near value of it, to wit, 0.240,831; and we ſhall have $z = 0.240,831 + e$.

Therefore zz will be $= \overline{0.240,831 + e}^2$ $(= \overline{0.240,831}^2 + 2 \times 0.240,831 \times e + e^2) = 0.057,999,770,561 + 0.481,662 \times e + e^2$, and z^3 will be $= \overline{0.240,831 + e}^3$ $(= \overline{0.240,831}^3 + 3 \times \overline{0.240,831}^2 \times e + 3 \times 0.240,831 \times e^2 + \&c = \overline{0.240,831}^3 + 3 \times 0.057,999,770,561 \times e + 0.722,493 \times e^2 + \&c) = 0.013,968,141,843,976,191 + 0.173,999,311,683 \times e + 0.722,493 \times e^2 + \&c$, and z^4 will be $= \overline{0.240,831 + e}^4$ $(= \overline{0.240,831}^4 + 4 \times \overline{0.240,831}^3 \times e + 6 \times \overline{0.240.831}^2 \times e^2 + \&c = \overline{0.240,831}^4 + 4 \times 0.013,968,141,843,976,191 \times e + 6 \times 0.057,999,770,561 \times e^2 + \&c) = 0.003,363,961,568,426,630,054,721 + 0.055,872,567,375,904,764 \times e + 0.347,998,623,366 \times e^2 + \&c$, and conſequently $34z$ will be $= 34 \times \overline{0.240,831 + e}$ $(= 34 \times 0.240,831 + 34e) = 8.188,254 + 34e$, and $5zz$ will be $= 5 \times 0.057,999,770,561 + 0.481,662 \times e + e^2$ $(= 5 \times 0.057,999,770,561 + 5 \times 0.481,662 \times e + 5 \times e^2) = 0.289,998,852,805 + 2.408,310 \times e + 5e^2$, and $34z^3$ will be $= 34 \times$

0.013,

$\overline{0.013,968,141,843,976,191 + 0.173,999,311,683 \times e + 0.722,493 \times e^2}$
$+ \&c\ (= 34 \times 0.013,968,141,843,976,191 + 34 \times 0.173,999,311,683$
$\times e + 34 \times 0.722,493 \times e^2 + \&c) = 0.474,916,822,695,190,494 +$
$5.915,976,597,222 \times e + 24.564,762 \times e^2 + \&c.$

Therefore the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ will be $=$ the multinomial quantity.

$$\left\{\begin{array}{llll} 8.188,254 & + 34e & & \\ + 0.289,998,852,805 & + 2.408,310 \times e & + 5e^2 & \\ - 0.474,916,822,695,190,494 & - 5.915,976,597,222 \times e & & \\ & - 24.564,762 \times e^2 - \&c & & \\ - 0.003,363,961,568,426,630,054,721 & & & \\ & - 0.055,872,567,375,904,764 \times e & & \\ & - 0.347,998,623,366 \times e^2 - \&c & & \end{array}\right\}$$

$$= \left\{\begin{array}{ll} 8.478,252,852,805 & \\ - 0.478,280,784,263,617,124,054,721 & \\ + 36.408,310 & \times e \\ - 5.971,849,164,597,904,764 & \times e \\ + 5.000,000,000,000 & \times e^2 \\ - 24,912,760,623,366 & \times e^2 - \&c \end{array}\right\}$$

$$= 7.999,972,068,541,382,875,945,279$$
$$+ 30.436,460,835,402,095,236 \times e$$
$$- 19.912,760,623,366 \times e^2 - \&c.$$

But the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ is $= 8$.

Therefore the quantity $7.999,972,068,541,382,875,945,279 + 30.436,460,835,402,095,236 \times e - 19.912,760,623,366 \times e^2 - \&c$ will alſo be $= 8$; and conſequently (ſubtracting $7.999,972,068,541,382,875,945,279$ from both ſides,) we ſhall have $30.436,460,835,402,095,236 \times e - 19.912,760,623,366 \times e^2 - \&c = 0.000,027,931,458,617,124,054,721$, and (adding $19.912,760,623,366 \times e^2 + \&c$ to both ſides) we ſhall have $30.436,460,835,402,095,236 \times e = 0.000,027,931,458,617,124,054,721 + 19.912,760,623,366 \times e^2 + \&c$, and conſequently $e = \frac{0.000,027,931,458,617,124,054,721}{30.436,460,835,402,095,236}$
$+ \frac{19.912,760,623,366, \times e^2}{30.436,460,835,402,095,236} + \&c = 0.000,000,917,697,3 + \&c.$ Therefore

fore z, or 0.240,831 + e, will be (= 0.240,831, + 0.000,000,917,697,3 + &c) = 0.240,831,917,697,3 + &c, that is, the leſſer root of the propoſed equation $34z + 5zz - 34z^3 - z^4 = 8$ will be = 0.240,831,917,697,3 + &c, or ſomewhat greater than 0.240,831,917,697,3. Q. E. I.

Of this number we may be almoſt certain that the firſt eleven figures 0.240,831,917,69, are exact. But this may be fully aſcertained in the following manner.

A Determination of the Number of Figures in the laſt near Value of the leſſer Root of the ſaid Equation, to wit, 0.240,831,917,697,3, *that are exact.*

Article 13. The true value of e is $0.000,000,917,697,3 + \frac{19.912,760,623,366 \times e^2}{30.436,460,835,402,095,23}$ + &c. Now this term $\frac{19.912, \&c \times e^2}{30.436, \&c}$, (which ought to be added to 0.000,000,917,697,3 in order to obtain a more accurate value of e,) is leſs than $\frac{19.92 \times e^2}{30.4}$, that is, than $0.65 \times e^2$, and (becauſe e, or 0.000,000,917,697,3 &c, is leſs than 0.000,000,92) is, *à fortiori*, leſs than $0.65 \times \overline{0.000,000,92}^2$, or $0.65 \times 0.000,000,000,000,846,4$, or 0.000,000,000,000,55,160; of which the firſt, or higheſt, figure 5 enters only in the thirteenth place of decimal fractions. Therefore the firſt, or higheſt, figure of the accurate value of the term $\frac{19.912, \&c \times e^2}{30.436, \&c}$ cannot enter before the thirteenth place of decimal fractions, and cannot be greater than 5 in the ſaid 13th place; and therefore it's addition to the number 0.000,000,917,697,3, or the value of e already obtained, cannot make any change in the figures 0.000,000,917,697, but can only, at the moſt, increaſe the laſt figure, 3, of the number 0.000,000,917,697,3 from 3 to 3 + 5, or 8. And conſequently the value of z, or 0.240,831 + e, obtained by the laſt proceſs of Mr. Raphſon's Method of Approximation, to wit, the number 0.240,831,917,697,3, will not only be exact in the firſt eleven figures 0.240,831,917,69, but in the firſt twelve figures 0.240,831,917,697; which is a very great degree of exactneſs.

The

The ſame concluſion concerning the exactneſs of the firſt 12 figures 0.240,831,917,697 of the number 0.240,831,917,697,3, obtained by the laſt proceſs of Mr. Raphſon's Method of Approximation for the value of 0.240,831 + e, or z, may alſo be proved in the following manner.

Another Determination of the ſame Point, by the Method propoſed by Mr. Ivory.

In the latter part of the laſt proceſs of Approximation we came to this equation, 30.436, &c $\times e$ — 19.912, &c $\times e^2$ — &c = 0.000.027,931,458, &c. Therefore e will be $= \frac{0.000,027,931,458, \&c}{30.436, \&c - 19.912, \&c \times e}$. But $\frac{0.000,027,931,458, \&c}{30.436, \&c - 19.912, \&c \times e}$ is greater than $\frac{0.000,027,931, \&c}{30.436, \&c - 0}$, or $\frac{0.000,027,931, \&c}{30.436, \&c}$. Therefore e muſt be greater than $\frac{0.000,027,931, \&c}{30.436, \&c}$, that is, than 0.000,000,917,697,3.

Further, the fraction $\frac{0.000,027,931, \&c}{30.436, \&c - 19.912, \&c \times e}$, (to which e is equal,) will be leſs than a fraction of which the numerator ſhall be the ſame with the numerator of the former fraction, to wit, 0.000,027,931, &c, but the denominator ſhall be leſs than the denominator of the former fraction, to wit, 30.436, &c — 19.912, &c $\times e$. Now, ſince the higheſt figure of the value of e is 9 in the ſeventh place of decimal fractions, it is evident that e muſt be leſs than 1 in the ſixth place of decimal fractions, or than 0.000,001. Therefore, if we ſubſtitute 0.000,001 inſtead of e in the quantity 30.436, &c — 19.912, &c $\times e$, the new quantity thence ariſing, to wit, 30.436, &c — 19.912, &c $\times$ 0.000,001, will be leſs than 30.436, &c — 19.912, &c $\times e$, and conſequently the new fraction $\frac{0.000,027,931, \&c}{30.436, \&c - 19.912, \&c \times 0.000,001}$ thereby produced, will be greater than the former fraction $\frac{0.000,027,931, \&c}{30.436, \&c - 19.912, \&c \times e}$, or than the true value of e. But this new fraction $\frac{0.000,027,931, \&c}{30.436, \&c - 19.912, \&c \times 0.000,001}$ will be $=$

$$\frac{0.000,027,931, \&c}{30.436, \&c - 19.912,760,623,366 \times 0.000,001} = \frac{0.000,027,931, \&c}{30.436, \&c - 0.000,019,912,760,623,366}$$

$= \frac{0.000,027,931,458,617,124,054,721}{30.436,460,835,402,095,236 - 0.000,019,912,760,623,366} =$

$\frac{0.000,027,931,458,617,124,054,721}{30.436,440,922,641,471,870} = 0.000,000,917,697,9$. Therefore 0.000, 000,917,697,9 will be greater than the true value of *e*. Therefore the true value of *e* will be greater than 0.000,000,917,697,3, but leſs than 0.000, 000,917,697,9, and conſequently the ſix firſt ſignificant figures of it muſt be 917,697 in the ſeventh and eighth, and the four following places of decimal fractions. Therefore the number 0.240,831,917,697,3 found above for the value of $0.240,831 + e$, or z, will be true in the firſt 12 figures 0.240, 831, 917,697. Q. E. D.

This laſt Method of demonſtrating that theſe twelve firſt figures of the number 0.240,831,917,697,3, here found for the value of z will be exact, is in ſubſtance the ſame with the Method invented for this purpoſe by Mr. Ivory, which has been deſcribed in his own words, and illuſtrated by an example, in the tract which immediately preceeds the preſent tract in this volume.

The Computation of the Values of the three Quantities a, b, and c, that are deriveable from the foregoing Value of z, or the leſſer Root of the Equation $34z + 5zz - 34z^3 - z^4 = 8$.

Article 14. Having thus found the value of z, or the leſſer root of the biquadratick equation $34z + 5zz - 34z^3 - z^4 = 8$, to be $= 0.240,831,917,697,3$, it remains that we derive from it the values of the numbers *a*, *b*, and *c*, that are required by the Problem to be found, by computing the expreſſions to which Mr. Ivory's Solution above-deſcribed has ſhewn them to be equal. Theſe expreſſions are either $a = \frac{2-z-zz}{2} \times \sqrt{\frac{2}{z-z^3}}$, $b = \sqrt{\frac{2}{z-z^3}}$, and $c = \frac{2+z-zz}{2} \times \sqrt{\frac{2}{z-z^3}}$, or $b = \sqrt{\frac{2}{z-z^3}}$, and $a = \frac{2-z-zz}{2} \times b$, and $c = \frac{2+z-zz}{2} \times b$; which latter expreſſions are rather more conciſe and convenient than the former.

We

We will therefore, in the firſt place, compute the value of b, or $\sqrt{\frac{z}{z - z^3}}$.

Now, ſince z is $=$ 0.240,831,917,697,3, we ſhall have z^2 ($=$ $\overline{0.240,831,917,697,3}^2$) $=$ 0.058,000,012,581,759,080,934,427,29, and z^3 ($=$ $\overline{0.240,831,917,697,3}^3$) $=$ 0.013,968,254,256,532,567,466,856,882,683,391,079,317. Therefore $z - z^3$ will be ($=$ 0.240,831,917,697,3 $-$ 0.013,968,254,256,532,567,466,856,882,683,391,079,317) $=$ 0.226,863,663,440,767,432,533,143,117,316,608,920,683, and $\frac{2}{z - z^3}$ will be ($=$ $\frac{2.000,000,000,000,000,000,000,000,000,000,000,000,000}{0.226,863,663,440,767,432,533,143,117,316,608,920,683}$) $=$ 8.815,867,511,203,203, and conſequently $\sqrt{\frac{2}{z - z^3}}$ will be ($=$ $\sqrt{}$ 8.815,867,511,203,203) $=$ 2.969,152,658,790,585, that is, b, or the ſecond of the three numbers a, b, and c, which the Problem requires us to find, will be $=$ 2.969,152,658,790,585.

And, ſince zz is $=$ 0.058,000,012,581,759,080,934,427,29, we ſhall have $z + zz$ ($=$ 0.240,831,917,697,3 $+$ 0.058,000,012,581,759,080,934,427,29) $=$ 0.298,831,930,279,059,080,934,427,29, and $2 - z - zz$ ($=$ $2 -$ 0.298,831,930,279,059,080,934,427,29) $=$ 1.701,168,069,720,940,919,065,572,71, and $\frac{2 - z - zz}{2}$ will be ($=$ $\frac{1.701,168,069,720,940,919,065,572,71}{2}$) $=$ 0.850,584,034,860,470,459,532,786,355; and conſequently $\frac{2 - z - zz}{2} \times \sqrt{\frac{2}{z - z^3}}$, or $\frac{2 - z - zz}{2} \times b$, will be ($=$ 0.850,584,034,860,470,459,532,786,355 $\times$ 2.969,152,658,790,585 $=$, nearly, 0.850,584,034,860,470 $\times$ 2.969,152,658,790,585) $=$ 2.525,513,848,630,788; that is, a, or the firſt number which the Problem requires us to find, will be $=$ 2.525,513,848,630,788.

And, laſtly, $2 + z - zz$ will be ($=$ 2 $+$ 0.240,831,917,697,3 $-$ 0.058,000,012,581,759,080,934,427,29 $=$ 2.182,831,905,115,540,919,065,572,71, and $\frac{2 + z - zz}{2}$ will be ($=$ $\frac{2.182,831,905,115,540,919,065,572,71}{2}$) 1.091,415,952,557,

952,557,770,459,532,786,355; and conſequently $\frac{2+z-zz}{2} \times \sqrt{\frac{2}{z-z^3}}$, or $\frac{2+z-zz}{2} \times b$, will be (= 1.091,415,952,557,770,459,532,786,355 × 2.969,152,658,790,585 =, nearly, 1.091,415,952,557,770 × 2.969,152,658,790,585) = 3 240,580,577,383,361; that is, c, or the third number which the Problem requires us to find, will be = 3.240,580,577,383,361.

Therefore the three numbers required by the Problem

are a = 2.525,513,848,630,788,
and b = 2.969,152,658,790,585,
and c = 3.240,580,577,383,361.

Theſe values of a, b, and c agree with thoſe found for them by Dr. Wallis in their firſt ſeven figures, thoſe values being as follows;

a = 2.525,513,986,744,158,
and b = 2.969,152,768,619,848,
and c = 3.240,580,681,617,174,

which anſwer the conditions of the Problem to a very great degree of exactneſs, making $aa + bc$ to be = 16.000,000,000,000,000, and $bb + ac$ to be = 17.000,000,000,000,002, and $cc + ab$ to be = 17.999,999,999,999,997. See Art. 18 of the account given of Dr. Wallis's Solution of Colonel Titus's Problem in the "Tracts on the Reſolution of Algebräick Equations by Approximation," in pages 231, 232. I expected to find the values of a, b, and c here obtained by means of Mr. Ivory's Solution of the Problem, and the ſubſequent reſolution of the equation $34z + 5zz - 34z^3 - z^4 = 8$, to agree with the aforeſaid very exact values of them found by Dr. Wallis to about twelve places of figures, inſtead of ſeven places; and, as they do not agree ſo far, I ſuppoſe I muſt have made ſome miſtake in ſome part of the computation of the expreſſions $\frac{2-z-zz}{2} \times \sqrt{\frac{2}{z-z^3}}$, and $\sqrt{\frac{2}{z-z^3}}$, and $\frac{2+z-zz}{2} \times \sqrt{\frac{2}{z-z^3}}$, which are equal to a, b, and c, and that the ſaid miſtake muſt have been about the eighth place of decimal fractions, ſo as to render all the following figures erroneous. But upon carefully looking over the

ſeveral

ſeveral operations of the calculation, I have not been able to diſcover any ſuch miſtake.

The Inveſtigation of the greater Root of the Equation

$$34z + 5zz - 34z^3 - z^4 = 8.$$

Article 15. I will now proceed to inveſtigate the greater root of the equation $34z + 5zz - 34z^3 - z^4 = 8$, and in this inquiry I will make uſe of the knowledge we have already acquired, that it's leſſer root is $= 0.240,831,917,697,3$, or (neglecting all the figures of this number, except the two firſt,) $= 0.24$, and this number I ſhall denote by the Letter g, in order to diſtinguiſh it from the greater root of this equation, which I ſhall now call z. Then, ſince g, or 0.24, is a root of this equation $34z + 5zz - 34z^3 - z^4 = 8$, we ſhall have $34g + 5gg - 34g^3 - g^4 = 8$; and, ſince z is alſo a root of it, we ſhall have $34z + 5zz - 34z^3 - z^4 = 8$. And therefore $34z + 5zz - 34z^3 - z^4$ will be $= 34g + 5gg - 34g^3 - g^4$. Therefore (adding $34z^3 + z^4$ to both ſides,) we ſhall have $34z + 5zz = 34g + 5gg + 34z^3 + z^4 - 34g^3 - g^4$, and (ſubtracting $34g + 5gg$ from both ſides,) $34z + 5zz - 34g - 5gg = 34z^3 + z^4 - 34g^3 - g^4$, or $34z - 34g + 5zz - 5gg = 34z^3 - 34g^3 + z^4 - g^4$, or $34 \times z - g + 5 \times zz - gg = 34 \times z^3 - g^3 + z^4 - g^4$. Therefore (dividing both ſides of the equation by the binomial quantity $z - g$, or the exceſs of the greater root z above the leſſer root g,) we ſhall have $34 + 5 \times \frac{zz - gg}{z - g} = 34 \times \frac{z^3 - g^3}{z - g} + \frac{z^4 - g^4}{z - g}$; that is, $34 + 5 \times z + g$ will be $= 34 \times zz + zg + gg + z^3 + zzg + zgg + g^3$, or $34 + 5z + 5g$ will be $= 34zz + 34zg + 34gg + z^3 + zzg + zgg + g^3$, and conſequently (ſubtracting $5z$ from both ſides,) $34 + 5g$ will be $= 34zz + 34zg + 34gg + z^3 + zzg + zgg - 5z + g^3$, and (ſubtracting $34gg + g^3$ from both ſides,) $34 + 5g - 34gg - g^3$ will be $= 34zz + 34gz + z^3 + gzz - 5z + ggz$, or (ranging the terms according to the powers of z,) we ſhall have $z^3 + 34zz + gzz + 34gz + ggz - 5z = 34 + 5g - 34gg - g^3$, or $z^3 + 34 + g \times zz + 34g + gg \times z - 5 \times z = 34 + 5g - 34gg - g^3$,

or

or (ſubſtituting for g it's numeral value 0.24,) we ſhall have $z^3 + \overline{34 + 0.24} \times zz + \overline{34 \times 0.24 + \overline{0.24}^2} \times z - 5z = 34 + 5 \times 0.24 - 34 \times \overline{0.24}^2 - \overline{0.24}^3$, or $z^3 + 34.24 \times zz + \overline{34 \times 0.24 + 0.0576} \times z - 5z = 34 + 1.20 - 34 \times 0.0576 - 0.013,824$, or $z^3 + 34.24 \times zz + \overline{8.16 + 0.0576} \times z - 5z = 35.20 - 1.9584 - 0.013,824 = 35.20 - 1.972,224 = 33.227,776$, or $z^3 + 34.24 \times zz + 8.2176 \times z - 5z = 33.227,776$, or $z^3 + 34.24 \times zz + 3.2176 \times z = 33.227,776$; which is a cubick equation that has only one root. We muſt therefore now endeavour to reſolve this cubick equation $z^3 + 34.24 \times zz + 3.2176 \times z = 33.227,776$; which may be done as follows.

In the firſt place, it is eaſy to diſcover that the root of this equation muſt be ſomewhat leſs than 1. For, if z were $= 1$, we ſhould have $zz = 1$, and z^3 alſo $= 1$, and conſequently $z^3 + 34.24 \times zz + 3.2176 \times z (= 1 + 34.24 + 3.2176) = 38.4576$; which is greater than 33.227,776, or the abſolute term of the equation $z^3 + 34.24 \times zz + 3.2176 \times z = 33.227,776$. Therefore 1 muſt be greater than the true value of z in that equation.

We will, therefore, ſuppoſe that z, or the root of the cubick equation $z^3 + 34.24 \times zz + 3.2176 \times z = 33.227,776$, is $= \frac{9}{10}$, or 0.9. And we ſhall then have $zz = 0.81$, and $z^3 = 0.729$, and $34.24 \times zz (= 34.24 \times 0.81) = 27.7344$, and $3.2176 \times z (= 3.2176 \times 0.9) = 2.895,84$, and conſequently $z^3 + 34.24 \times zz + 3.2176 \times z (= 0.729 + 27.7344 + 2.895,84) = 31.359,24$; which is leſs than 33.227,776, or the abſolute term of the equation $z^3 + 34.24 \times zz + 3.2176 \times z = 33.227,776$. Therefore 0.9 muſt be leſs than the true value of z in that equation.

We will, therefore, in the next place, ſuppoſe z to be $= 0.94$. And we ſhall then have $zz (= \overline{0.94}^2) = 0.8836$, and $z^3 (= \overline{0.94}^3) = 0.830,584$, and $34.24 \times zz (= 34.24 \times 0.8836) = 30.254,464$, and $3.2176 \times z (= 3.2176 \times 0.94) = 3.024,544$, and conſequently $z^3 + 34.24 \times zz + 3.2176 \times z (= 0.830,584 + 30.254,464 + 3.024,544) = 34.109,592$; which is greater than 33.227,776, or the abſolute term of the equation $z^3 +$

$34.24 \times zz + 3.2176 \times z = 33.227,776$. Therefore 0.94 muſt be greater than the true value of z in that equation.

We will therefore, in the next place, ſuppoſe z to be $= 0.93$, and try the effect of this conjecture.

Now, if z is $= 0.93$, we ſhall have zz $(= \overline{0.93}^2) = 0.8649$, and z^3 $(= \overline{0.93}^3) = 0.804,357$, and $34.24 \times zz$ $(= 34.24 \times 0.8649) = 29.614,176$, and $3.2176 \times z$ $(= 3.2176 \times 0.93) = 2.992,368$, and conſequently $z^3 + 34.24 \times zz + 3.2176 \times z$ $(= 0.804,357 + 29.614,176 + 2.992,368) = 33.410,901$; which is greater than $33.227,776$, or the abſolute term of the equation $z^3 + 43.24 \times zz + 3.2176 \times z = 33.227,776$. Therefore 0.93 will be greater than the true value of z in that equation. But, as the difference between the numbers 33.410,901 and 33.227,776 is but ſmall, the difference between 0.93 and the true value of z in this cubick equation will be but ſmall likewiſe, and 0.93 may be conſidered as a tolerably good firſt near value of z in this cubick equation, and therefore of the value of z, or the greater root of the biquadratick equation $34z + 5zz - 34z^3 - z^4 = 8$, from which this cubick equation was derived.

We will therefore now ſubſtitute 0.93 inſtead of z in the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$, in order to diſcover whether the value of the ſaid quantity reſulting from ſuch ſubſtitution will be greater than the abſolute term 8 of the ſaid biquadratick equation, or leſs than the ſaid abſolute term.

Now, if z is $= 0.93$, we ſhall have zz $(= \overline{0,93}^2) = 0.8649$, and z^3 $(= \overline{0.93}^3) = 0.804,357$, and z^4 $(= \overline{0.93}^4) = 0.748,052,01$, and $34z$ $(= 34 \times 0.93) = 31.62$, and $5zz$ $(= 5 \times 0.8649) = 4.3245$, and $34z^3$ $(= 34 \times 0.804,357) = 27.348,138$, and conſequently $34z + 5zz - 34z^3 - z^4$ $(= 31.62 + 4.3245 - 27.348,138 - 0.748,052,01 = 35.9445 - 28.096,190,01) = 7.848,309,99$; which is leſs than 8, or the abſolute term of the biquadratick equation $34z + 5zz - 34z^3 - z^4 = 8$. Therefore 0.93 muſt be greater than the true value of z, or the greater root of the ſaid equation. But it will be near enough to the ſaid true value to be made the

baſis,

baſis, or foundation, of a further approach to it by Mr. Raphſon's Method of Approximation.

A further Approach to the Value of z, or the greater Root of the Equation $34z + 5zz - 34z^3 - z^4 = 8$, *by a Proceſs of Mr. Raphſon's Method of Approximation.*

Article 16. We will therefore put e for the unknown exceſs of 0.93 above the true value of z, or the greater root of the equation $34z + 5zz - 34z^3 - z^4 = 8$. And we ſhall then have $z = 0.93 - e$, and conſequently $zz = \overline{0.93 - e}^2$ $(= \overline{0.93}^2 - 2 \times 0.93 \times e + \&c) = 0.8649 - 1.86 \times e + \&c$, and $z^3 = \overline{0.93 - e}^3$ $(= \overline{0.93}^3 - 3 \times \overline{0.93}^2 \times e + \&c = \overline{0.93}^3 - 3 \times 0.8649 \times e + \&c) = 0.804,357 - 2.5947 \times e + \&c$, and $z^4 = \overline{0.93 - e}^4$ $(= \overline{0.93}^4 - 4 \times \overline{0.93}^3 \times e + \&c = \overline{0.93}^4 - 4 \times 0.804,357 \times e + \&c) = 0.748,052,01 - 3.217,428 \times e + \&c$. Therefore $34z$ will be $= 34 \times \overline{0.93 - e}$ $(= 34 \times 0.93 - 34 \times e) = 31.62 - 34e$, and $5zz$ will be $= 5 \times \overline{0.8649 - 1.86 \times e + \&c}$ $(= 5 \times 0.8649 - 5 \times 1.86 \times e + \&c) = 4.3245 - 9.30 \times e + \&c$; and $34z^3$ will be $= 34 \times \overline{0.804,357 - 2.5947 \times e + \&c}$ $(= 34 \times 0.804,357 - 34 \times 2.5947 \times e + \&c) = 27.348,138 - 88.2198 \times e + \&c$. Therefore the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ will be $=$ the multinomial quantity

$$\left\{\begin{array}{llll} 31.62 & - 34e & & \\ + \ 4.3245 & - \ 9.30 & \times e + \&c & \\ - \ 27.348,138 & + \ 88.2198 & \times e - \&c & \\ - \ 0.748,052,01 & & & \\ & + \ 3.217,428 & \times e - \&c & \end{array}\right\} =$$

$$\left\{\begin{array}{lll} 35.944,500,00 & - \ 43.300,000 & \times e + \&c \\ - \ 28.096,190,01 & + \ 91.437,228 & \times e - \&c \end{array}\right\}$$

$$= 7.848,309,99 + 48.137,228 \times e \ \&c.$$

But

But the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ is $= 8$.

Therefore the quantity 7.848,309,99 + 48.137,228 × e &c will alſo be $= 8$; and conſequently the quantity 48.137,228 × e will be ($= 8 -$ 7.848,309,99) $=$ 0.151,690,01, and e will be $= \frac{0.151,690,01}{48.137,228} =$ 0.0031. Therefore z, or 0.93 — e, will be ($=$ 0.93 — 0,0031 $=$ 0.9269, that is, the greater root of the equation $34z + 5zz - 34z^3 - z^4 = 8$) will be nearly $=$ 0.9269. Q. E. I.

Now let this number 0.9299 be ſubſtituted inſtead of z in the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$, in order to diſcover whether the value of the ſaid quantity reſulting from the ſaid ſubſtitution will be greater, or will be leſs, than 8, or the abſolute term of the propoſed equation, and conſequently whether the ſaid number 0.9269 will be leſs, or will be greater, than the true value of z in the ſaid equation. This ſubſtitution may be made as follows.

If z is $=$ 0.9269, we ſhall have zz ($= \overline{0.9269}^2$) $=$ 0.859,143,61, and z^3 ($= \overline{0.9269}^3$) $=$ 0.796,340,212,109, and z^4 ($= \overline{0.9269}^4$) $=$ 0.738,127,742, 603,832,1, and conſequently $34z$ ($= 34 \times$ 0.9269) $=$ 31.5146, and $5zz$ ($= 5 \times$ 0.859,143,61) $=$ 4.295,710,05, and $34z^3$ ($= 34 \times$ 0.796,340,212, 109) $=$ 27.075,567,211,706. Therefore the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ will be ($=$ 31.5146 + 4.295,710,05 — 27.075,567,211, 706 — 0.738,127,742,603,832,1 $=$ 35.810,318,05 — 27.813,694,954,309, 832,1) $=$ 7.996,623,095,690,167,9; which is ſomewhat leſs than 8, or the abſolute term of the propoſed equation. Therefore 0.9269 will be ſomewhat greater than the true value of z, or the greater root of that equation.

A further Approach to the Value of z, or the greater Root of the Equation $34z + 5zz - 34z^3 - z^4 = 8$, by a ſecond Proceſs of Mr. Raphſon's Method of Approximation.

Article 17. We will therefore put e for the exceſs of 0.9269 above z; and we ſhall then have $z =$ 0.9269 — e. Therefore zz will be $= \overline{0.9269 - e}^2$ ($= \overline{0.9269}^2$ — 2 × 0.9269 × e + &c) $=$ 0.859,143,61 — 1.8538 × e + &c, and z^3

will be $= \overline{0.9269 - e}^3$ $(= \overline{0.9269}^3 - 3 \times \overline{0.9269}^2 \times e +$ &c $= \overline{0.9269}^3 - 3 \times 0.859,143,61 \times e +$ &c$)$ $= 0.796,340,212,109 - 2.577,430,83 \times e +$ &c, and z^4 will be $= \overline{0.9269 - e}^4$ $(= \overline{0.9269}^4 - 4 \times \overline{0.9269}^3 \times e +$ &c $= \overline{0.9269}^4 - 4 \times 0.796,340,212,109 \times e +$ &c$) = 0.738,127,742,603,832,1 - 3.185,360,848,436 \times e +$ &c; and $34z$ will be $= 34 \times \overline{0.9269 - e}$ $(= 34 \times 0.9269 - 34e) = 31.5146 - 34 \times e$; and $5zz$ will be $(= 5 \times 0.859,143,61 - 5 \times 1,8538 \times e +$ &c$) = 4.295,718,05 - 9.2690 \times e +$ &c; and $34z^3$ will be $(= 34 \times 796,340,212,109 - 34 \times 2.577,430,83 \times e +$ &c$) = 27.075,567,211,706 - 87.632,648,22 \times e +$ &c; and the quadrinomial quantity $34z + 5zz - 34z^3 + z^4$ will be $=$ the multinomial quantity

$$\left\{\begin{array}{rrl} 31.514,600,00 & - & 34e \\ + \ 4.295,718,05 & - & 9.2690 \times e + \&c \\ - 27.075,567,211,706 & & \\ & + & 87.632,648,22 \times e - \&c \\ - \ 0.738,127,742,603,832,1 & & \\ & + & 3.185,360,848,436 \times e - \&c \end{array}\right\} =$$

$$\left\{\begin{array}{rrrl} + 35.810,318,050,000,000,0 & - & 43.2690 & \times e + \&c \\ - 27.813,694,954,309,832,1 & + & 90.818,009,068,436 & \times e - \&c \end{array}\right\}$$

$$= 7.996,623,095,690,167,9 + 47.549,009,068,436 \times e \ \&c.$$

But the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ is $= 8$.

Therefore the quantity $7.996,623,095,690,167,9 + 47.549,009,068,436 \times e$ &c will also be $= 8$. And consequently $47.549,009,068,436 \times e$ will be $(= 8 - 7.996,623,095,690,167,9) = 0.003,376,904,309,832,1$, and e will be $= \frac{0.003,376,904,309,832,1}{47.549,009,068,436} = 0.000,071,019$. Therefore z, or $0.9269 - e$, will be $= 0.9269 - 0.000,071,019 = 0.926,828,981$; that is, the greater root of the equation $34z + 5zz - 34z^3 - z^4 = 8$ will be, nearly, equal to $0.926,828,981$. Q. E. I.

Of this number $0.926,828,981$ I believe the firſt ſeven figures $0.926,828,9$ to be exact. I will, therefore, make uſe of them as the foundation of another proceſs of Mr. Raphſon's Method of Approximation, by which we ſhall obtain another

another near value of z that will be exact to almoſt twice as many figures as are exact in the preſent value of it. And with this view I ſhall begin this new computation by ſubſtituting the number 0.926,828,9 inſtead of z in the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$, in order to diſcover whether the value of the ſaid quantity reſulting from ſuch ſubſtitution will be greater than 8, or the abſolute term of the equation $34z + 5zz - 34z^3 - z^4 = 8$, or will be leſs than the ſaid abſolute term, and conſequently whether the ſaid number, 0.926,828,9, will be leſs, or will be greater, than the true value of z, or the greater root of the ſaid equation.

Now, if $\dot{z}$ is $=$ 0.926,828,9, we ſhall have

zz $(= \overline{0.926,828,9}^2) = 0.859,011,809,875,21,$

and z^3 $(= \overline{0.926,828,9}^3) = 0.796,156,970,833,650,021,569,$

and z^4 $(= \overline{0.926,828,9}^4) = 0.737,901,289,505,083,932,475,772,544,1,$

and $34z$ $(= 34 \times 0.926,828,9) = 31.512,182,6,$

and $5zz$ $(= 5 \times 0.859,011,809,875,21) = 4.295,059,049,376,05,$

and $34z^3$ $(= 34 \times 0.796,156,970,833,650,021,569)$

$= 27.069,337,008,344,100,733,346.$

Therefore the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ will be $(= 31.512,182,6 + 4.295,059,049,376,05 - 27.069,337,008,344,100,733,346 - 0.737,901,289,505,083,932,475,772,544,1 = 35.807,241,649,376,05 - 27.807,238,297,849,184,665,821,772,544,1) = 8.000,003,351,526,865,334,178,227,455,9$; which is a very little greater than 8, or the abſolute term of the equation $34z + 5zz - 34z^3 - z^4 = 8$. Therefore the number 0.926,828,9 will be a very little leſs than the true value of z, or the greater root of that equation.

A further Approach to the Value of z, or the greater Root of the Equation $34z + 5zz - 34z^3 - z^4 = 8$, *by a third Proceſs of Mr. Raphſon's Method of Approximation.*

Article 18. Now let e be put for the exceſs of this root z, above the number 0.926,828,9; and we ſhall have $z = 0.926,828,9 + e$. Therefore zz will be $= \overline{0.926,828,9 + e}^2$ $(= \overline{0.926,828,9}^2 + 2 \times 0.926,828,9 \times e + e^2$

$= \overline{0.926,828,9}^2 + 1.853,657,8 \times e + e^2) = 0.859,011,809,875,21 + 1.853,657,8 \times e + e^2$, and z^3 will be $= \overline{0.926,828,9 + e}^3 (= \overline{0.926,828,9}^3 + 3 \times \overline{0.926,828,9}^2 \times e + 3 \times 0.926,828,9 \times e^2 + \&c = \overline{0.926,828,9}^3 + 3 \times 0.859,011,809,875,21 \times e + 2.780,486,7 \times e^2 + \&c = \overline{0.926,828,9}^3 + 2.577,035,429,625,63 \times e + 2.780,486,7 \times e^2 + \&c)$ $0.796,156,970,833,650,021,569 + 2.577,035,429,625,63 \times e + 2.780,486,7 \times e^2 + \&c$, and z^4 will be $= \overline{0.926,828,9 + e}^4 (= \overline{0.926,828,9}^4 + 4 \times \overline{0.926,828,9}^3 \times e + 6 \times \overline{0.926,828,9}^2 \times e^2 + \&c = \overline{0.926,828,9}^4 + 4 \times 0.796,156,970,833,650,021,569 \times e + 6 \times 0.859,011,809,875,21 \times e^2 + \&c = \overline{0.926,828,9}^4 + 3.184,627,883,334,600,086,276 \times e + 5.154,070,859,251,26 \times e^2 + \&c) = 0.737,901,289,505,083,932,475,772,544,1 + 3.184,627,883,334,600,086,276 \times e + 5.154,070,859,251,26 \times e^2 + \&c$. And consequently

$34z$ will be $= 34 \times \overline{0.926,828,9 + e}$ $(= 34 \times 0.926,828,9 + 34 \times e) = 31.512,182,6 + 34 \times e$;

and $5zz$ will be $(= 5 \times 0.859,011,809,875,21 + 5 \times 1.853,657,8 \times e + 5 \times e^2) = 4.295,059,049,376,05 + 9.268,289,0 \times e + 5e^2$;

and $34z^3$ will be $(= 34 \times 0.796,156,970,833,650,021,569 + 34 \times 2.577,035,429,625,63 \times e + 34 \times 2.780,486,7 \times e^2 + \&c) = 27.069,337,008,344,100,733,346 + 87.619,204,607,271,42 \times e + 94.536,547,8 \times e^2 + \&c$.

Therefore the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ will be $=$

$$\left\{\begin{array}{l} 31.512,182,600,000,00 + 34.000,000,0 \times e \\ + 4.295,059,049,376,05 + 9.268,289,0 \times e + 5 \times e^2 \\ - 27.069,337,008,344,100,733,346 \\ \qquad - 87.619,204,607,271,42 \times e \\ \qquad - 94.536,547,8 \times e^2 + \&c \\ - 0.737,901,289,505,083,932,475,772,544,1 \\ \qquad - 3.184,627,883,334,600,086,276 \times e \\ \qquad - 5.154,070,859,251,26 \times e^2 - \&c \end{array}\right\}$$

$= 35.807,$

$$= \left\{ \begin{array}{l} 35.807,241,649,376,05 \quad + 43.268,289,0 \times e + 5 \times e^2 \\ - 27.807,238,297,849,184,665,821,772,544,1 \\ \qquad\qquad - 90.803,832,490,606,020,086,276 \times e \\ \qquad\qquad - 99.690,618,659,251,26 \times e^2 - \&c. \end{array} \right\}$$

$$\left\{ \begin{array}{l} = 8.000,003,351,526,865,334,178,227,455,9 \\ \qquad - 47.535,543,490,606,020,086,276 \times e \\ \qquad - 94.690,618,659,251,26 \times e^2 - \&c. \end{array} \right\}$$

But the quadrinomial quantity $34z + 5zz - 34z^3 - z^4$ is $= 8$.

Therefore the quantity 8.000,003,351,526,865,334,178,227,455,9 — 47.535,543,490,606,020,086,276 × e — 94.690,618,659,251,26 × e^2 — &c will alſo be = 8. And conſequently (by adding to both ſides the quantity 47.535,543,490,606,020,086,276 × e) we ſhall have 8.000,003,351,526,865,334,178,227,455,9 — 94.690,618,659,251,26 × e^2 = 8 + 47.535,543,490,606,020,086,276 × e, and (ſubtracting 8 from both ſides,) we ſhall have 47.535,543,490,606,020,086,276 × e = 0.000,003,351,526,865,334,178,227,455,9 — 94.690,618,659,251,26 × e^2 — &c, and conſequently $e = \frac{0.000,003,351,526,865,334,178,227,455,9}{47.535,543,490,606,020,086,27} - \frac{94.690,618,659,251,26 \times e^2}{47.535,543,490,606,020,086,27} - \&c =$ 0.000,000,070,505,702 — &c. Therefore z, or 0.926,828,9 + e, will be (= 0.926,828,9 + 0.000,000,070,505,702 — &c) = 0.926,828,970,505,702, — &c, or the greater root of the equation $34z + 5zz - 34z^3 - z^4 = 8$ will be very nearly equal to, but ſomewhat leſs than, 0.926,828,970,505,702. Q. E. I.

Of this value of z, conſiſting of fifteen figures, I believe the firſt twelve figures, to wit, the figures 0.926,828,970,505, to be exact. But this may be aſcertained in the following manner.

A Determination

A Determination of the Number of Figures that will be exact in the Value last-obtained of z, or the greater Root of the Equation $34z + 5zz - 34z^3 - z^4 = 8$, *to wit*, 0.926,828,970,505,702.

Article 19. The more accurate value of z, or $0.926,828,9 + e$, is 0.926, 828,9 + 0.000,000,070,505,702 — the term $\frac{94.690,618,659,251,26 \times e^2}{47.535,543,490,606,020,086,27}$. We will therefore inquire in what place of decimal fractions the higheſt figure of the value of this term, if it were known, would enter, and how far the ſubtraction of the ſaid term from 0.000,000,070,505,702, or the value of e already found, would alter the latter figures of this value.

Now the term $\frac{94.690,618,659,251,26 \times e^2}{47.535,543,490,606,020,086,27}$ is leſs than $\frac{94.691 \times e^2}{47.535}$, that is, than $1.99 \times e^2$, and, *à fortiori*, leſs than $2e^2$, or than $2 \times \overline{0.000,000,070,505,702}^2$, and, therefore, again *à fortiori*, will be leſs than $2 \times \overline{0.000,000,071}^2$, or than $2 \times 0.000,000,000,000,005,041$, or than 0.000, 000,000,000,010,082; of which number the firſt, or higheſt, figure is 1 in the fourteenth place of decimal fractions. Therefore the firſt figure of the value of the ſaid term, if it were known, could not enter the calculation in any higher place of decimal fractions than the fourteenth place, and, if it entered in that place, it could only be an unit, and could only, by being ſubtracted from the number 0.000,000,070,505,702, change the ſaid number into the number 0.000,000,070,505,69, &c, of which the firſt twelve figures 0.000,000,070,505, are the ſame as in the number 0.000,000,070,505,702 already obtained for the value of e. Therefore in the number hereby obtained for the value of $0.926,828,9 + e$, or z, to wit, the number 0.926,828,970, 505,702, the twelve firſt figures 0.926,828,970,505, will certainly be exact.

Q. E. D.

The number of figures that are exact in the number 0.926,828,970,505,702 here found for the laſt near value of z, or the greater root of the equation $34z - 5zz + 34z^3 - z^4 = 8$, may alſo be aſcertained in the manner deſcribed by Mr. Ivory, by proceeding as follows.

Another

Another Determination of the ſame Point, by the Method propoſed by Mr. Ivory.

In the foregoing inveſtigation of e, or the number that was to be added to 0.926,828,9 in order to obtain a ſtill nearer value of z, we came to this equation, $47.535,543,490,606,020,086,276 \times e = 0.000,003,351,526,865,334,178,227,455,9 - 94.690,618,659,251,26 \times e^2 -$ &c. Therefore (adding $94.690,618,659,251,26 \times e^2 +$ &c to both ſides,) we ſhall have $47.535,543,490,606,020,086,276 \times e + 94.690,618,659,251,26 \times e^2 +$ &c $= 0.000,003,351,526,865,334,178,227,455,9$, and conſequently $e = \frac{0.000,003,351,526,865,334,178,227,455,9}{47.535,543,490,606,020,086,276 + 94.690,618,659,251,26 \times e +}$ &c.

Now this fraction (which expreſſes the true value of e) is leſs than any fraction that has the ſame numerator with itſelf, but a ſmaller denominator, and is greater than any fraction that has the ſame numerator with itſelf, but a greater denominator. It therefore muſt be leſs than the fraction $\frac{0.000,003,351,526,865,334,178,227,455,9}{47.535,543,490,606,020,086,276}$, or than the decimal fraction 0.000,000,070,505,702, but will be greater than the fraction $\frac{0.000,003,351,526,865,334,178,227,455,9}{47.535,543,490,606,020,086,276 + 94.690,618,659,251,26 \times f}$, if the quantity f be greater than e. Now e is leſs than 0.000,000,070,505,702, and, *à fortiori*, is leſs than 0.000,000,071. Therefore, if we take $f = 0.000,000,071$, the true value of e will be greater than the fraction $\frac{0.000,003,351,526,865,334,178,227,455,9}{47.535,543,490,606,020,086,276, + 94.690,6.8,659,251,26 \times 0.000,000,071}$, and, *à fortiori*, greater than $\frac{0.000,003,351,526,865,334,178,227,455,9}{47.535,543,490,606,020,086,276 + 94.691 \times 0.000,000,071,}$, or than $\frac{0.000,003,351,526,865,334,178,227,455,9}{47.535,543,490,606,020,086,276 + 0.000,006,723,061}$, or than $\frac{0.000,003,351,526,865,334,178,227,455,9}{47.535,550,213,667,020,086,276}$, or than 0.000,000,070,505,692. Therefore the true value of e is greater than 0.000,000,070,505,692, but leſs than 0.000,000,070,505,702; and conſequently the true

value

value of z, or 0.926,828,9 + e, will be greater than (0.926,828,9 + 0.000,000,070,505,692, or) 0.926,828,970,505,692, but leſs than (0.926,828,9 + 0.000,000,070,505,702, or) 0.926,828,970,505,702, and therefore the firſt twelve figures of it muſt be the ſame with the firſt twelve figures of each of the two numbers 0.926,828,970,505,692 and 0.926,828,970,505,702 between which it lies, or muſt be 0.926,828,970,505; or, in other words, the laſt near value of z, or the greater root of the equation $34z + 5zz - 34z^3 - z^4 = 8$, obtained by the foregoing inveſtigation, to wit, the number 0.926,828,970,505,702, will be true in the firſt twelve figures 0.926,828,970,505. Q. E. D.

The equation $34z + 5zz - 34z^3 - z^4 = 8$ can have only two roots, which we have now found to be nearly equal to 0.240,831,917,697,3 and 0.926,828,970,505,702; and therefore it is now compleatly reſolved.

The Computation of the Values of the three Quantities a, b, and c, that are deriveable from the foregoing Value of z, or the greater Root of the Equation $34z + 5zz - 34z^3 - z^4 = 8$.

Article 20. This greater root, 0.926,828,970,505,702, or (dropping the two laſt figures of it,) 0.926,828,970,505,7, of the equation $34z + 5zz - 34z^3 - z^4 = 8$, will enable us, if it be ſubſtituted inſtead of z in the three expreſſions $\sqrt{\frac{2}{z - z^3}}$, and $\frac{2 - z - zz}{2} \times \sqrt{\frac{2}{z - z^3}}$, and $\frac{2 + z - zz}{2} \times \sqrt{\frac{2}{z - z^3}}$ given us by Mr. Ivory's Solution of Colonel Titus's Problem, to find a ſecond ſet of numeral values of the three quantities, b, a, and c, which will anſwer the conditions of the Problem. Theſe ſubſtitutions may be made in the following manner.

If z is = 0.926,828,970,505,7, we ſhall have zz (= $\overline{0.926,828,970,505,7}^2$) = 0.859,011,940,568,655,720,513,732,49, and z^3 (= $\overline{0.926,828,970,505,7}^3$) = 0.796,157,152,529,350,734,073,919,752,426,029,820,193. Therefore $z - z^3$

$- z^3$ will be $=$ 0.926,828,970,505,700,000,000,000,000,000,000,000,000, $-$ 0.796,157,152,529,350,734,073,919,752,426,029,820,193, $=$ 0.130,671,817,976,349,265,926,080,247,573,970,179,807, and $\frac{2}{z - z^3}$ will be $(= \frac{2.000,000,000,000,000,000,000,000,000,000,000,000,000}{0.130,671,817,976,349,265,926,080,247,573,970,179,807}) = 15.305,519,055,087$, and b, or $\surd \frac{2}{z - z^3}$, will be $(= \surd 15.305,519,055,087) = 3.912,226,866,515$.

And $z + zz$ will be $(= 0.926,828,970,505,7 + 0.859,011,940,568,655,720,513,732,49) = 1.785,840,911,074,355,720,513,732,49$, and $z - zz$ will be $(= 0.926,828,970,505,7 - 0.859,011,940,568,655,720,513,732,49) = 0.067,817,029,937,044,279,486,267,51$; and $2 - z - zz$ will be $(= 2 - 1.785,840,911,074,355,720,513,732,49) = 0.214,159,038,925,644,279,486,267,51$, and $\frac{2-z-zz}{2}$ will be $(= \frac{0.214,159,088,925,644,279,486,267,51}{2}) = 0.107,079,544,462,822,139,743,133,755$; and $2 + z - zz$ will be $= 2.067,817,029,937,044,279,486,267,51$, and $\frac{2+z-zz}{2}$ will be $(= \frac{2.067,817,029,937,044,279,486,267,51}{2}) = 1.033,908,514,968,522,139,793,133,755$. Therefore a, or $\frac{2-z-zz}{2} \times \surd \frac{2}{z - z^3}$, or $\frac{2-z-zz}{2} \times b$, will be $(= 0.107,079,544,462,822,139,743,133,755 \times 3.912,226,866,515 =$, nearly, $0.107,079,544,462,822 \times 3.912,226,866,515) = 0.418,919,470,701,639$, &c; and c, or $\frac{2+z-zz}{2} \times \surd \frac{2}{z - z^3}$, or $\frac{2+z-zz}{2} \times b$, will be $(= 1.033,908,514,968,522,139,793,133,755 \times 3.912,226,866,515 =$, nearly, $1.033,908,514,968,522 \times 3.912,226,866,515) = 4.044,884,669,778$, &c.

These three values of a, b, and c, or the three Numbers fought in this Problem, to wit, 0.418,919,470,701,639, 3.912,226,866,515, and 4.044,884,669,778, agree with the values of these Numbers found by Dr. Wallis's Solution, which are 0.418,919,470, 3.912,226,866 and 4.044,884,670, and which are found to anſwer the conditions of the Problem, or to make $aa + bc = 16$, and $bb + ac = 17$, and $cc + ab = 18$. See the *Tracts on the Reſolu-*

tion of Algebräick Equations, by the Methods of Approximation, pages 232, 233, 234, - - - 236.

A Scholium.

Article 21. It now appears that the two values of z in the biquadratick equation $34z + 5zz - 34z^3 - z^4 = 8$, produced by Mr. Ivory's Solution of Colonel Titus's Problem, to wit, the two Numbers 0.240,831,917,697,3 and 0.926,828,970,505,7, will enable us, (by means of the three expreſſions, $\sqrt{\frac{2}{z-z^3}}$, $\frac{2-z-zz}{2} \times \sqrt{\frac{2}{z-z^3}}$, and $\frac{2+z-zz}{2} \times \sqrt{\frac{2}{z-z^3}}$ given us by Mr. Ivory for the values of b, a, and c,) to find both the ſets of Numbers, or values of b, a, and c, that will anſwer the conditions of the Problem. And this equation has no more than theſe two roots, and therefore is exactly fitted to ſolve the Problem that produced it, and no other Problem whatſoever: whereas the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, produced by Dr. Wallis's Solution of the ſame Problem, admits of four real and affirmative roots, of which the leaſt, and leaſt but one, are fitted to ſolve the Problem, and the other two roots are fitted to ſolve two other Problems, a little different from that of Colonel Titus, but which would produce the ſame final equation; and the equation $x^4 - 11.54x^3 + 21.76x^2 - 11.24x = 0.02$, produced by Mr. Frend's Solution of the ſame Problem, has three real and affirmative roots, of which the middle root and the third, or greateſt, root, are fitted to ſolve the Problem from which it was derived, but the firſt, or leaſt, root, has no relation to that Problem, but is fitted to ſolve another Problem a little different from it, but which would produce the ſame final equation. Now, when the final equation, derived from the conditions of a Problem that is required to be ſolved, is exactly commenſurate to the Problem that produced it, and has no ſuperfluous roots that are of no uſe in ſolving the Problem that has been propoſed, but relate to ſome other Problems differing a little in their conditions from the Problem under conſideration, ſuch a Solution ſeems to be ſomewhat preferable, in point of perſpicuity and elegance, to another Solution that produces a final equation that admits of more roots than are wanted for the Solution of the propoſed Problem. And in that reſpect Mr. Ivory's Solution of this Problem of Colonel Titus ſeems to

be preferable to either of the two former Solutions of it by Dr. Wallis and Mr. Frend.

End of the Reſolution of the Biquadratick Equation $34z + 5zz - 34z^3 - z^4 = 8$, *obtained by Mr. Ivory's Solution of Colonel Titus's Problem.*

Article 22. I have alſo received another Letter from the ſame learned gentleman, Mr. Ivory, which relates to the Reſolution of Algebräick Equations by the Methods of Approximation; and likewiſe contains ſome curious and valuable remarks on Dr. Halley's Nautical Problem, of which a Solution has been given in the fourth volume of this Collection of Tracts, intitled *Scriptores Logarithmici.* This Letter is as follows.

A Letter from Mr. JAMES IVORY *to Mr. Baron* MASERES,

dated, Douglaſtown, Feb. 6, 1802.

SIR,

I Duly received your Letter of laſt January the 13th.

—— I believe that moſt of the methods that have been propoſed for the numerical reſolution of equations, (as Sir Iſaac Newton's, and Mr. Raphſon's, Methods,) as well as that which you very properly call the *Differential* Method, will be found ultimately to depend on one common principle, or property, of variable quantities, and all theſe methods may be conſidered as only different applications of the property I allude to. This property of variable quantities may be thus deſcribed. "Let X repreſent any variable quantity that depends upon another variable quantity called x. In the modern Algebra X is called a *Function* of x; and you may conſider X as the ordinate of a Curve of which x is the Abſciſs; or, if the inquiry is concerning equations, you may

conceive X as repreſenting the amount of all that ſide of an equation which contains the unknown quantity x in all it's terms. Now let x receive a finite increment, denoted by $\dot{x}$, and let $\dot{X}$ be the correſpondent increment of X; ſo that, when x becomes $x + \dot{x}$, X ſhall become $X + \dot{X}$. Upon theſe ſuppoſitions, I ſay that, whatever be the nature of the Function X, provided that the finite increment $\dot{x}$ of the radical quantity x be not too great, the increment $\dot{X}$ of the Function X may be conſidered, without much error, as proportional to $\dot{x}$; and in all caſes $\dot{x}$ may be taken ſo ſmall that the error ariſing from the ſuppoſition that $\dot{X}$ is proportional to $\dot{x}$ will be inconſiderable." This property of two variable quantities, of which one depends on the other, appears to me to be the common foundation of all the Methods of Approximation: and I believe they will all be found of the ſame degree of accuracy when the ſame *data* are uſed. If the increment $\dot{X}$ were exactly proportional to $\dot{x}$, and not only nearly ſo, the *Differential* Method would no longer be an approximation, but would give a reſult that would be rigidly accurate. And in Sir Iſaac Newton's, and Mr. Raphſon's Methods, when you reject all the terms that contain $\dot{x}^2$, $\dot{x}^3$, &c, and retain only the term that contains $\dot{x}$, you in effect retain only that part of the increment $\dot{X}$ which is proportional to $\dot{x}$, and reject the reſt.

Dr. Halley's Nautical Problem is a Problem of great difficulty; and I believe you have formed a right judgement of the Solution of it which you have publiſhed. That Solution may, however, be rendered more perfect and eaſier in practice by making a few alterations in ſome parts of it: for all the coefficients C, D, E, &c, are really finite quantities, depending on the co-ſine and tangent of the Latitude ſailed-from. I have found other Serieſes for the Solution of this Problem: but I am not compleatly ſatisfied with any of them. I have, however, found a rule for the particular caſe when the place ſailed-from is in the Equator, which is remarkable for it's ſimplicity and accuracy. It is as follows.

Let φ denote the given difference of Longitude, or the length of the arch of the Equator between the place ſailed-from, and the Meridian paſſing through the place come-to: and let d denote the given diſtance, or the length

of

of the Rhumb-line paſſing through the two places. I ſuppoſe that both φ and d are expreſſed in Nautical miles, 60 to a degree; and I further put $R = 3437', \frac{74}{100}$, which is the number of nautical miles in the radius of the terreſtrial ſphere. Then will the Sine of the Courſe be $= \frac{\varphi}{d} \times \sqrt{\frac{3R^2 - d^2}{3R^2 - \varphi^2}}$ and the co-ſine of the Courſe will be $= \frac{R}{d} \times \sqrt{\frac{3 \times \overline{d^2 - \varphi^2}}{3R^2 - \varphi^2}}$, and the Latitude come-to will be (in nautical miles,) $= R \times \sqrt{\frac{3 \times \overline{d^2 - \varphi^2}}{3R^2 - \varphi^2}}$. This rule is not indeed ſtrictly accurate, but may be looked-upon as ſuch in practice: for, provided the Latitude come-to does not exceed 80 degrees, the error in the Courſe will ſeldom exceed a few ſeconds of a degree. In Snellius's example (in page 170 of the 4th volume of the *Scriptores Logarithmici*,) the angle of the Courſe comes-out 67°, 22′, 32″ by my rule; and, according to Dr. Mackay's calculation, in page 283, it is 67°, 22′, 38″.

My rule fails altogether when $3R^2 - \varphi^2$ is $= 0$, or φ is $= R \times \sqrt{3}$, or when φ is $= 5954' = 99°, 14'$. And in this caſe the Problem is almoſt indeterminate; that is to ſay, the diſtance ſailed-over will remain very nearly the ſame, in whatever Latitude the place come-to is ſituated. Thus, ſuppoſing φ to be 99°, 14′, and the latitude come-to to be $= 80°$, I find d to be $= 5890$ miles, which is leſs than φ, or than 5954 miles, by only 64 miles; ſo that a difference of Latitude of 80 degrees has only produced a variation of 64 miles in a diſtance of 5954 miles.

I am, &c.

JAMES IVORY.

A

CONVENIENT METHOD

OF

RESOLVING BIQUADRATICK EQUATIONS OF ALL KINDS,

WITHOUT FIRST TAKING AWAY THEIR SECOND TERMS;

WHICH MAY ALSO BE APPLIED TO THE

RESOLUTION OF ALL KINDS OF CUBICK EQUATIONS,

AND OF

ALL EQUATIONS OF THE FIFTH, SIXTH, AND SEVENTH, AND ALL OTHER HIGHER POWERS.

BY FRANCIS MASERES, F.R.S.

CURSITOR BARON OF THE COURT OF EXCHEQUER IN ENGLAND.

CAPSTICK BARON-OF THE COURT OF EXCHEQUER IN ENGLAND.

BY FRANCIS MASERES ESQ.

HIGHER POWERS.

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RESOLVING BIQUADRATICK EQUATIONS OF ALL

OF

CONTINUAL METHOD.

V

A CONVENIENT METHOD

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RESOLUTION OF ALL KINDS OF CUBICK EQUATIONS,

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ALL EQUATIONS OF THE FIFTH, SIXTH, AND SEVENTH, AND ALL OTHER HIGHER POWERS.

BY FRANCIS MASERES, F.R.S.
CURSITOR BARON OF THE COURT OF EXCHEQUER IN ENGLAND.

ARTICLE I.

THIS Method, (which is in ſubſtance Mr. Raphſon's Method,) may be beſt explained by applying it to a particular example.

The Reſolution of the Biquadratick Equation $x^4 - 6x^3 + 10xx + 1.242,639, \&c \times x = 1.863,954, \&c.$

Let it therefore be required to find the root of the biquadratick equation $x^4 - 6x^3 + 10xx + 3 \times \overline{\sqrt{2} - 1} \times x = \frac{\sqrt{2}-1}{2} \times 9$, or $x^4 - 6x^3 + 10xx + 3 \times .414,213, \&c \times x = \frac{.414,213, \&c}{2} \times 9$, or $x^4 - 6x^3 + 10xx + 1.242,$

$+ 1.242,639$, &c $\times x = .207,106$, &c $\times 9$, or $x^4 - 6x^3 + 10xx +$ $1.242,639$, &c $\times x = 1.863,954$, &c, to six places of figures. To resolve this equation I proceed in the following manner.

Conjectural Approaches to the Value of the Root of this Equation.

Article 2. In the first place I know, from the nature of the Problem which produced this equation, that x must be less than 1.5. I therefore suppose it to be equal to 1. And I find upon this supposition that the compound quantity $x^4 - 6x^3 + 10xx + 1.242,639$ &c x will be $6.242,639$,&c, which is much greater than $1.863,954$,&c, or the value of the compound quantity $x^4 - 6x^3 + 10xx + 1.242,639$,&c $\times x$ in the proposed equation. I therefore conclude that 1 is much greater than the true value of x.

Article 3. I therefore, in the next place, suppose x to be equal $\frac{1}{2}$, or 0.5. And I find, upon this supposition, that the compound quantity $x^4 - 6x^3 + 10xx + 1.242,639$ &c, x is $= 2.4338$, which is still considerably greater than $1.863,954$, &c, or the value of that compound quantity in the proposed equation: from which I conclude that 0.5 is somewhat greater than the value of x in that equation.

Article 4. I therefore, in the third place, suppose x to be equal to 0.4; and upon this supposition I find the compound quantity $x^4 - 6x^3 + 10xx + 1.242,639$,&c x to be equal to 1.7386, which is less than $1.863,954$,&c. Therefore 0.4 is less than the true value of x in the proposed equation.

A further Approach to the Value of the said Root, by a Process of Mr. Raphson's Method of Approximation.

Article 5. I therefore, in the fourth place, suppose x to be equal to $0.4 + z$, and substitute this binomial quantity instead of x in the proposed equation, which thereby becomes $\overline{.4 + z}^4 - 6 \times \overline{.4 + z}^3 + 10 \times \overline{.4 + z}^2 +$ $1.242,639$, &c $\times \overline{.4 + z} = 1.863,954$, &c, or (neglecting all the terms in which

which the ſquare of z and all higher powers of it are involved, as being inconſiderable in compariſon of thoſe which involve the ſimple power of z,) $\overline{.4}^4 + 4 \times \overline{.4}^3 z - 6 \times \overline{\overline{.4}^3 + 3 \times \overline{.4}^2 z} + 10 \times \overline{\overline{.4}^2 + 2 \times .4\, z} +$ 1.242,639, &c $\times \overline{.4 + z} = 1.863{,}954$; that is, $.0256 + 4 \times .064z - 6 \times \overline{.064 + 3 \times .16\, z} + 10 \times \overline{.16 + .8\, z} + .497{,}0556 + 1.242{,}639, \&c\, z$, $= 1.863{,}954$ &c; or $.0256 + .256\, z - .384 - 2.88\, z + 1.6 + 8\, z +$ will be $.497{,}055{,}6 + 1.242{,}639, \&c\, z = 1.863{,}954$, &c; or $.0256 + 1.6 + .497{,}055{,}6 - .384 + .256\, z + 8z + 1.242{,}639$ &c $z - 2.88\, z$ will be $= 1.863{,}954$, &c; or $2.122{,}655{,}6 - .384 + 9.498{,}639\, z - 2.88\, z$ will be $= 1.863{,}954$, &c; or $1.738{,}655{,}6 + 6.618{,}639\, z$ will be $= 1.863{,}954$, &c. Therefore $6.618{,}639z$ will be $= 1.863{,}954$, &c $- 1.738{,}655{,}6 = 0.125{,}298$, &c; and z will be $= \frac{125{,}298, \&c}{6.618{,}639} = .018$. Conſequently $.4 + z$, or x, will be $= .418$.

Article 6. This value of x is ſtill ſomewhat leſs than the truth. For, if we ſubſtitute .418 inſtead of x in the compound quantity $x^4 - 6x^3 + 10xx +$ 1.242,639, &c x, it will become $(= .030{,}528{,}212, - 6 \times .073{,}034{,}632, + 10 \times .174{,}724 + 1.242{,}639 \times .418 = .030{,}528{,}212 - .438{,}207{,}792 + 1.747{,}240, + .519{,}423{,}102 = 2.297{,}191{,}314, - .438{,}207{,}792) = 1.858{,}983$, &c; which is leſs than 1.863,954 &c, or the true value of that compound quantity in the propoſed equation. Therefore .418 is leſs than the true value of x.

A ſecond Proceſs of Mr. Raphſon's Method of Approximation.

Article 7. In order therefore to find the value of x to a greater degree of exactneſs, I ſuppoſe, in the fifth place, that it is equal to $.418 + v$, and ſubſtitute this binomial quantity inſtead of it in the propoſed equation, neglecting, as before, all the powers of v except the ſimple power. Then we ſhall have

$$\begin{aligned} xx &(= .174{,}724 + 2 \times .418\, v + \&c) \\ &= .174{,}724 + .836\, v + \&c; \\ \text{and conſequently } 10\, xx &= 1.747{,}240 + 8.36\, v + \&c; \\ \text{and } x^3 &(= .073{,}034{,}632 + 3 \times .174{,}724\, v + \&c) \\ &= .073{,}034{,}632 + .524{,}172, v + \&c; \end{aligned}$$

and consequently $6x^3 = .438,207,792 + 3.145,032,v + \&c$;
and x^4 $(= .030,528,212, + 4 \times .073,034,632, v + \&c)$
$= .030,528,212, + .292,138,528, v + \&c$;
and $1.242,639\,x$ $(= 1.242,639, \times .418 + 1.242,639\,v)$
$= .519,423,102 + 1.242,639\,v$;
and consequently x^4 $- 6x^3 + 10xx + 1.242,639, x$

$$= \left\{ \begin{array}{llll} .030,528,212, & + .292,138,528, v & + \&c \\ - .438,207,792, & - 3.145,032, v & - \&c \\ + 1.747,240, & + 8.360, v & + \&c \\ + 0.519,423,102 & + 1.242,639\,v & + \&c \end{array} \right\}$$

$$= \left\{ \begin{array}{lll} 2.297,191,314, & + 9.894,777,528\,v & + \&c \\ - .438,207,792, & - 3.145,032, & v - \&c \end{array} \right\}$$

$$= 1.858,983,522, + 6.749,745,528\,v + \&c.$$

Therefore 1.863,954 (which is $= x^4 - 6x^3 + 10xx + 1.242,639\,x$) will also be $= 1.858,983,522 + 6.749,745,528\,v + \&c$; and, (subtracting 1.858, 983,522 from both sides,) 6.749,745,528, v will be $= 0.004,970,478$; and v will be $= \frac{0.004,970,478}{6.749,745,528} = .000,736$. Therefore $.418 + v$, or x, is $=$ .418,736. Q. E. I.

Article 8. This value of x is very nearly equal to, but still somewhat less than, the true value of x in the equation $x^4 - 6x^3 + 10xx + 1.242,639\,x = 1.863,954$, as will appear by substituting it instead of x in the first, or left-hand, side of that equation, as follows.

If x is taken $= .418,736$, we shall have $xx = .175,339,837$, and $x^3 = .073,421,086$, and $x^4 = .030,744,015$; and consequently $10xx = 1.753,398$, and $6x^3 = .440,526$, and $1.242,639 \times x$ $(= 1.242,639 \times .418,736) = .520,337$, and $x^4 - 6x^3 + 10xx + 1.242,639, \times x$ $(= .030,744 - .440,526, + 1.753,398, + .520,337, = 2.304,479, - .440,526,) = 1.863,953$; which is less than 1.863,954, or the true value of the compound quantity $x^4 - 6x^3 + 10xx + 1.242,639\,x$ in the proposed equation, by only an unit in the seventh place of figures. Therefore .418,736 is less than, but very nearly equal to, the true value of x in the proposed equation.

Article 9.

Article 9. If it were required to find the value of x to a ſtill greater degree of exactneſs, we might obtain it to double the number of figures already known, that is, to twelve places of figures, or, at leaſt, to eleven places of figures, by ſuppoſing x to be equal to .418,736, $+ w$, and ſubſtituting that quantity inſtead of x in the propoſed equation, and reſolving the transformed equation thence ariſing in the ſame manner as we did the laſt transformed equation, that is, by conſidering it as a ſimple equation, and neglecting all the terms that involve the ſquare and other higher powers of w. And thus we may proceed to determine the value of x to greater and greater degrees of exactneſs *ad infinitum.*

A Method of diſcovering how many of the Figures of the Number 0.418,736, *juſt now obtained for the Value of* x, *will be exact.*

Article 10. It remains that we ſhew in what manner we ought to proceed in order to know how many places of figures the number .418,736 gives the value of x exact. Now this may be determined in the following manner.

Let .418,736 $+ w$ be ſuppoſed to be equal to x; and, for the ſake of avoiding intricate and unneceſſary calculations, let us ſuppoſe a to be equal to .418,736.

Then we ſhall have $a + w = x$, and conſequently

$$\begin{aligned}
xx &= aa + 2aw + ww,\\
\text{and } x^3 &= a^3 + 3aaw + 3aww + w^3,\\
\text{and } x^4 &= a^4 + 4a^3w + 6aaww + 4aw^3 + w^4,\\
\text{and } 10xx &= 10aa + 20aw + 10ww,\\
\text{and } 6x^3 &= 6a^3 + 18aaw + 18aww + 6w^3,\\
\text{and } 1.242,639,x &= 1.242,639,a + 1.242,639,w;\\
\text{and conſequently } x^4 - 6x^3 + 10xx &+ 1.242,639,x\\
= \Big\{ &\begin{array}{r} a^4 + 4a^3w + 6aaww + 4aw^3 + w^4 \\ -6a^3 - 18aaw - 18aww - 6w^3 \\ +10aa + 20aw + 10ww \\ +1.242,639,a + 1.242,639,w. \end{array} \Big\}
\end{aligned}$$

But

But $x^4 - 6x^3 + 10xx + 1.242,639, x$ is $= 1.863,954$. Therefore 1.863,954 will alſo be $=$

$$\left\{\begin{array}{rrrrr} a^4 + & 4a^3w & + 6aaww & + 4aw^3 & + w^4 \\ - 6a^3 & - 18aaw & - 18aww & - 6w^3 & \\ + 10aa & + 20aw & + 10ww & & \\ + 1.242,639,a & + 1.242,639w. & & & \end{array}\right\}$$

But $a^4 - 6a^3 + 10aa + 1.242,639, a$, or $\overline{.418,736}^4 - 6 \times \overline{.418,736}^3 + 10 \times \overline{.418,736}^2 + 1.242,639 \times .418,736$, has been already ſhewn, in Art. 8, to be equal to 1.863,953. Therefore 1.863,954 is $=$

$$\left\{\begin{array}{rrrrr} 1.863,953 + & 4a^3w & + 6aaww & + 4aw^3 & + w^4 \\ & - 18aaw & - 18aww & - 6w^3 & \\ & + 20aw & + 10ww & & \\ & + 1.242,639,w & & & \end{array}\right\}$$

and, ſubtracting 1.863,953 from both ſides, 0.000,001 is $=$

$$\left\{\begin{array}{rrrrr} & 4a^3w & + 6aaww & + 4aw^3 & + w^4 \\ - 18aaw & - 18aww & - 6w^3 & & \\ + 20aw & + 10ww & & & \\ + 1.242,639w. & & & & \end{array}\right\}$$

Now, ſince a is equal to .418,736, and conſequently is greater than 0.4, but leſs than 0.42, it follows that $20a$ will be greater than 20×0.4, or 8, and that $18aa$ will be leſs than $18 \times \overline{0.42}^2$, or $18 \times .1764$, or 3.1752. But 8 is greater than 3.1752. Therefore, *à fortiori*, $20a$ (which is greater than 8,) will be greater than 3.1752, and conſequently than $18aa$; and $20aw$ will be greater than $18aaw$. Therefore, again, *à fortiori*, $4a^3w + 20aw + 1.242,639, w$ (being greater than $20aw$) will be greater than $18aaw$.

Further, ſince a is leſs than .42, the quantity $18a$ will be leſs than $18 \times .42$, or 7.56, and conſequently, *à fortiori*, leſs than 10. Therefore $10ww$, and, ſtill more, $10ww + 6aaww$, will be greater than $18aww$.

Laſtly, ſince a is leſs than .42, $4a$ will be leſs than $4 \times .42$, or 1.68, and therefore, *à fortiori*, leſs than 6.

Therefore, if p be put $= 4a^3 - 18aa + 20a + 1.242,639$,
and $q = 6aa - 18a + 10$,
and $r = 6 - 4a$,

we

we shall have 0.000,001 $= p\,w + q\,w\,w - r\,w^3 + w^4$. Add $r\,w^3$ to both sides of the equation; and we shall have 0.000,001 $+ r\,w^3 = p\,w + q\,w\,w + w^4$; and, subtracting $q\,w\,w + w^4$ from both sides, we shall have

$$p\,w = 0.000,001 + r\,w^3 - q\,w\,w - w^4,$$

or $p\,w = 0.000,001, -q\,w\,w + r\,w^3 - w^4$; and (dividing both sides by p,) $w = \frac{0.000,001}{p} \quad \frac{-q\,w\,w + r\,w^3 - w^4}{p}$. Now, since a is $=$ .41 &c, we shall have $a\,a = .1681$ &c, and $6\,a\,a = 6 \times .1681$ &c $= 1.0086$, and $18\,a = 7.38$, and $6\,a\,a - 18\,a + 10$ $(= 1.0086 - 7.38 + 10 = 11.0086 - 7.38) = 3.6286$; that is, q is nearly $= 3.6286$. Also $4\,a$ is $= 4 \times .41$ &c $= 1.64$ &c, and, consequently, $6 - 4\,a$ is $= 6 - 1.64 = 4.36$; that is, r is $= 4.36$. Therefore, if w were equal to .5, that is, if it were greater than a, or x, instead of being a very small quantity necessary to compleat the value of the latter, $r\,w$ would be $(= 4.36 \times .5 = \frac{4.36}{2}) = 2.18$, and consequently less than 3.6286, or q. Therefore, *à fortiori*, as w is less than .5, $r\,w$ will be less than q. Consequently $r\,w^3$ will be less than $q\,w\,w$, and, *à fortiori*, less than $q\,w\,w + w^4$. Therefore the quantity $\frac{0.000,001}{p} \quad \frac{-q\,w\,w + r\,w^3 - w^4}{p}$, which is equal to the accurate value of w, is less than $\frac{0.000,001}{p}$, or $\frac{0.000,001}{p}$ is greater than the accurate value of w; that is, $\frac{0.000,001}{4\,a^3 - 18\,a\,a + 20\,a + 1.242,639}$, or $\frac{0.000,001}{6.749\text{ \&c}}$, or 0.000,000,14, (of which decimal fraction the first significant figure, 1, enters in the seventh place of decimal fractions,) is greater than the accurate value of w. Therefore the first figure of the accurate value of w cannot enter in any higher place of figures than the seventh place of decimal fractions, and therefore cannot affect the first six figures of the value of x already found, to wit, .418,736; but those figures must be concluded to be exact. And thus, by calculating the next correction of the value of x in a coarse manner to only one figure, so as to discover in what place of figures the said first figure of such correction will enter, we may always determine with certainty to how many places the figures we have already found for the root of the proposed equation, are exact; which is all that was wanting to make this method of resolving a biquadratick equation scientifick and satisfactory.

A Scholium.

A Scholium.

Article 11. This method of resolving a biquadratick equation may be applied to the resolution of an equation of the fifth, or sixth, or seventh power, or any higher power whatsoever, and even to that of an equation of an infinite number of terms, and is, I believe, the very best method that can be taken for the resolution of biquadratick and all other higher equations whatsoever, and perhaps also for the resolution even of cubick equations. It is, substantially, the same with Mr. Raphson's Method, and is, in my opinion, preferable to Dr. Halley's Method of resolving high equations, which proceeds by resolving the transformed equations, (obtained by the substitution of the binomial quantities $.4 + z$, $.418 + v$, and $.418,736 + w$, &c instead of x in the original equation,) as if they were quadratick equations, and neglecting only the cubes and fourth powers, and other higher powers, of the unknown quantity, but not the squares. For, though this method triples the number of the figures of the value of x already known, by every new process, whereas the above method of Mr. Raphson only doubles them, yet this advantage is dearly bought by the greater intricacy and difficulty of the computations necessary to be gone-through in the resolution of these quadratick equations, and the loss of that simplicity of conception which is peculiar to simple equations; so that, upon the whole, I believe it will be found easier to make two processes by Mr. Raphson's Method, and thereby to quadruple the number of figures already known of the value of x, than to perform one process by Dr. Halley's Method, by which we should only triple them. And this will more especially be found to be true when the number of figures of the value of x already known is considerable, as, for instance, five or six. For, in these cases the difference of the labour attending the two methods will be found to be very considerable, and the only advantage belonging to Dr. Halley's Method,—that of tripling the figures already known instead of doubling them,—will be thrown-away, since a determination of the value of x, or the root of the proposed equation, to ten or twelve figures is, on most occasions, as satisfactory as a determination of it to fifteen or eighteen figures. If, therefore, Dr. Halley's Method is at any time to be preferred to Mr. Raphson's, it is only in the first processes of the resolution, when the value of x is known only to one or two, or at most three, figures. But, as the processes in Mr. Raphson's Method, by the resolution of only simple equations, are in these cases so very short and simple, and may be so easily repeated, (as we have seen above in Art. 5, where x is supposed equal to $.4 + z$) I should be inclined to use his method in preference to Dr. Halley's, even on these occasions.

Jan. 13, 1777.

The

The Reſolution of the Biquadratick Equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, *reſulting from Dr. Wallis's Solution of Colonel Titus's Problem.*

Article 12. As a further illuſtration of this method of reſolving biquadratick and other high equations, without taking away any of their terms, by ſubſtituting in their terms, inſtead of their roots, binomial quantities, conſiſting of near values of their roots, together with the unknown complements of the ſaid near values to the true values of the roots, and conſidering the transformed equations reſulting from ſuch ſubſtitutions as if they were mere ſimple equations, by neglecting all the terms of them that involve either the fourth power, the cube, or the ſquare, of the new unknown quantity, I will here ſubjoin the reſolution of a celebrated biquadratick equation, which has four real roots, and which has been reſolved by Dr. Wallis, Mr. Raphſon, and Dr. Halley, and is conſidered by the latter as one of the moſt difficult biquadratick equations that he had ever met with. It is an equation that reſulted from a very difficult Arithmetical Problem that was propoſed to Dr. Wallis by Colonel Silas Titus, but had been originally propoſed by Dr. Pell to Colonel Titus, and which Dr. Wallis, by a moſt laborious Algebräical Proceſs, reduced to this biquadratick equation, $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, the roots of which he inveſtigates and aſſigns with great exactneſs. The Problem is as follows.

Problem.

Article 13. Suppoſing $aa + bc$ to be $= 16$, and $bb + ac$ to be $= 17$, and $cc + ab$ to be $= 18$, to find the numbers a, b, and c.

To this queſtion Dr. Wallis, (in a Letter of June 12, 1662,) returned the following anſwer, to wit, that the numbers are $a = 2.5255 +$ &c, $b = 3.9692 -$ &c, and $c = 3.2406 -$ &c, very nearly, whoſe ſquares and rectangles anſwer the conditions of the Problem.

For aa is $= 6.3782 +$ &c and bb is $= 8.8161 -$ &c
and bc is $= 9.6220 -$ &c and ac $= 8.1841 -$ &c
and $aa+bc$ is $= 16.0002 -$ &c and $bb+ac$ is $= 17.0002 -$ &c
and cc is $= 10.5015 -$ &c
and ab is $= 7.4987 -$ &c
and $cc+ab$ is $= 18.0002 -$ &c.

Article 14. By a long proceſs (for which I refer the reader to Wallis's Algebra, pages 225 *et ſeq.*) Dr. Wallis reduces this Problem to the following equation of the eighth power, to wit, $e^8 - 80e^6 + 1998e^4 - 14{,}937e^2 + 5000 = 0$, in which ee is $= 2aa$. This equation, though it riſes to the eighth power of e, is of the form of a biquadratick equation, and, if we ſubſtitute x inſtead of ee, will appear as ſuch. For then it will be $x^4 - 80x^3 + 1998xx - 14{,}937x + 5000 = 0$, or, as I rather chuſe to ſtate it, $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$. Of this equation I now propoſe to find the roots.

The Inveſtigation of the firſt, or leaſt, Root of this Equation.

Article 15. In the firſt place then, as I am ignorant of the limits of theſe roots, ſuggeſted by the conditions of the Problem that produced this equation, I ſuppoſe x to be equal to 1, and try what will be the value of the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$ reſulting from this ſuppoſition. And I find that it will be $(= 14{,}937 - 1998 + 80 - 1 = 15{,}017 - 1999) = 13{,}018$; which is conſiderably greater than 5000, or it's value in the propoſed equation. I therefore conclude that 1 is very far from being equal to x.

Article 16. In the next place I conſider that, if x be ſuppoſed to decreaſe from 1 to 0, the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$ will likewiſe decreaſe from 13,018 to 0; and that this decreaſe will be gradual and continual, ſo as to make the ſaid compound quantity paſs ſucceſſively through all degrees of magnitude between 13,018 and 0. There will therefore be a point of time, during the decreaſe of x from 1 to 0 at which the ſaid quantity $14{,}937x - 1998xx + 80x^3 - x^4$ will be equal to 5000, or x will have a certain magnitude leſs than 1, which will make that compound quantity equal to 5000; or, in other words, one of the roots of the equation $14{,}937x - 1998xx + 80x^3 - x^4$ will be leſs than 1. Let us therefore ſuppoſe it to be 5.

Then we ſhall have $xx = .25$,

$x^3 = .125$,

$x^4 = .0625$,

and $14{,}937x = 7468.5$,

$1998xx = 499.5$,

$80x^3 = 10$,

and $14{,}937x - 1998xx + 80x^3 - x^4 = 7468.5 - 499.5 + 10 - .0625$ $= 7478.5000 - 499.5625 = 6978.9375$, which is ſtill greater than it's true value 5000. Therefore .5 is greater than the true value of x. And, as 6978.9375 exceeds 5000 very nearly in the proportion of 7 to 5, I conjecture that .4, as well as .5, will be greater than the true value of x.

Article 17. I therefore, in the third place, ſuppoſe x to be equal to .3, and try what will be the value of the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$ reſulting from this ſuppoſition.

Now, if x is $= .3$, we ſhall have $xx = .09$, and $x^3 = .027$, and $x^4 = .0081$, and $14{,}937x = 4481.1$, and $1998xx = 179.82$, and $80x^3 = 2.160$, and conſequently $14{,}937x - 1998xx + 80x^3 - x^4$ ($= 4481.1000 - 179.8200 + 2.1600 - .0081 = 4483.2600 - 179.8281$) $= 4303.4319$; which is leſs than 5000. Therefore .3 is leſs than the true value of x.

A further Approach to the Value of the firſt, or leaſt, Root of this Equation by a Proceſs of Mr. Raphſon's Method of Approximation.

Article 18. In the fourth place therefore, let us ſuppoſe x to be equal to $.3 + z$, and ſubſtitute this binomial quantity inſtead of x in the original equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$; and we ſhall thereby transform it into another equation, in which z will be the unknown quantity, and in which we may neglect all the powers of z above the ſimple power, becauſe we know that x is leſs than .5, and conſequently that z is leſs than ($.5 - .3$, or) .2, and therefore that the higher powers of it are conſiderably leſs than it's lower powers, every new power being leſs than the next before it in the proportion of 0.2 to 1, or of 1 to 5.

Now, ſince x is $= .3 + z$, we ſhall have

$$xx = \overline{.3 + z}^2 \ (= .09 + 2 \times .3z + \&c)$$
$$= .09 + .6z \quad + \&c,$$

$$\text{and } x^3 = \overline{.3 + z}^3 \ (= .027 + 3 \times .09z + \&c)$$
$$= .027 + \quad .27z + \&c,$$

and

and $x^4 = \overline{.3 + z}^4$ ($= .0081 + 4 \times .027z +$ &c)
$= .0081 + .108z +$ &c,
and conſequently $14{,}937x$ ($= 14{,}937 \times \overline{.3 + z}$)
$= 4481.1 + 14{,}937z$,
and $1998xx$ ($= 1998 \times \overline{.09 + .6z +}$ &c)
$= 179.82 + 1198.8z +$ &c,
and $80x^3$ ($= 80 \times \overline{.027 + .27z +}$ &c)
$= 2.160 + 21.60z +$ &c,
and conſequently $14{,}937x - 1998xx + 80x^3 - x^4 =$

$$\left\{\begin{array}{rrr} 4481.1000 & + & 14{,}937z \\ - \ 179.8200 & - & 1198.8z - \text{\&c} \\ + \ 2.1600 & + & 21.60z + \text{\&c} \\ - \ .0081 & - & .108z - \text{\&c} \end{array}\right\} = 4303.4319, + 14{,}958.60z - 1{,}198.908z$$

$= 4303.4319 + 13{,}759.692 \times z$. Therefore 5000 (which is $= 14{,}937x - 1998xx + 80x^3 - x^4$) will alſo be $= 4303.4319 + 13{,}759.692 \times z$, and conſequently $13{,}759.692 \times z$ will be $= 5000 - 4303.4319 = 696.56681$, and z will be $= \frac{696.56681}{13{,}759.692} = .05$. Therefore $.3 + z$, or x, will be $= .35$.

Article 19. This value of x is very nearly equal to, but ſtill ſomewhat leſs than, the truth. For, if we ſubſtitute .35 inſtead of x in the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$, we ſhall have $xx = .1225$, $x^3 = .042{,}875$, and $x^4 = .015{,}006{,}25$, and conſequently $14{,}937x$ ($= 14{,}937 \times .35$) $= 5227.95$, and $1998xx$ ($= 1998 \times .1225$) $= 244.7550$, and $80x^3$ ($= 80 \times .042{,}875$) $= 3.430$, and $14{,}937x - 1998xx + 80x^3 - x^4$ ($= 5227.950$, $- 244.7550$, $+ 3.430$, $- .015{,}006{,}25 = 5231.380$, $- 244.770{,}006{,}25$) $= 4986.609{,}993{,}75$; which is leſs than 5000.

A ſecond Proceſs of Mr. Raphſon's Method of Approximation.

Article 20. We will therefore, in the fifth place, ſuppoſe x to be equal to $.35 + v$, and ſubſtitute $.35 + v$ inſtead of x in the original equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, omitting, as before, all the terms that riſe beyond the ſimple power of v.

Now, ſince x is $= .35 + v$, we ſhall have

$xx =$

$$xx = \overline{.35 + v}^2 \ (= .1225 + 2 \times .35\,v + \&c)$$
$$= .1225 + .70\,v + \&c,$$
$$\text{and } x^3 = \overline{.35 + v}^3 \ (= .042,875 + 3 \times .1225\,v + \&c)$$
$$= .042,875 + .3675\,v + \&c$$
$$\text{and } x^4 = \overline{.35 + v}^4 \ (= .015,006,25 + 4 \times .042,875\,v + \&c)$$
$$= .015,006,25 + .171,500\,v + \&c,$$

and consequently $14,937\,x$ $(= 14,937 \times \overline{.35 + v})$

$$= 5227.950 + 14,937\,v,$$
$$\text{and } 1998\,xx \ (= 1998 \times \overline{.1225 + .70\,v + \&c})$$
$$= 244.7550 + 1398.6\,v + \&c,$$
$$\text{and } 80\,x^3 \ (= 80 \times \overline{.042,875 + .3675\,v + \&c})$$
$$= 3.430 + 29.400\,v,$$

and, consequently, $14,937\,x - 1998\,xx + 80\,x^3 - x^4 =$

$$\left\{\begin{array}{rrrr} & 5227.950 & + 14,937\,v & \\ - & 244.7550 & - 1398.6\,v - \&c & \\ + & 3.430 & + 29.4\,v + \&c & \\ - & .015,006,25 & - .171,500\,v - \&c & \end{array}\right\} = 4,986.609,993,75, + 14,966.4000\,v - 1398.7715\,v$$

$$= 4986.609,993,75 + 13,567.6285\,v.$$

Therefore 5000 (which is $= 14,937\,x - 1998\,xx + 80\,x^3 - x^4$) will alſo be $= 4986.609,993,75 + 13,567.6285\,v$. Therefore $13,567.6285 \times v$ will be $= 5000 - 4986.609,993,75$, $= 13.390,006,25$, and v will be $= \frac{13.390,006,25}{13,567.6285} = .000,9$. Conſequently $.35 + v$, or x, is .3509.

Article 21. This value of x is ſtill ſomewhat leſs than the truth. For, if x is $= .3509$, we ſhall have $xx = .123,130,81$, and $x^3 = .043,196,601,229$, and $x^4 = .015,167,687,371,256,1$, and conſequently $14,937\,x$ $(= 14,937 \times .3509) = 5241.3933$, and $1998\,xx$ $(= 1998 \times .123,130,81) = 246.015,358,38$, and $80\,x^3$ $(= 80 \times .043,196,601,229) = 3.455,728,098,320$; and conſequently $14,937\,x - 1998\,xx + 80\,x^3 - x^4$ $(= 5241.3933 - 246.015,358,38, + 3.455,728,098,320 - .015,167,687,371,256,1 = 5244.849,028,098,320 - 246.030,526,067,371,256,1,) = 4998.818,502,030,948,743,9$; which is leſs than 5000, or the true value of the compound quantity $14,937x - 1998\,xx + 80\,x^3 - x^4$ in the original equation.

A third

A third Proceſs of Mr. Raphſon's Method of Approximation.

Article 22. We will therefore, in the ſixth place, ſuppoſe x to be equal to $.3509 + w$, and ſubſtitute $.3509 + w$ inſtead of x in the original equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, omitting, as before, all the terms that riſe beyond the ſimple power of w.

Now, ſince x is $= .3509 + w$, we ſhall have

$xx = \overline{.3509 + w}^2$ $(= .123{,}130{,}81 + 2 \times .3509w +$ &c)
$= .123{,}130{,}81 + .7018w +$ &c,

and $x^3 = \overline{.3509 + w}^3$ $(= .043{,}196{,}601{,}229 + 3 \times .123{,}130{,}81 \times w +$ &c)
$= .043{,}196{,}601{,}229 + .369{,}392{,}43w +$ &c,

and $x^4 = \overline{.3509 + w}^4$ $(= .015{,}167{,}687{,}371{,}256{,}1 + 4 \times .043{,}196{,}601{,}229, w +$ &c)
$= .015{,}167{,}687{,}371{,}256{,}1 + .172{,}786{,}404{,}916, w +$ &c,

and consequently $14{,}937x$ $(= 14{,}937 \times \overline{.3509 + w})$ $= 5241.3933 + 14{,}937w$,

and $1998xx$ $(= 1998 \times .123{,}130{,}81 + .7018w +$ &c)
$= 246.015{,}358{,}38 + 1402.1964w +$ &c,

and $80x^3$ $(= 80 \times .043{,}196{,}601{,}229 + .369{,}392{,}43w +$ &c)
$= 3.455{,}728{,}098{,}320 + 29.551{,}394{,}40w +$ &c,

and consequently $14{,}937x - 1998xx + 80x^3 - x^4 =$

$$\left\{\begin{array}{ll} 5241.3933 & + 14{,}937w \\ -\ 246.015{,}358{,}38 & -\ 1402.1964w - \&c \\ +\ 3.455{,}728{,}098{,}320 & + 29.551{,}394{,}40w + \&c \\ -\ .015{,}167{,}687{,}371{,}256{,}1 & -\ .172{,}786{,}404{,}916w - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{ll} 4998.818{,}502{,}030{,}948{,}743{,}9 & + 14{,}966.551{,}394{,}40w + \&c \\ & -\ 1402.369{,}186{,}404{,}916w - \&c \end{array}\right\}$$

$$= 4998.818{,}502{,}030{,}948{,}743{,}9 + 13{,}564.182{,}207{,}995{,}084w.$$

Therefore 5000 (which is $= 14{,}937x - 1998xx + 80x^3 - x^4$,) will alſo be $= 4998.818{,}502{,}030{,}948{,}743{,}9 + 13{,}564.182{,}207{,}995{,}084, w$. Conſequently

quently 13,564.182,207,995,084, w will be ($=$ 5000 $-$ 4998.818,502,030,948,743,9) $=$ 1.181,497,969,051,256,1, and w will be ($=$ $\frac{1.181,497,969,051,256,1}{13,564.182,207,995,084,}$) $=$ .000,087,10. Therefore .3509 $+$ w, or x, will be $=$ .350,987,10.

Article 23. This value of x is somewhat greater than the truth. For, if we suppose x to be equal to .350,987,10, or .350,987,1 we shall have xx $=$ .123,191,944,366,41,
and x^3 $=$ .043,238,783,296,527,583,311,
and x^4 .015,176,255,156,776,656,536,336,288,1;
and consequently 14,937 x ($=$ 14,937 $\times$ .350,987,1) $=$ 5242.694,312,7
and 1998 xx ($=$ 1998 $\times$ .123,191,944,366,41,) $=$ 246.137,504,844,087,18,
and 80 x^3 ($=$ 80 $\times$ .043,238,783,296,527,583,311,) $=$ 3.459,102,663,722,206,664,880;
and consequently 14,937 x $-$ 1998 xx $+$ 80 x^3 $-$ x^4 ($=$ 5242.694,312,7
$-$ 246.137,504,844,087,18 $+$ 3.459,102,663,722,206,664,880,
$-$ .015,176,255,156,776,656,536,336,288,1 $=$ 5246.153,415,363,722,206,664,880 $-$ 246.152,681,099,243,956,656,536,336,288,1)
$=$ 5000.000,734,264,478,250,008, 343,663,711,9; which is greater than 5000, or the true value of the compound quantity 14,937 x $-$ 1998 xx $+$ 80 x^3 $-$ x^4 in the proposed equation.

A fourth Process of Mr. Raphson's Method of Approximation.

Article 24. To find therefore the value of x to a still greater degree of exactness, we will suppose, in the seventh place, that x is equal to .350,987,1 $-$ y, instead of 350,987,1 $+$ y, and will substitute .350,987,1 $-$ y instead of x in the original equation 14,937 x $-$ 1998 xx $+$ 80 x^3 $-$ x^4 $=$ 5000, omitting, as before, all the terms that rise beyond the simple power of y.

In doing this it will be convenient, in order to avoid the tedious trouble of copying-over the long numbers that compose the powers of .350,987,1, to put k for .350,987,1, and consequently $k - y$ for x.

Then we shall have xx ($=$ $\overline{k-y}^2$) $=$ kk $-$ $2ky$ $+$ &c,
and x^3 ($=$ $\overline{k-y}^3$) $=$ k^3 $-$ $3kky$ $+$ &c,

and

and $x^4 \; (= \overline{k - y}^4) = k^4 - 4k^3y \; +$ &c,

and conſequently $14{,}937\,x = 14{,}937\,k - 14{,}937\,y$,

and $1998\,x\,x = 1998\,k\,k - 3996\,k\,y \; +$ &c,

and $80\,x^3 = \quad 80\,k^3 - 240\,k\,k\,y \; +$ &c,

and conſequently $14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4 =$

$$\left\{\begin{array}{rrr} 14{,}937\,k & - & 14{,}937\,y \\ - \; 1998\,k\,k & + & 3996\,k\,y \\ + \; 80\,k^3 & - & 240\,k\,k\,y \\ - \; k^4 & + & 4\,k^3y \end{array}\right\} = \begin{array}{l} 5000.000{,}734{,}264{,}478{,}250{,}008{,}343{,}663{,}711{,}9 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 711{,}9 \\ - \; 14{,}937\,y \\ + \; 3996 \times .350{,}987{,}1\,,y \\ - \; 240 \times .123{,}191{,}944{,}366{,}41\,,y \\ + \; 4 \times .043{,}238{,}783{,}296{,}527{,}583{,}311\,,y \end{array}$$

$$= \left\{\begin{array}{l} 5000.000{,}734{,}264{,}478{,}250{,}008{,}343{,}663{,}711{,}9 \\ - \; 14{,}937\,y \\ + \; 1402.544{,}451{,}6\,y \\ - \quad 29.566{,}066{,}647{,}938{,}40\,y \\ + \quad .172{,}955{,}133{,}186{,}110{,}333{,}244\,,y \end{array}\right\}$$

$$= \left\{\begin{array}{l} 5000.000{,}734{,}264{,}478{,}250{,}008{,}343{,}663{,}711{,}9, \\ - \; 14{,}966.566{,}066{,}647{,}938{,}40\,y \\ + \; 1402.727{,}406{,}733{,}186{,}110{,}333{,}244\,,y, \end{array}\right\}$$

$$= 5000.000{,}734{,}264{,}478{,}250{,}008{,}343{,}663{,}711{,}9 - 13{,}563.838{,}659{,}914{,}752{,}289{,}666{,}756\,,y.$$

Therefore 5000 (which is $= 14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4$) will alſo be $= 5000.000{,}734{,}264{,}478{,}250{,}008{,}343{,}663{,}711{,}9 - 13{,}563.838{,}659{,}914{,}752{,}289{,}666{,}756\,,y$;

and $5000 + 13{,}563.838{,}659{,}914{,}752{,}289{,}666{,}756\,,y$ will be $=$ $5000.000{,}734{,}264{,}478{,}250{,}008{,}343{,}663{,}711{,}9$;

and conſequently $13{,}563.838{,}659{,}914{,}752{,}289{,}666{,}756\,y$ will be $=$ $.000{,}734{,}264{,}478{,}250{,}008{,}343{,}663{,}711{,}9$, and y will be

$$= \frac{.000{,}734{,}264{,}478{,}250{,}008{,}343{,}663{,}711{,}9}{13{,}563.838{,}659{,}914{,}752{,}289{,}666{,}756}$$

$.000{,}000{,}054{,}133{,}97$. Therefore $.350{,}987{,}1 - y$, or x, will be $(= .350{,}987{,}1 - .000{,}000{,}054{,}133{,}97) = .350{,}987{,}045{,}866{,}03$; which, if no error has been made in the calculation, muſt be true in all the figures, except the laſt. Therefore the firſt, or leaſt, root of the biquadratick equation $14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4 = 5000$ will be $= 0.350{,}987{,}045{,}866{,}0$.

Q. E. I.

Article 25.

Article 25. Dr. Wallis, who does not carry the computation of this root of the propoſed equation quite ſo far, has made it equal to .350,987,046, which is as near to .350,987,045,866,03 as can be expreſſed in nine places of figures.

The Reduction of this Equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, *from a Biquadratick to a Cubick Equation, by means of the Value of the firſt, or leaſt, Root, which is now known.*

Article 26. Having thus found one of the roots of the biquadratick equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, we may reduce it to a cubick equation that ſhall involve all the other roots of it. This may be done in the following manner.

Let d be put for .350,987,045,866,03, or the root of the propoſed equation already diſcovered, which is evidently the leaſt root of it. Then will $14{,}937d - 1998dd + 80d^3 - d^4$ be $= 5000$; and conſequently, if the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ has any other roots beſides d, and x be ſuppoſed to repreſent any one of thoſe roots, we ſhall have $14{,}937d - 1998dd + 80d^3 - d^4 = 14{,}937x - 1998xx + 80x^3 - x^4$.

Add $1998xx + x^4$ to both ſides; and we ſhall then have $14{,}937d + 1998xx - 1998dd + 80d^3 + x^4 - d^4 = 14{,}937x + 80x^3$.

But, ſince d is leſs than x, the binomial quantity $14{,}937d + 80d^3$ will be leſs than the binomial quantity $14{,}937x + 80x^3$, and conſequently than the other ſide of the laſt equation, and therefore may be ſubtracted from both ſides. Let them be ſo ſubtracted; and we ſhall then have $1998xx - 1998dd + x^4 - d^4 = 14{,}937x - 14{,}937d + 80x^3 - 80d^3$, or (ranging the terms according to the rank of their powers,) $x^4 - d^4 + 1998xx - 1998dd = 80x^3 - 80d^3 + 14{,}937x - 14{,}937d$; that is, $x^4 - d^4 + 1998 \times \overline{xx - dd}$ will be $= 80 \times \overline{x^3 - d^3} + 14{,}937 \times \overline{x - d}$. But $x^4 - d^4$, $x^3 - d^3$, and $xx - dd$ are, all of them, diviſible without a remainder by the binomial quantity $x - d$. Let them be ſo divided; and we ſhall then have $x^3 + dxx + ddx + d^3 + 1998 \times \overline{x + d} = 80 \times \overline{xx + dx + dd} + 14{,}937$, or $x^3 + dxx + ddx$

$+ ddx + d^3 + 1998x + 1998d = 80xx + 80dx + 80dd + 14{,}937.$
Subtract $80xx + 80dx$ from both sides; and we shall have

$$\left.\begin{array}{l} x^3 + dxx + ddx + d^3 \\ -80xx - 80dx \\ \qquad + 1998x + 1998d \end{array}\right\} = 80dd + 14{,}937.$$

Lastly, since $1998dd + d^4$ are less than $14{,}937d + 80d^3$, it is evident that $1998d + d^3$ must be less than $80dd + 14{,}937$, and therefore may be subtracted from both sides of the equation. Let them be so subtracted; and we shall then have

$$\left.\begin{array}{l} x^3 + dxx + ddx \\ -80xx - 80dx \\ \qquad + 1998x \end{array}\right\} = 14{,}937 + 80dd - d^3 - 1998d,$$

which is a cubick equation whose roots are the remaining values of x in the original biquadratick equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, or the remaining roots of the said biquadratick equation. We must therefore now endeavour to find the roots of this cubick equation.

Article 27. Now, in order to avoid the labour of computing the values of dd and d^3, upon a supposition that d is equal to the long decimal fraction .350,987, 045,866,03, we will suppose d to be equal only to .3509, whose powers we have already had occasion to compute above in Art. 21. By this means we shall obtain the roots of the foregoing cubick equation, or the remaining values of x in the original biquadratick equation, exact to about three places of figures, after which it will be easy to investigate them to a greater number of figures by the same method of approximation as has been already made use of to discover the value of d.

Now, if we take $d = .3509$, we shall have $dd = .123,130,81$, and $d^3 = .043,196,601,229$. Consequently $80d$ will be $(= 80 \times .3509) = 28.0720$, and $80dd$ will be $(= 80 \times .123,130,81) = 9.850,464,80$,
and $1998d$ will be $(= 1998 \times .3509) = 701.0982$.
Therefore $80 - d$ will be $= 80 - .3509 = 79.6491$,
and $1998 + dd - 80d$ will be $(= 1998 + .123,130,81 - 28.0720$
$= 1998.123,130,81 - 28.0720)$
$= 1970.051,130,81$,
and $14{,}937 + 80dd - d^3 - 1998d$ will be $(= 14{,}937$
$+ 9.850,464,80 - .043,196,601,229 - 701.0982$
$= 14{,}946.850,464,80 - 701.141,396,601,229) =$

14.245,

14,245.709,068,198,771. Therefore the cubick equation

$$\left\{\begin{array}{r} x^3 + dxx + ddx \\ -80xx - 80dx \\ +1998x \end{array}\right\} = \begin{array}{l} 14{,}937 + 80dd \\ -d^3 - 1998d \end{array}$$

will become $x^3 - 79.6491xx + 1970.051{,}130{,}81, x =$ 14,245.709,068,198,771; which is a cubick equation properly prepared for resolution.

Of the Number of Roots in the Cubick Equation just obtained.

Article 28. Let p be put for 79.6491, and q for 1970.051,130,81, and r for 14,245.709,068,198,771. Then will pp be = 6343.979,130,81; and $\frac{pp}{3}$ will be = 2114.659,710,27; and $\frac{pp}{4}$ be = 1585.994,782,70; and $3q$ be = 5910.153,392,43; and $pp - 3q$ be = 433.825,738,38; and $2pp - 6q$ be = 867.651,476,76; and $\sqrt{pp - 3q}$ will be = 20.8284; and $2pp - 6q \times \sqrt{pp - 3q}$ will be $(= 867.651{,}476{,}76 \times 20.8284) = 18{,}071.792{,}018{,}547{,}984$; and $\frac{2pp - 6q \times \sqrt{pp - 3q}}{27}$ will be $(= \frac{18071.792{,}018{,}547{,}984}{27}) =$ 669.325,630,316,592; and pq will be $(= 79.6491 \times 1970.051{,}130{,}81) =$ 155,762.799,522,998,771; and $\frac{pq}{3}$ will be = 51,920.933,174,332,923; and $\frac{p}{3}$ will be $(= \frac{79.6491}{3}) = 26.5497$; and $\frac{pp}{9}$ will be $(= \overline{26.5497}^2) =$ 704.886,570,09; and $\frac{p^3}{27}$ will be $(= \overline{26.5497}^3) = 18{,}714.526{,}969{,}918{,}473$; and $\frac{2p^3}{27}$ will be = 37,429.053,939,836,946; and $\frac{pq}{3} - \frac{2p^3}{27}$ will be $(= 51{,}920.933{,}174{,}332{,}923, - 37{,}429.053{,}939{,}836{,}946,) = 14{,}491.879{,}234{,}495{,}977$; and $\frac{pq}{3} - \frac{2p^3}{27} - \frac{2pp - 6q \times \sqrt{pp - 3q}}{27}$ will be $(= 14{,}491.879{,}234{,}495{,}977, - 669.325{,}630{,}316{,}592,) = 13{,}822.553{,}604{,}179{,}385$.

Article 29. It appears therefore that q, or 1970.051,130,81, is less than $\frac{pp}{3}$,

or 2114.659,710,27, but greater than $\frac{pp}{4}$, or 1585.994,782,70; and that r, or 14,245.709,068,198,771, is greater than $\frac{pq}{3} - \frac{2p^3}{27} - \frac{\sqrt{pp-3q} \times 2\sqrt{pp-6q}}{27}$, or 13,822.553,604,179,385, but less than $\frac{pq}{3} - \frac{2p^3}{27}$, or 14,491.879,234,495,977. Therefore the equation $x^3 - 79.6491xx + 1970.051,130,81x = 14,245.709,068,198,771$, comes under the eleventh case of the equation $x^3 - pxx + qx = r$ set down in the 16th chapter of the Dissertation on the use of the negative sign in Algebra, Art. 249, page 210, and consequently has three roots; of which the least is less than $\frac{p - \sqrt{pp-3q}}{3}$, (or $\frac{79.6491 - 20.8284}{3}$, or $\frac{58.8207}{3}$,) or 19.6069, and, *à fortiori*, less than $\frac{p}{3}$, (or $\frac{79.6491,}{3}$,) or 26.5497, and therefore may be supposed to be equal to $\frac{p}{3} - y$, and the other two are greater than $\frac{p}{3}$, or 26.5497, and therefore may be supposed to be equal to $\frac{p}{3} + y$.

The Investigation of the least of the three Roots of the foregoing Cubick Equation $x^3 - 79.6491xx + 1970.051,130,81x = 14.245,709,068,198,771$.

Article 30. In order therefore to find the least of these three roots, let x be put $= \frac{p}{3} - y$. Then will xx be $= \frac{pp}{9} - \frac{2py}{3} + yy$, and x^3 $(= \frac{p^3}{27} - 3 \times \frac{pp}{9} \times y + 3 \times \frac{p}{3} \times yy - y^3) = \frac{p^3}{27} - \frac{ppy}{3} + pyy - y^3$, and consequently pxx will be $= \frac{p^3}{9} - \frac{2ppy}{3} + pyy$; and qx will be $= \frac{pq}{3} - qy$. Therefore $x^3 - pxx + qx$ will be $(= \frac{p^3}{27} - \frac{ppy}{3} + pyy - y^3 - \frac{p^3}{9} + \frac{2p^2y}{3} - pyy + \frac{pq}{3} - qy) = \frac{p^3}{27} - \frac{ppy}{3} - y^3 - \frac{p^3}{9} + \frac{2p^2y}{3} + \frac{pq}{3} - qy) = \frac{ppy}{3} - qy - y^3 + \frac{pq}{3} - \frac{2p^3}{27}$. Therefore $\frac{ppy}{3} - qy - y^3 + \frac{pq}{3}$

$\frac{pq}{3} - \frac{2p^3}{27}$ will be $= r$; and consequently, adding $y^3 + qy$ to both sides, $\frac{ppy}{3} + \frac{pq}{3} - \frac{2p^3}{27}$ will be $= r + y^3 + qy$; and, subtracting r from both sides, $\frac{ppy}{3} + \frac{pq}{3} - \frac{2p^3}{27} - r$ will be $= y^3 + qy$; and, lastly, subtracting $\frac{ppy}{3}$ (which is evidently less than $\frac{ppy}{3}$ together with the excess of $\frac{pq}{3} - \frac{2p^3}{27}$ above r, or than $\frac{ppy}{3} + \frac{pq}{3} - \frac{2p^3}{27} - r$, and consequently than the other side of the equation,) from both sides, $\frac{pq}{3} - \frac{2p^3}{27} - r$ will be $= y^3 + qy - \frac{ppy}{3}$, or $y^3 - \left\{\frac{pp}{3} - q\right\} \times y$ will be $= \frac{pq}{3} - \frac{2p^3}{27} - r$; that is, in numbers, $y^3 - \{2114.659,710,27 - 1970.051,030,81\} \times y$ will be $= 14,491.879,234,495,977, - 14,245.709,068,198,771$, or $y^3 - 144.608,579,46, y$ will be $= 246.170,166,297,206$; which is a binomial cubick equation properly prepared for resolution, and which (as the square of the unknown quantity y is wanting in it,) is capable of being resolved either by Cardan's second rule, or by the trisection of a circular arc.

Article 31. In order to determine whether this cubick equation $y^3 - 144.608,579,46 \times y = 246.170,166,297,206$ does, or does not, come under Cardan's second rule, we will put $c = 144.608,579,46$, and $d = 246.170,166,297,206$; and we shall then have $y^3 - cy = d$.

Since c is $= 144.608,579,46$, we shall have $\sqrt{c} = 12.0253$, and $c\sqrt{c} = 144.608,579,46 \times 12.0253 = 1478.666,150,580,338$; and $2c\sqrt{c} = 2957.332,301,160,676$; and $\frac{2c\sqrt{c}}{3\sqrt{3}} = \frac{2c\sqrt{c}}{3 \times 1.732,050,8} = \frac{2957.332,301,160,676}{5.196,152,4} = 569.138,86$. Therefore d, or $246.170,166,297,206$, is less than $\frac{2c\sqrt{c}}{3\sqrt{3}}$; and consequently this equation $y^3 - cy = d$ does not come under Cardan's second rule, but it's root will be equal to the sum of the two roots of the opposite equation $cy - y^3 = d$, of which the roots may be found by the trisection of a circular arc. The roots of this latter equation $cy - y^3 = d$, or

144.608,

$144.608,579,46 \times y - y^3 = 246.170,166,297,206$, muſt therefore now be ſought.

Article 32. Theſe latter roots are equal to the chords of two circular arches which are the third parts of the two arches whereof $\frac{3d}{c}$ is the common chord, in a circle whoſe radius is $\frac{\sqrt{c}}{\sqrt{3}}$. Now, ſince c is $=$ 144.608,579,46, and d is $=$ 246.170,166,297,206, we ſhall have $3d = 738.510,498,891,618$, and $\frac{3d}{c} = \frac{738.510,498,891,618}{144.608,579,46} = 5.10696$, and $\frac{c}{3} = \frac{144.608,579,46}{3} = 48.202,859,82$; and $\frac{\sqrt{c}}{\sqrt{3}} = \sqrt{\frac{c}{3}} = \surd 48.202,859,82 = 6.9428$. Say therefore, as the radius $\frac{\sqrt{c}}{\sqrt{3}}$, or 6.9428, is to the chord $\frac{3d}{c}$, or 5.10696, ſo is 10,000, the tabular radius, to a fourth proportional; and that fourth proportional will be the tabular chord of the ſame arch, or it's chord in a circle whoſe radius is called 10,000. Therefore this tabular chord is $= \frac{5.10696 \times 10000}{6.9428} = 7355.76$. Therefore $\frac{7355.76}{2}$, or 3677.88, is the tabular ſine of half the leſſer of the two arches whoſe common chord is 7355.76. But 3677.88 appears by the tables of ſines to be the ſine of an arch of 21 degrees and 35 minutes. Therefore the leſſer of the two circular arches of which $\overline{7355.76}$ is the common chord in a circle whoſe radius is 10,000, is $= 2 \times \overline{21^\circ, 35'}$, or $43^\circ, 10'$; and conſequently the greater of thoſe arches is $360^\circ - 43^\circ, 10'$, or $316^\circ, 50'$. Therefore the third part of the leſſer of thoſe arches is $14^\circ, 23'$; and the third part of the greater of them is $105^\circ, 36'$. And the halves of theſe laſt arches are $7^\circ, 11'$, and $52^\circ, 48$, of which the tabular ſines are 1250.446, and 7965.299. Therefore 2×1250.446 and 2×7965.299, that is, 2500.892 and 15930.598 are the tabular chords of the third parts of the two arches whoſe common chord is 7355.76; and conſequently $2500.892 \times \frac{6.9428}{10000}$ and $15930.598 \times \frac{6.9428}{10000}$, or $0.250,089,2 \times 6.9428$ and $1.593,059,8 \times 6.9428$, that is, 1.736,319 and 11.060,295 are the chords of the third parts of the two arches whoſe common chord is $\frac{3d}{c}$, or 5.106,96, in the circle whoſe

whose radius is $\frac{\sqrt{c}}{\sqrt{3}}$ or 6.9428. Therefore 1.736,319 and 11.060,295 are the two roots of the equation $cy - y^3 = d$, or $144.608,579,46\,y - y^3 = 246.170,166,297,206$; and their sum 12.796,614, is the root of the opposite equation $y^3 - cy = d$, or $y^3 - 144.608,579,46\,y = 246.170,166,297,206$, which was to be found.

Article 33. This value of y in the equation $y^3 - 144.608,579,46\,y = 246.170,166,297,206$, is somewhat less than the truth. For even 12.8, which is greater than 12.796,614, is somewhat less than the true value of y in that equation, as will appear by trying it as follows. Let y be supposed to be equal to 12.8. Then we shall have $yy = 163.84$, and $y^3 = 2097.152$, and $144.608,579,46\,y\ (= 144.608,579,46 \times 12.8) = 1850.989,817,088$; and consequently $y^3 - 144.608,579,46\,y\ (= 2097.152,000,000 - 1850.989,817,088,) = 246.162,182,912$; which is somewhat less than 246.170,297,206, or the true value of the compound quantity $y^3 - 144.608,579,46\,y$ in the foregoing equation. Therefore 12.8, and, *à fortiori*, 12.796,614, is less than the true value of y in that equation.

The first near Value of the least Root of the Cubick Equation

$x^3 - 79.6491\,xx + 1970.051,130,81 \times x =$
14,245.709,968,198,771, *or of the least Root but one, or second Root, of the Biquadratick Equation* $14,937x - 1998\,xx + 80\,x^3 - x^4 = 5000$.

Article 34. It follows therefore that $\frac{p}{3} - 12.8$, or $26.5497 - 12.8$, that is, 13.7497, is greater than the true value of the least root of the cubick equation $x^3 - 79.6491\,xx + 1970.051,130,81\,x = 14,245.709,068,198,771$, or of the least root but one of the original biquadratick equation $14,937x - 1998\,xx + 80\,x^3 - x^4 = 5000$.

A Proof

A Proof that the ſaid firſt near Value of the ſecond Root of the Biquadratick Equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ *is greater than the truth.*

Article 35. That this value of x in the aforeſaid biquadratick equation is greater than the truth, may be ſhewn alſo in the following manner.

Let x be ſuppoſed to be $= 13.7$. Then we ſhall have $xx = 187.69$, and $x^3 = 2571.353$, and $x^4 = 35{,}227.5361$; and conſequently $14{,}937x$ $(= 14{,}937 \times 13.7) = 204{,}636.9$, and $1998xx$ $(= 1998 \times 187.69) = 375{,}004.62$, and $80x^3$ $(= 80 \times 2571.353) = 205{,}708.240$; and conſequently $14{,}937x - 1998xx + 80x^3 - x^4$ $(= 204{,}636.9 - 375{,}004.62 + 205{,}708.240 - 35{,}227.5361 = 410{,}345.1400 - 410{,}232.1561) = 112.9839$, which is much leſs than 5000, or the true value of the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$ in the original biquadratick equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$. Therefore 13.7 is not the exact value of the ſaid root of that equation. But, whether it is greater or leſs than that root, does not as yet appear. This we will therefore now proceed to enquire.

Article 36. We have ſeen, above, in Art. 15, that when x is $= 1$, the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$ is $= 13{,}018$. And it has juſt now appeared, upon trial, that, when x is $= 13.7$, the ſame compound quantity is equal to 112.9839. It follows therefore that, while x increaſes from 1 to 13.7, the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$ will decreaſe from 13,018 to 112.9839. Conſequently there will be a point of time during the increaſe of x from 1 to 13.7 at which that quantity will be equal to 5000, or, in other words, there will be a certain value of x, greater than 1, and leſs than 13.7, which, being ſubſtituted inſtead of x in the quantity $14{,}937x - 1998xx + 80x^3 - x^4$, will make it equal to 5000; that is, one of the roots of the biquadratick equation $14{,}937x - 1998xx + x^3 - x^4 = 5000$ will be greater than 1 and leſs than 13.7.

A further

A further Approach to the Value of the second Root, or least Root but one of the Biquadratick Equation $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$, *by a Process of Mr. Raphson's Method of Approximation.*

Article 37. In order therefore to find this value of x more exactly, let us suppose it to be equal to $13.7 - z$. Then we shall have

$$xx = \overline{13.7 - z}^2 \,(= 187.69 - 2 \times 13.7z + \&c)$$
$$= 187.69 - 27.4z + \&c,$$

$$\text{and } x^3 = \overline{13.7 - z}^3 \,(= 2571.353 - 3 \times 187.69 \times z + \&c)$$
$$= 2571.353 - 563.07 \times z + \&c,$$

$$\text{and } x^4 = \overline{13.7 - z}^4 \,(= 35{,}227.5361 - 4 \times 2571.353 \times z + \&c)$$
$$= 35{,}227.5361 - 10{,}285.412 \times z + \&c;$$

$$\text{and } 14{,}937\,x \,(= 14{,}937 \times \overline{13.7 - z}) = 204{,}636.9 - 14{,}937\,z;$$

$$\text{and } 1998\,xx \,(= 1998 \times \overline{187.69 - 27.4z + \&c})$$
$$= 375{,}004.62 - 54{,}745.2\,z + \&c;$$

$$\text{and } 80\,x^3 \,(= 80 \times \overline{2571.353 - 563.07\,z + \&c})$$
$$= 205{,}708.240 - 45{,}045.60\,z + \&c;$$

and consequently $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 =$

$$\left\{\begin{array}{llll} 204{,}636.9 & - 14{,}937\,z & & \\ - 375{,}004.62 & + 54{,}745.2 & \times z - \&c & \\ + 205{,}708.240 & - 45{,}045.60 & \times z + \&c & \\ - 35{,}227.5361 & + 10{,}285.412 & \times z - \&c & \end{array}\right\}$$

$$= 112.9839 - 59{,}982.60\,z + 65{,}030.612 \times z - \&c$$

$= 112.9839 + 5048.012 \times z - \&c$. Therefore 5000 (which is $= 14{,}937\,x - 1998\,xx + 80\,x^3 - x^4$,) will also be $= 112.9839 + 5048.012 \times z - \&c$; and consequently $5048.012 \times z$ will be $= 5000 - 112.9839 = 4887.0161$; and z will be $= \frac{4887.0161}{5048.012} = .96$.

Therefore $13.7 - z$, or x, will be $(= 13.7 - .96) = 12.74$.

Article 38. This value of x is ſomewhat leſs than it's true value. For, if we ſuppoſe x to be equal to 12.74, we ſhall have $x\,x = 162.3076$, and $x^3 = 2067.798,824$, and $x^4 = 26,343.757,017,76$; and conſequently $14,937\,x$ $(= 14,937 \times 12.74) = 190,297.38$, and $1998\,x\,x$ $(= 1998 \times 162.3076) = 324,290.5848$, and $80\,x^3$ $(= 80 \times 2067.798,824) = 165,423.905,920$; and $14,937\,x - 1998\,x\,x + 80\,x^3 - x^4$ will be $(= 190,297.38 - 324,290.5848, + 165,423.905,920 - 26,343.757,017,76 = 355,721.285,920 - 350,634.341,817,76) = 5086.944,102,24$; which is greater than 5000, or the true value of the compound quantity $14,937\,x - 1998\,x\,x + 80\,x^3 - x^4$ in the propoſed biquadratick equation. Therefore 12.74 is leſs than the true value of x in that equation.

A further Approach to the Value of the ſaid ſecond Root, by a ſecond Proceſs of Mr. Raphſon's Method of Approximation.

Article 39. Let us therefore, in the next place, ſuppoſe x to be equal to $12.74 + v$.

Then we ſhall have $x\,x = \overline{12.74 + v}^2$

$$(= \overline{12.74}^2 + 2 \times 12.74 \times v + \&c)$$
$$= 162.3076 + 25.48 \times v + \&c,$$

and $x^3 = \overline{12.74}^3 + 3 \times \overline{12.74}^2 \times v + \&c$

$$(= 2067.798,824 + 3 \times 162.3076 \times v + \&c)$$
$$= 2067.798,824 + 486.9228 \times v + \&c,$$

and $x^4 = \overline{12.74}^4 + 4 \times \overline{12.74}^3 \times v + \&c$

$$(= 26,343.757,017,76 + 4 \times 2067.798,824 \times v + \&c)$$
$$= 26,343.757,017,76 + 8271.195,296, \times v + \&c;$$

and $14,937\,x$ $(= 14,937 \times \overline{12.74 + v}) = 190,297.38 + 14,937\,v$,

and $1998\,x\,x$ $(= 1998 \times \overline{162.3076 + 25.48\,v + \&c})$

$$= 324,290.5848 + 50,909.04\,v + \&c,$$

and $80\,x^3$ $(= 80 \times \overline{2067.798,824 + 486.9228 \times v + \&c})$

$$= 165,423.905,920 + 38,953.8240 \times v + \&c,$$

and

and conſequently $14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4 =$

$$\left\{\begin{array}{lll} 190{,}297.38 & + & 14{,}937\ v \\ -\ 324{,}290.5848 & - & 50{,}909.04\ \times v - \&c \\ +\ 165{,}423.905{,}920 & + & 38{,}953.8240 \times v + \&c \\ -\ 26{,}343.757{,}017{,}76 & - & 8271.195{,}296, \times v - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{lll} 5086.944{,}102{,}24, & + & 53{,}890.8240 \times v + \&c \\ & - & 59{,}180.235{,}296 \times v - \&c \end{array}\right\}$$

$$= 5086.944{,}102{,}24 - 5289.411{,}296 \times v - \&c.$$

Therefore 5000 (which is $= 14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4$,) will alſo be $=$ 5086.944,102,24 $-$ 5289.411,296, $\times\ v -$ &c; and conſequently 5000 $+$ 5289.411,296, $\times\ v$ will be $=$ 5086.944,102,24 $-$ &c; and 5289.411,296 $\times\ v$ will be $=$ 86.944,102,24 $-$ &c; and conſequently v will be $= \frac{86.944{,}102{,}24}{5289.411{,}296} = .0164$. Therefore 12.74 $+\ v$, or x_2 will be $=$ 12.74 $+$ .0164, or 12.7564.

Article 40. This value of x is leſs than the truth. For, if we ſuppoſe x to be equal to 12.7564, we ſhall have $x\,x = 162.725{,}740{,}96$, and $x^3 = 2075.794{,}641{,}982{,}144$, and $x^4 = 26{,}479.666{,}770{,}981{,}021{,}721{,}6$;
and conſequently $14{,}937\,x$ ($= 14{,}937 \times 12.7564$) $= 190{,}542.3468$;
and $1998\,x\,x$ ($= 1998 \times 162.725{,}740{,}96$,) $= 325{,}126.030{,}438{,}08$;
and $80\,x^3$ ($= 80 \times 2075.794{,}641{,}982{,}144$,) $= 166{,}063.571{,}358{,}571{,}520$.
Therefore the compound quantity $14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4$ will be ($=$ 190,542.3468 $-$ 325,126.030,438,08 $+$ 166,063.571,358,571,520, $-$ 26,479.666,770,981,021,721,6 $=$ 356,605.918,158,571,520, $-$ 351,605.697,209,061,021,721,6,) $=$ 5000.220,949,510,498,278,4; which is greater than 5000. Conſequently 12.7564 is leſs than the true value of x.

A further Approach to the Value of the ſaid ſecond Root, by a third Proceſs of Mr. Raphſon's Method of Approximation.

Article 41. Let us therefore, in the next place, ſuppoſe x to be equal to 12.7564 + w. And, to avoid the repetition of long numbers, let k be ſuppoſed equal to 12.7564.

Then we ſhall have $x = k + w$, and

conſequently $xx = \overline{k + w}^2$ $(= k^2 + 2kw +$ &c,
$= k^2 + 2 \times 12.7564 \times w +$ &c.)
$= k^2 + 25.5128 \times w +$ &c,

and $x^3 = \overline{k + w}^3$ $(= k^3 + 3k^2w +$ &c
$= k^3 + 3 \times 162.725,740,96 \times w +$ &c)
$= k^3 + 488.177,222,88 \times w +$ &c,

and $x^4 = \overline{k + w}^4$ $(= k^4 + 4k^3w +$ &c
$= k^4 + 4 \times 2075.794,641,982,144, \times w +$ &c)
$= k^4 + 8303.178,567,928,576, \times w +$ &c;

and conſequently $14{,}937x = 14{,}937k + 14{,}937w$,

and $1998xx$ $(= 1998 \times \overline{k^2 + 25.5128 \times w +}$ &c)
$= 1998k^2 + 50{,}974.5744w +$ &c,

and $80x^3$ $(= 80 \times \overline{k^3 + 488.177,222,88 \times w +}$ &c,)
$= 80k^3 + 39{,}054.177,830,40 \times w +$ &c;

and conſequently $14{,}937x - 1998xx + 80x^3 - x^4 =$

$$\left\{\begin{array}{l} 14{,}937k + 14{,}937w \\ -1998kk - 50{,}974.5744w - \&c \\ +80k^3 + 39{,}054.177,830,40 \times w + \&c \\ -k^4 - 8{,}303.178,567,928,576, \times w - \&c \end{array}\right\}$$

$= 5000.220,949,510,498,278,4$
$+ 53{,}991.177,830,40 \times w +$ &c
$- 59{,}277.752,967,928,576, \times w -$ &c
$= 5000.220,949,510,498,278,4 - 5286.575,137,528,576, w -$ &c.

Therefore

Therefore 5000 (which is $= 14{,}937x - 1998xx + 80x^3 - x^4$) will alſo be $=$ 5000. 220,949,510,498,278,4, $-$ 5286.575,137,528,576, w — &c; and conſequently 5000 + 5286.575,137,528,576, w will be $=$ 5000.220,949,510,498,278,4; and, (ſubtracting 5000 from both ſides,) 5286.575,137,528,576, $\times w$ will be $=$ 0.220,949,510,498,278,4. Therefore w will be $= \frac{0.220,949,510,498,278,4}{5286.575,137,528,576,}$ $=$ .000,041,794,4, and conſequently 12.7564 + w, or x, will be $=$ 12.756, 441,794,4; which agrees with the values of x found by Dr. Wallis, Mr. Raphſon, and Dr. Halley, (who have, all of them, reſolved this equation,) and therefore may be concluded to be exact in all it's figures.

Q. E. I.

Article 42. The value of x aſſigned by Dr. Wallis is 12.756,441,794,460, 744; that of Mr. Raphſon is 12.756,441,794,481,11; and that aſſigned by Dr. Halley is 12.756,441,794,480,744.02.

A Proof that the laſt near Value of the ſaid ſecond Root of the Equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ *is exact in all it's Figures.*

Article 43. It remains that we ſhew that the number 12.756,441,794, is exact in all it's figures. And for this purpoſe it will be neceſſary to reſume the conſideration of the transformed equation reſulting from the ſuppoſition that x was equal to 12.7564 + w, or $k + w$, and to inquire into the magnitude of thoſe of it's terms which involve the ſquare, cube, and fourth power of w.

Article 44. Now, if x is $= k + w$, we ſhall have

$$xx = kk + 2kw + ww,$$
$$\text{and } x^3 = k^3 + 3k^2w + 3kww + w^3,$$
$$\text{and } x^4 = k^4 + 4k^3w + 6kkww + 4kw^3 + w^4;$$

and conſequently $14{,}937x = 14{,}937k + 14{,}937w$,

and $1998xx$ $(= 1998kk + 2 \times 1998kw + 1998ww)$
$= 1998kk + 3996kw + 1998ww$,

and $80x^3$ $(= 80k^3 + 3 \times 80kkw + 3 \times 80kww + 80w^3)$
$= 80k^3 + 240kkw + 240kww + 80w^3$.

Therefore

Therefore $14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4$ is =

$$\left\{\begin{array}{rrrrr} 14{,}937\,k & + & 14{,}937\,w & & \\ - 1998\,k\,k & - & 3996\,k\,w & - & 1998\,w\,w \\ + 80\,k^3 & + & 240\,k\,k\,w & + 240\,k\,w\,w & + 80\,w^3 \\ - k^4 & - & 4\,k^3\,w & - 6\,k\,k\,w\,w & - 4\,k\,w^3 - w^4 \end{array}\right\}.$$

But we have already seen that $14{,}937\,k - 1998\,k\,k + 80\,k^3 - k^4$ is = 5000.220,949,510,498,278,4, and that $3996\,k + 4\,k^3 - 240\,k\,k - 14{,}937$ is = 5286.575,137,528,576, when k is supposed to be = 12.7564. Therefore $14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4$ will, upon this supposition, be =

$$\begin{array}{rrrrr} 5000.220{,}949{,}510{,}498{,}278{,}4 & - & 5286.575{,}137{,}528{,}576\,w & & \\ & - & 1998\,w\,w & & \\ & + & 240\,k\,w\,w & + 80\,w^3 & \\ & - & 6\,k\,k\,w\,w & - 4\,k\,w^3 - w^4. & \end{array}$$

But $14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4$ is = 5000. Therefore 5000 will also be = 5000.220,949,510,498,278,4

$$\begin{array}{rrrrr} & & - 1998\,w\,w & & \\ - 5286.575{,}137{,}528{,}576\,w & + & 240\,k\,w\,w & + 80\,w^3 & \\ & & - 6\,k\,k\,w\,w & - 4\,k\,w^3 - w^4. & \end{array}$$

Therefore $5000 + 5286.575{,}137{,}528{,}576,\,w$ will be =

$$\begin{array}{rrrrr} 5000.220{,}949{,}510{,}498{,}278{,}4, & - & 1998\,w\,w & & \\ & + & 240\,k\,w\,w & + 80\,w^3 & \\ & - & 6\,k\,k\,w\,w & - 4\,k\,w^3 - w^4; & \end{array}$$

and consequently (subtracting 5000 from both sides)

$$\begin{array}{rrrrr} 5286.575{,}137{,}528{,}576,\,w \text{ will be} = .220{,}949{,}510{,}498{,}278{,}4 & & & & \\ & - & 1998\,w\,w & & \\ & + & 240\,k\,w\,w & + 80\,w^3 & \\ & - & 6\,k\,k\,w\,w & - 4\,k\,w^3 - w^4; & \end{array}$$

and w will be accurately equal to $\frac{220{,}949{,}510{,}498{,}278{,}4}{5286.575{,}137{,}528{,}576} + \frac{240\,k\,w\,w}{5286.575{,}137{,}528{,}576} - \frac{\overline{6\,k\,k + 1998} \times w\,w}{5286.575{,}137{,}528{,}576} + \frac{80\,w^3}{5286.575{,}137{,}528{,}576} - \frac{4\,k\,w^3}{5286.575{,}137{,}528{,}576} - \frac{w^4}{5286.575{,}137{,}528{,}576} = .000{,}041{,}794{,}451{,}935{,}2 + \frac{240\,k\,w\,w}{5286.\,\&c} - \frac{\overline{6\,k\,k + 1998} \times w\,w}{5286.\,\&c} + \frac{80\,w^3}{5286.\,\&c} - \frac{4\,k\,w^3}{5286.\,\&c} - \frac{w^4}{5286.\,\&c}$. We must therefore inquire in the first place,

place, whether this whole set of terms, which expresses the accurate value of w, is greater or less than it's first term, or the number .000,041,794,451,935,2, and, secondly, in what place of figures it may happen that the first figure of the difference of the second and third terms, $\frac{240\,k\,w\,w}{5286.\,\&c}$ and $\frac{\overline{6kk+1998}\times w\,w}{5286.\,\&c}$, shall enter, and what is the greatest possible magnitude of such first figure. For, when this is ascertained, we need only add, or subtract, the figure so found to, or from, the number .000,041,794,451,935,2, and all the figures of that number which remain unaltered, after such addition or subtraction, may be concluded to be exact.

Article 45. Now, since k is $= 12.7564$, we shall have $kk = 162.725,740,96$. Therefore $240k$ is $(= 240 \times 12.7564) = 3061.5360$; and $6kk$ is $(= 6 \times 162.725,740,96) = 976.354,445,76$; and $6kk + 1998$ is $(= 976.354,445,76 + 1998) = 2974.354,445,76$; which is less than 3061.5360. Therefore $240k$ is greater than $6kk + 1998$.

And since w is less than 1, w^4 will be less than w^3. Consequently $4kw^3 + w^4$ will be less than $4kw^3 + w^3$, or than $4 \times 12.7564 \times w^3 + w^3$, or $51.0256\,w^3 + w^3$, or $52.0256\,w^3$. But $52.0256\,w^3$ is less than $80\,w^3$. Therefore, *à fortiori*, $4kw^3 + w^4$ will be less than $80\,w^3$.

Therefore $.000,041,794,451,935,2 + \frac{240\,k\,w\,w}{5286.\,\&c} - \frac{\overline{6kk+1998}\times w\,w}{5286.\,\&c} + \frac{80\,w^3}{5286.\,\&c} - \frac{4kw^3}{5286.\,\&c} - \frac{w^4}{5286.\,\&c}$, or the accurate value of w, is greater than .000,041,794,451,935,2, or it's near value; which was the first thing we were to determine.

Article 46. In the next place we are to inquire in what place of figures the highest figure of the quantity $\frac{240\,k\,w\,w}{5286.\,\&c} - \frac{\overline{6kk+1998}\times w\,w}{5286.\,\&c} + \frac{80\,w^3}{5286.\,\&c} - \frac{4kw^3}{5286.\,\&c} - \frac{w^4}{2586.\,\&c}$, (which is to be added to .000,041,794,451,935,2, in order to express the accurate value of w,) may possibly enter. Now, this may be determined in the following manner.

Article 47. Let m be put $= 240k - 6kk - 1998$, and $n = 80 - 4k$.

And

And we shall have w accurately equal to .000,041,794,451,935,2 $+ \frac{m\,w\,w}{5286.\,\&c}$ $+ \frac{n\,w^3}{5286.\,\&c}, - \frac{w^4}{5286.\,\&c}$, of which it is evident the third and fourth terms, which involve w^3 and w^4, will be much less than the second term $\frac{m\,w\,\dot{w}}{5286.\,\&c}$, and consequently their highest figures will enter in much lower places than the highest figure of $\frac{m\,w\,w}{5286.\,\&c}$. We therefore need only determine in what place of figures the highest figure of $\frac{m\,w\,w}{5286.\,\&c}$ may possibly enter. Now, since $240k$ is $= 3061.5360$, and $6kk + 1998$ is $= 2974.354,445,76$, we shall have $240k - 6kk - 1998 = 3061.5360 - 2974.354,445,76 = 87.181,554,24$, that is, m will be $= 87.181,554,24$. Therefore $\frac{m\,w\,w}{5286.\,\&c}$, is $= \frac{87.181,554,24,\,w\,w}{5286.\,\&c}$. But, since w is equal to .000,041,794,451,935,2, together with the quantity, $\frac{m\,w\,w}{5286.\,\&c} + \frac{n\,w^3}{5286.\,\&c} - \frac{w^4}{5286.\,\&c}$, (which involves the square, cube, and fourth power of w, and therefore will be less than w in a very great proportion, so as to enter in a place of decimal figures many places lower than the first figure of the value of w, or of it's first and principal member, .000,041,794,451,935,2,) we may safely conclude that it is less than $\overline{.000,042}^{q}$, or .000,000,001,764, and $m\,w\,w$ will be less than $m \times$ .000,000,001,764, or $87.181,554,24 \times$ .000,000,001,764, and, *à fortiori*, less than $87.182 \times$ .000,000,001,764, or than .000,000,153,789,048. Therefore $\frac{m\,w\,w}{5286.\,\&c}$ will be less than $\frac{.000,000,153,789,048,}{5286.\,\&c}$, or .000,000,000,029. Therefore the two highest figures of $\frac{m\,w\,w}{5286}$ cannot be greater than 29 in the eleventh and twelfth places of decimal fractions, and consequently cannot, by their addition to the number .000,041,794,451,935,2, or the first member of the accurate value of w, make it greater than .000,041,794,451,935,2 + .000,000,000,029, or 000,041,794,480,935,2, or alter the first six significant figures of it, to wit, .000,041,794,4. Therefore those figures may be concluded to be exact. Therefore 12.756,441,794,4 is the true value of $k + w$, or 12.7564 $+ w$, or x, as far as it can be expressed by twelve figures. Q. E. D.

An

An easy Method of obtaining the Value of the said second Root of the Equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$ *exact to three or four more Figures than in the last near Value of it,* 12.756,441,794,4, *without a new Process of Mr. Raphson's Method of Approximation.*

Article 48. If we are desirous of finding the value of x to a still greater degree of exactness, it may be done, without raising the powers of the long number 12.756,441,794,4, by computing the expression $\frac{m \times ww}{5286.575,137,528,576,}$ to four or five places of figures, upon a supposition that w is equal to .000,041,794,4, than which we know it to be somewhat greater. For, if this be done, the result will be the true value of w in all the figures preceeding that place of figures in which the first figure of $\frac{n\,w^3}{5286.\,\&c}$, or $\frac{80 - 4k \times w^3}{5286.\,\&c}$, would enter.

Now, if w be supposed $=$ .000,041,794,4, we shall have $ww =$ .000,000,001,746,771,871,36, and $m\,w\,w$, or 87 181,554,24 $\times\, ww =$ .000,000,152,286,286,647,878,142,566,4, and consequently $\frac{87.181,554,24 \times ww}{5280.575,137,528,576} = \frac{.000,000,152,286,286,647,878,142,566,4}{5286.575,137,528,576} =$ .000,000,000,028,806,2. Therefore w is $=$ .000,041,794,451,935,2 + .000,000,000,028,806,2 $=$ .000,041,794,480,741,4, and 12.7564 + w, or x, is $=$ 12.756,441,794,480,741,4; which agrees with Dr. Halley's number in the first sixteen figures, 12.756,441,794,480,74, and differs from it only in the two last figures, which are 14 instead of 40.

The Investigation of the third and fourth Roots, or the two greater Roots, of the Biquadratick Equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$.

Article 49. Having thus found two of the roots of the biquadratick equation $14,937x - 1998xx + 80x^3 - x^4 = 5000$ to be .350,987,045,866,03

and 12.756,441,794,480,74, we may find the remaining roots by the reſolution of a quadratick equation. But, as the doing this to the ſame degree of exactneſs as we have aſſigned the two roots already found, namely, to the fourteenth place of decimal fractions, would require the raiſing the powers of the long numbers that expreſs thoſe roots, which would be exceedingly troubleſome, I think it will be better to ſuppoſe thoſe roots to be equal only to .3509 and 12.7564, in performing the operations neceſſary to depreſs the aforeſaid biquadratick equation to a quadratick equation, and ſo to determine, by the reſolution of the quadratick equation thereby obtained, the two remaining roots of the biquadratick equation to about three places of figures, after which it will be eaſy to determine them with greater exactneſs by the proceſſes of approximation that have been before deſcribed.

The Reduction of the Biquadratick Equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ *to a Quadratick Equation, by means of the firſt and ſecond Roots of the ſaid Equation, which have been already inveſtigated.*

Article 50. Now we have already ſeen that, if the leaſt root of the biquadratick equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ be called d, the remaining roots of that equation will be the ſame with thoſe of the cubick equation $\left.\begin{array}{r} x^3 + dxx + ddx \\ -80xx - 80dx \\ +1998x \end{array}\right\} =$ $14{,}937 + 80dd - d^3 - 1998d$; and that, if .3509 be ſubſtituted inſtead of d in this equation, it will be converted into this, to wit, $x^3 - 79.6491xx + 1970.051{,}130{,}81x = 14{,}245.709{,}068{,}198{,}771$.

Article 51. Now let c be put for the leaſt root of this cubick equation, or the leaſt root but one of the original biquadratick equation, $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, which we have found to be $= 12.7564$ &c; and let p be $= 79.6491$, and $q = 1970.051{,}130{,}81$, and $r = 14{,}245.709{,}068{,}198{,}771$, as above in Art. 27. And, ſince c is one of the values of x in the equation $x^3 - pxx + qx = r$, we ſhall have $c^3 - pcc + qc = r$. Therefore, if x repreſents either of the other two roots of the equation $x^3 - pxx + qx$

$+ qx = r$, the trinomial quantity $c^3 - pcc + qc$ will be equal to the trinomial quantity $x^3 - pxx + qx$. Therefore, (adding pxx to both ſides) $c^3 + pxx - pcc + qc$ will be $= x^3 + qx$; and, ſubtracting $c^3 + qc$ (which are leſs than $x^3 + qx$, becauſe c is leſs than x, and therefore are leſs than the other ſide of the equation,) from both ſides, $pxx - pcc$ will be $= x^3 - c^3 + qx - qc$, or $p \times xx - cc$ will be $= x^3 - c^3 + q \times x - c$; and, (dividing both ſides by $x - c$,) $p \times x + c$ will be $= xx + cx + cc + q$, that is, $px + pc$ will be $= xx + cx + cc + q$. Therefore, ſubtracting $xx + cx$ from both ſides, we ſhall have $px - cx - xx + pc = cc + q$. But pc, or 79.6491×12.7564, is $= 1016.035,779,24$, which is leſs than $1970.051,130,81$, or q, and therefore, *à fortiori*, leſs than $cc + q$, and conſequently than the other ſide of the equation. Let it be ſubtracted from both ſides; and we ſhall have $px - cx - xx = cc + q - pc$, that is, in numbers, $79.6491,x - 12.7564x - xx = 162.725,740,96 + 1970.051,130,81 - 1016.035,779,24$, or $66.8927x - xx = 2132.776,871,77 - 1016.035,779,24$, or $66.8927x - xx = 1116.741,092,53$; which is a quadratick equation of the third form, and therefore has two roots, which may be found as follows.

Article 52. Subtract both ſides of this equation from the ſquare of half the co-efficient of x, that is, from the ſquare of half 66.8927, or the ſquare of 33.4463, which is $= 1118.654,983,69$; and we ſhall have $1118.654,983,69 - 66.8927x + xx = 1118.654,983,69 - 1116.741,092,53, = 1.913,891,16$.

Therefore 33.4463 — the leſſer value of x is $= \sqrt{1.913,891,16} = 1.3834$; and 33.4463 is $= 1.3834 +$ the leſſer value of x; and the leſſer value of x is $= 33.4463 - 1.3834 = 32.0629$.

Alſo the exceſs of the greater value of x above 33.4463 will be $= \sqrt{1.913,891,16} = 1.3834$; and conſequently the greater value of x in this equation will be $= 1.3834 + 33.4463, = 34.8297$.

Therefore the two roots of the quadratick equation $66.8927x - xx = 1116.741,092,53$, and conſequently the two greateſt roots of the biquadratick equation $14,937x - 1998xx + 80x^3 - x^3 = 5000$, (from which that quadratick equation was derived,) are 32.0629 and 34.8297, or, (neglecting the two laſt figures as probably not exact, becauſe, in depreſſing the biquadratick

equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ to a cubick equation, we made uſe of only the four firſt figures, .3509, of it's leaſt root .350,987, 045,866,03,) 32.06 and 34.82, or rather, (becauſe the next figure of this greateſt root 34.8297 is a 9,) 32.06 and 34.83.

Article 53. We will now proceed to determine theſe two greateſt roots of the biquadratick equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ to the ſame degree of exactneſs as we have already found it's other roots. And firſt we will conſider the leſſer of the two, which is 32.06.

A further Inveſtigation of the Value of the third Root of the Equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$.

Article 54. Now, if x be ſuppoſed equal to 32.06, we ſhall have $xx = 1027.8436$, and $x^3 = 32{,}952.665{,}816$, and $x^4 = 1{,}056{,}462.466{,}060\,96$, and $14{,}937x$ $(= 14{,}937 \times 32.06) = 478{,}880.22$, and $1998xx$ $(= 1998 \times 1027.8436) = 2{,}053{,}631.5128$, and $80x^3$ $(= 80 \times 32{,}952.665{,}816,) = 2{,}636{,}213.265{,}280$; and conſequently $14{,}937x - 1998xx + 80x^3 - x^4$ $(= 478{,}880.22 - 2{,}053{,}631.5128, + 2{,}636{,}213.265{,}280 - 1{,}056{,}462.466{,}060{,}96, = 3{,}115{,}093.485{,}280, - 3{,}110{,}093.978{,}860{,}96) = 4{,}999.506{,}419{,}04$; which is very nearly equal to 5000. Therefore 32.06 is very nearly equal to this value of x in the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$. But, whether it is greater or leſs than the true value of x, remains ſtill to be determined.

Article 55. Now, in order to diſcover whether the true value of x in the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ is greater, or is leſs, than 32.06, we muſt inquire, whether, when x is about the magnitude of 32.06, the compound quantity $14{,}937x - 1998xx + 80x^3 - x^4$ (which we have before obſerved to increaſe at ſome times, and at other times to decreaſe, while x increaſes,) is in a ſtate of increaſing or of decreaſing. For, if it is in a ſtate of increaſing, it is evident that 32.06 will be leſs than the true value of x; but, if it is in a ſtate of decreaſing, 32.06 will be greater than that true value. Now, whether or no that compound quantity is in a ſtate of increaſing with the increaſe of x when x is about the magnitude of 32.06, will appear by ſuppoſ-

ing

ing x to increaſe from 32.06 to ſome other value a little greater, as, for inſtance, to 32.1, and trying whether the compound quantity $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4$ will be increaſed, or will be diminiſhed thereby. Let us therefore ſuppoſe x to be equal to 32.1.

Article 56. Then we ſhall have $x\,x = 1030.41$, and $x^3 = 33{,}076.161$, and $x^4 = 1{,}061{,}744.7681$; and conſequently $14{,}937\,x$ $(= 14{,}937 \times 32.1) = 479{,}477.7$, and $1998\,x\,x$ $(= 1998 \times 1030.41) = 2{,}058{,}759.18$, and $80\,x^3$ $(= 80 \times 33{,}076.161) = 2{,}646{,}092.880$; and conſequently $14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4$ $(= 479{,}477.7 - 2{,}058{,}759.18 + 2{,}646{,}092.880 - 1{,}061{,}744.7681 = 3{,}125{,}570.580 - 3{,}120{,}503.9481,) = 5066.6319$. It appears therefore that while x increaſes from 32.06 to 32.1, the compound quantity $14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4$ will increaſe from 4999.506,419,04 to 5066.6319. Therefore that compound quantity is in an increaſing ſtate when x is about the magnitude of 32.06; and conſequently the value of x, when that compound quantity is equal to 5000, will be greater than when that quantity is equal to 4999.506,419,04, that is, the value of x in the equation $14{,}937\,x - 1998\,x\,x + 80\,x^3 - x^4 = 5000$ will be greater than 32.06.

A further Approach to the Value of the ſaid third Root, by a Proceſs of Mr. Raphſon's Method of Approximation.

Article 57. Let us therefore ſuppoſe x to be $= 32.06 + z$. Then we ſhall have $xx = \overline{32.06 + z}^2$ $(= \overline{32.00}^2 + 2 \times 32.06 \times z + \&c)$
$= 1027.8436 + 64.12 \times z + \&c$,
and $x^3 = \overline{32.06 + z}^3$ $(= \overline{32.06}^3 + 3 \times \overline{32.06}^2 \times z + \&c$,
$= 32{,}952.665{,}816 + 3 \times 1027.8436 \times z + \&c)$
$= 32{,}952.665{,}816 + 3083.5308\,z + \&c$;
and $x^4 = \overline{32.06 + z}^4$ $(= \overline{32.00}^4 + 4 \times \overline{32.00}^3 \times z + \&c$
$+ 4 \times 32{,}952.665{,}816 \times z + \&c)$
$= 1{,}056{,}462.466{,}060{,}96 + 131{,}810.663{,}264 \times z + \&c.$

Therefore

Therefore 14,937 x will be $(= 14{,}937 \times \overline{32.06 + z})$
$= 478{,}880.22 + 14.937\,z$;
and 1998 xx will be $(= 1998 \times \overline{1027.8436 + 64.12\,z + \&c};)$
$= 2{,}053{,}631.5128 + 128{,}111.76 \times z + \&c$;
and 80 x^3 will be $(= 80 \times \overline{32{,}952.665{,}816 + 3083.5308\,z + \&c})$
$= 2{,}636{,}213.265{,}280 + 246{,}682.4640 \times z + \&c$;
and consequently 14,937 x — 1998 xx + 80 x^3 — x^4 will be

$$= \left\{\begin{array}{lll} 478{,}880.22 & + & 14{,}937\,z \\ -\ 2{,}053{,}631.5128 & - & 128{,}111.76\,z - \&c \\ +\ 2{,}636{,}213.265{,}280 & + & 246{,}682.4640\,z + \&c \\ -\ 1{,}056{,}462.466{,}060{,}96 & - & 131{,}810.663{,}264\,z - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{lll} 4999.506{,}419{,}04 & + & 261{,}619.4640 \times z + \&c \\ & - & 259{,}922.423{,}264, \times z - \&c \end{array}\right\}$$

$= 4999.506{,}419{,}04 + 1697.040{,}736, \times z\ \&c.$

But 14,937 x — 1998 xx + 80 x^3 — x^4 is = 5000.

Therefore 5000 will be = 4999.506,419,04 + 1697.040,736, × z &c; and 1697.040,736, × z will be (= 5000.000,000,00 — 4999.506,419,04) = 0.493,580,96; and consequently z will be $= \frac{0.493{,}580{,}96}{1697.040{,}736} = .000{,}290.$
Therefore 32.06 + z, or x, will be = 32.060,290.

Article 58. This value of x is less than the truth. For, if we suppose x to be = 32.060,290, we shall have xx = 1027.862,194,884,1, and x^3 = 32,953.560,048,020,762,389, and x^4 = 1,056,500.691,671,959,568,210,432,81. Therefore 14,937 x will be (= 14,937 × 32.060,290) = 478,884.551,73; and 1998 xx will be (= 1998 × 1027.862,194,884,1,) = 2,053,668.665,378,431,8; and 80 x^3 will be (= 480 × 32,953.560,048,020,762,389,) = 2,636,284.803,841,660,991,120. Consequently 14,937 x — 1998 xx + 80 x^3 — will be (= 478,884.551,73 — 2.053,668.665,378,431,8 + 2,636,284.803,841,660,991,120, — 1,056,500.691,671,959,568,210,432,81, = 3,115,169.355,571,660,991,120, — 3,110,169.357,050,391,368,210,432,81) = 4999 998,521,269,622,909,567,19; which is less than 5000. Therefore 32.060,290 is less than the true value of x in the equation, 14,937 x — 1998 xx + 80 x^3 — x^4 = 5000.

A further

A further Approach to the Value of the said third Root by a second Process of Mr. Raphson's Method of Approximation.

Article 59. In order therefore to obtain this value of x to a still greater degree of exactness, we must suppose it to be equal to $32.060,290 + v$, or $32.060,29 + v$.

Let k be substituted for $32.060,29$, to avoid the unnecessary repetition of the very long numbers that constitute the powers of $32.060,29$. And we shall have $xx = \overline{k + v}^2$ $(= kk + 2kv +$ &c
$= kk + 2 \times 32.060,29, \times v +$ &c)
$= kk + 64.120,58, \times v +$ &c

and $x^3 = \overline{k + v}^3$ $(= k^3 + 3kkv +$ &c
$= k^3 + 3 \times 1027.862,194,884,1, \times v +$ &c)
$= k^3 + 3083.586,584,652,3, \times v +$ &c;

and $x^4 = k^4 + 4k^3v +$ &c
$(= k^4 + 4 \times 32,953.560,c48,020,762,389, v +$ &c)
$= k^4 + 131,814.240,192,083,049,556, v +$ &c

Therefore $14,937\,x$ will be $= 14,937\,k + 14,937\,v$;

and $1998\,xx$ will be $(= 1998 \times \overline{kk + 64.120,58\,v +$ &c$})$
$= 1998\,kk + 128,112.918,84\,v +$ &c

and $80\,x^3$ will be $(= 80 \times \overline{k^3 + 3083.586,584,652,3 \times v +$ &c$})$
$= 80k^3 + 246,686.926,772,184,0, v +$ &c;

and $14,937\,x - 1998\,xx + 80\,x^3 - x^4$ will be

$$= \left\{ \begin{array}{rrl} 14,937\,k & + & 14,937\,v \\ - 1998\,kk & - & 128,112.918,84\,v - \text{&c} \\ + 80\,k^3 & + & 246,686\,926,772,184, v + \text{&c} \\ - k^4 & - & 131,814.240,192,083,049,556, v - \text{&c} \end{array} \right\},$$

$$= \left\{ \begin{array}{rrl} 14.937\,k & + & 261,623.926,772,184, v \\ - 1998\,kk & - & 259,927.159,032,083,049,556, v \\ + 80\,k^3 & & \\ - k^4 & & \end{array} \right\}$$

$= 4999$

$= 4999.998,521,269,622\,909,567,19$
$+ 1696.767,740,100,950\,444, \times v.$

Therefore 5000 (which is $= 14,937\,x - 1998\,x\,x + 80\,x^3 - x^4$,) will also be $= 4999.998,521,269,622,909,567,19,$
$+ 1696.767,740,100,950,444, \times v$;

and consequently 1696.767,740,100,950,444 v will be $(= 5000 - 4999.998,521,269,622,909,567,19) = 0.001,478,730,377,090,432,81$; and v will be $= \frac{0.001,478,730,377,090,432,81}{1696.767,740,100,950,444} = 0.000,000,871,498,403$. Therefore $32.060,290, + v$, or z, will be $= 32\,060,290,871,498,403$, or (dropping the two laſt figures 03 as poſſibly not exact,) $= 32.060,290,871,498,4$, that is, the greateſt root but one of the biquadratickequation $14,937\,x - 1998\,x\,x + 80\,x^3 - x^4 = 5000$ will be $= 32.060,290,871,498,4$. Q. E. I.

Article 60. The number aſſigned for this root by Dr. Wallis is 32.060,290, 88, which differs from that found above only in the laſt, or tenth, figure, which ſhould be a 7 inſtead of an 8.

A further Inveſtigation of the Value of the fourth, or greateſt, Root of the Equation $14,937x - 1998\,x\,x + 80\,x^3 - x^4 = 5000$.

Article 61. It remains that we diſcover the fourth, or greateſt, root of the aforeſaid biquadratick equation to as great a degree of exactneſs as the other three. Now this root we have already found in Art. 52. to be nearly equal to 34.83.

Article 62. Now, if x be ſuppoſed equal to 34.83, we ſhall have $x\,x = 1213.1289$, and $x^3 = 42,253.279,587$, and $x^4 = 1,471,681.728,015,21$; and conſequently $14,937\,x\,(= 14,937 \times 34.83) = 520,255.71$, and $1998\,x\,x$ $(= 1998 \times 1213.1289) = 2,423,831.5422$, and $80\,x^3\,(= 80 \times 42,253.279,587) = 3,380,262.366,960$; and $14,937\,x - 1998\,x\,x + 80\,x^3 - x^4\,(= 520,255.71 - 2,423,831.5422 + 3,380,262.366,960, - 1,471,681.728,015,21 = 3,900,518.076,960 - 3,895,513.270,215,22) = 5004.806,744.78$; which differs but little from 5000, or the value of the compound quantity

$14,937\,x$

$14{,}937\,x - 1998\,xx + 80\,x^3 - x^4$ in the propoſed biquadratick equation. Therefore 34.83 muſt be nearly equal to the root of that equation.

Article 63. But whether 34.83 is greater, or is leſs, than the true value of x, remains ſtill to be determined. Now this will appear by ſuppoſing x to increaſe a little from 34.83 to ſome quantity a little greater than 34.83, as, for inſtance, to 34.9, and ſubſtituting that quantity inſtead of x in the compound quantity $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4$, ſo as to diſcover whether that quantity will have increaſed or decreaſed, while x increaſed from 34.83 to 34.9. For, if we ſhall find that it has increaſed, we may conclude that the true value of x in the equation $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$ is leſs than 34.83, which made the compound quantity $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4$ equal to 5004.806,744,78; but, if we ſhall find that it has decreaſed, we may conclude that the ſaid true value of x is greater than 34.83.

Article 64. Now, if x be ſuppoſed to be $= 34.9$, we ſhall have $xx = 1218.01$, and $x^3 = 42{,}508.549$, and $x^4 = 1{,}483{,}548.3601$, and conſequently $14{,}937\,x\ (= 14{,}937 \times 34.9) = 521{,}301.3$, and $1998\,xx\ (= 1998 \times 1218.01) = 2{,}433{,}583.98$, and $80\,x^3\ (= 80 \times 42{,}508.549) = 3{,}400{,}683.920$, and $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4\ (= 521{,}301.3 - 2{,}433{,}583.98 + 3{,}400{,}683.920. - 1{,}483{,}548.3601 = 3{,}921{,}985.220 - 3{,}917{,}132.3401) = 4852.8799$. It appears therefore that, while x increaſes from 34.83 to 34.9, the aforeſaid compound quantity decreaſes from 5004.806,744,78 to 4852.8799. Therefore, while x increaſes from 34.83 to ſome quantity leſs than 34.9, that compound quantity will decreaſe from 5004.806,744,78 to 5000, or, in other words, the true value of x in the biquadratick equation $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$ is greater than 34.83.

A further Approach to the Value of the ſaid fourth, or greateſt, Root, by a Proceſs of Mr. Raphſon's Method of Approximation.

Article 65. In order therefore to determine this value of x to a greater degree of exactneſs, we will ſuppoſe it to be equal to $34.83 + v$.

Then we ſhall have $xx = \overline{34.83 + v}^2$

$(= \overline{34.83}^2 + 2 \times 34.83 \times v + \&c)$

$= 1213.1289 + 69.66\,v + \&c,$

and $x^3 = \overline{34.83 + v}^3$ $(= \overline{34.83}^3 + 3 \times \overline{34.83}^2 \times v + \&c$

$= 42,253.279,587 + 3 \times 1213.1289 \times v + \&c)$

$= 42,253.279,587 + 3639.3867 \times v + \&c,$

and $x^4 = \overline{34.83 + v}^4$ $(= \overline{34.83}^4 + 4 \times \overline{34.83}^3 \times v + \&c$

$= 1,471,681.728,015,21 + 4 \times 42,253.279,587\,v + \&c)$

$= 1,471,681.728,015,21 + 169,013.118,348, \times v + \&c;$

and conſequently $14,937\,x$ $(= 14,937 \times \overline{34.83 + v}) =$

$520,255.71 + 14,937\,v;$

and $1998\,xx$ $(= 1998 \times \overline{1213.1289 + 69.66\,v + \&c})$

$= 2,423,831.5422 + 139,180.68\,v + \&c;$

and $80\,x^3$ $(= 80 \times \overline{42,253.279,587 + 3639.3867 \times v + \&c})$

$= 3,380,262.366,960 + 291,150.9360\,v + \&c.$

Therefore $14,937\,x - 1998\,xx + 80\,x^3 - x^4$ will be $=$

$$\left\{\begin{array}{lll} & 520,255.71 & + 14,937\,v \\ - & 2,423,831.5422 & - 139,180.68\,v - \&c \\ + & 3,380,262.366,960 & + 291,150.9360\,v + \&c \\ - & 1,471,681.728,015,21 & - 169,013.118,348\,v - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{lll} & 3,900,518.076,960, & + 306,087.9360\,v + \&c \\ - & 3,895,513.270,215,22 & - 308,193.798,348,\,v - \&c \end{array}\right\}$$

$$= 5004.806,744,78 - 2,105.862,348,\,v - \&c.$$

Therefore 5000 (which is $= 14,937\,x - 1998\,xx + 80\,x^3 - x^4$,) will alſo be $= 5004.806,744,78 - 2,105.862,348,\,v$ &c; and conſequently $5000 + 2,105.862,348,\,v$ will be $= 5004.806,744,78$; and $2,105.862,348,\,v$ will be $= 4.806,744,78$; and v will be $= \frac{4.806,744,78}{2,105.862,348} = .002,282.$ Therefore $34.83 + v$, or x, will be $= 34.832,282.$ Q. E. I.

Article 66. This value of x is ſomewhat greater than the truth, as will appear by ſubſtituting it inſtead of x in the compound quantity $14,937\,x -$

1998

$1998\,x\,x + 80\,x^3 - x^4$. For, if x be ſuppoſed equal to 34.832,282, we ſhall have $x\,x = 1213.287,869,327,524$, and $x^3 = 42,261.585,211,595,466,329,768$, and $x^4 = 1,472,067,453,857,322,953,119,983,970,576$. Therefore $14,937\,x$ will be $(= 14,937 \times 34.832,282) = 520,289.796,234$; and $1998\,x\,x$ will be $(= 1998 \times 1213.287,869,327,524) = 2,424,149.162,916,392,952$; and $80\,x^3$ will be $(= 80 \times 42,261.585,211,595,466,329,768,) = 3,380,926.816,927,637,306,381,440$; and $14,937\,x - 1998\,x\,x + 80\,x^3 - x^4$ will be $(= 520,289.796,234 - 2,424,149.162,916,392,952, + 3,380,926.816,927,637,306,381,440, - 1,472,067.453,857,322,953,119,983,970,576, = 3,901,216.613,161,637,306,381,440, - 3,896,216.616,773,714,953,119,983,970,576,) = 4999.996,387,922,353,261,456,029,424$; which is leſs than 5000, or the value of the compound quantity $14,937\,x - 1998\,x\,x + 80\,x^3 - x^4$ in the propoſed equation $14,937\,x - 1998\,x\,x + 80\,x^3 - x^4 = 5000$. It appears therefore that, while x increaſes from 34.83 to 34.832,282, the compound quantity $14,937\,x - 1998\,x\,x + 80\,x^3 - x^4$ will decreaſe from 5004.806,744,78 to 4999.996,387,922,353,261,456,029,424. Therefore, while x increaſes from 34.83 to ſome quantity a little leſs than 34.832,282, that compound quantity will decreaſe from 5004.806,744,78 to 5000; or, in other words, the true value of x in the equation $14,937\,x - 1998\,x\,x + 80\,x^3 - x^4 = 5000$ will be leſs than 34.832,282. Q. E. D.

A further Approach to the Value of the ſaid fourth, or greateſt, Root, by a ſecond Proceſs of Mr. Raphſon's Method of Approximation.

Article 67. In order therefore to determine this value of x to a ſtill greater degree of exactneſs, let us ſuppoſe it to be equal to $34.832,282 - w$, or, (putting $k = 34.832,282$, in order to avoid the needleſs repetition of long numbers,) to $k - w$.

Then we ſhall have $x\,x = \overline{k - w}^2$ $(= k^2 - 2\,k\,w + w\,w = k\,k - 2 \times 34.832,282 \times w + w\,w)$
$= k\,k - 69.664,564,\, w + w\,w$;

and $x^3 = \overline{k - w}^3$ $(= k^3 - 3\,k\,k\,w + 3\,k\,w\,w - w^3$
$= k^3 - 3 \times 1213.287,869,327,524,\times w + 3\,k\,w\,w - w^3)$
$= k^3 - 3639.863,607,982,572,\, w + 3\,k\,w\,w - w^3$;

and $x^4 = \overline{k - w}^4$ ($= k^4 - 4k^3w + 6kkww - 4kw^3 + w^4$
$= k^4 - 4 \times 42{,}261.585{,}211{,}595{,}466{,}329{,}768,\ w$
$+ 6kkww - 4kw^3 + w^4$)
$= k^4 - 169{,}046.340{,}846{,}381{,}865{,}319{,}072,\ w + 6kkww$
$- 4kw^3 + w^4$.

Therefore $14{,}937x$ is $= 14{,}937k - 14{,}937w$; and
$1998xx$ is ($= 1998kk - 1998 \times 69.664{,}564\ w$ + &c)
$= 1998kk - 139{,}189{,}798{,}872,\ w$ + &c;
and $80x^3$ is ($= 80k^3 - 80 \times 3639.863{,}607{,}982{,}572,\ w$ + &c,)
$= 80k^3 - 291{,}189.088{,}638{,}605{,}760,\ w$ + &c;

and consequently $14{,}937x - 1998xx + 80x^3 - x^4$ is $=$

$$\left\{\begin{array}{l} 14{,}937k - 14{,}937w \\ -1998kk + 139{,}189.798{,}872w - \&c \\ +\ 80k^3 - 291{,}189.088{,}638{,}605{,}760,\ w + \&c \\ -\ k^4 + 169{,}046.340{,}846{,}381{,}865{,}319{,}072,\ w - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{l} 14{,}937k + 308{,}236.139{,}718{,}381{,}865{,}319{,}072,\ w \\ -1998kk - 306{,}126.088{,}638{,}605{,}760,\ w \\ +\ 80k^3 \\ -\ k^4 \end{array}\right\}$$

$$= \left\{\begin{array}{l} 14{,}937k + 2{,}110.051{,}079{,}776{,}105{,}319{,}072,\ w \\ -1998kk \\ +\ 80k^3 \\ -\ k^4 \end{array}\right.$$

$= 4999.996{,}387{,}922{,}353{,}261{,}456{,}029{,}424$
$+ 2{,}110.051{,}079{,}776{,}105{,}319{,}072,\ w.$

Therefore 5000 (which is $= 14{,}937x - 1998xx + 80x^3 - x^4$) will be also $= 4999.996{,}387{,}922{,}353{,}261{,}456{,}029{,}424 + 2{,}110.051{,}079{,}776{,}105{,}319{,}072,\ w$; and, (subtracting $4999.996{,}387{,}922{,}353{,}261{,}456{,}029{,}424$ from both sides,) $2{,}110.051{,}079{,}776{,}105{,}319{,}072,\ w$ will be $= 0.003{,}612{,}077{,}646{,}738{,}543{,}970{,}576$; and consequently w will be $=$ $\frac{.003{,}612{,}077{,}646{,}738{,}543{,}970{,}576}{2{,}110.051{,}079{,}776{,}105{,}319{,}072} = .000{,}001{,}711{,}843{,}6$. Therefore $k - w$, or $34.832{,}282 - w$, or x, will be ($= 34.832{,}282 - .000{,}001{,}711{,}843{,}6$)
$= 34.832,$

$=$ 34.832,280,288,156,4, of which the first thirteen figures, 34.832,280,288, 15, are probably exact. Q. E. I.

Article 68. The value of this root of the biquadratick equation $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$, assigned by Dr. Wallis, is 34.832,280,28; of which all the figures are exact, if no mistake has been made in the foregoing calculations, by which it's more accurate value has been found to be 34.832, 280,288,156,4.

Article 69. It appears therefore, that the four roots of the proposed biquadratick equation $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$, are

0.350,987,045,866,03,
12.756,441,794,480,74,
32.060,290,871,498,4,
and 34.832,280,288,156,4.

Q. E. I.

Article 70. Of these four roots the second, to wit, 12.756,441,794,480,74, has been shewn with certainty (in Art. 47.) to be exact in the first twelve figures, to wit, 12.756,441,794,4, and with a high degree of probability to be exact in the four remaining figures .000,000,000,080,74, so that the whole sixteen figures, 12.756,441,794,480,74, may be safely concluded to be exact. See Art. 48. But the like inquiries have not been made concerning the other three roots, to wit, 0.350,987,045,866,03, and 32.060,290,871,498,4, and 34.832,280,288,156,4, so as to enable us to determine with certainty how many of their figures are exact. The Method of conducting such inquiries concerning those three roots has, however, been so fully explained in the case of the said second root, that it seems to be unnecessary to exhibit new applications of it to the cases of those other roots. And therefore I shall here conclude this paper.

Note. This Paper was drawn-up in the year 1777, and communicated to the learned Mr. Charles Hutton, (now Dr. Hutton,) Professor of Mathematicks at the Military Academy at Woolwich; who returned it to me with the following approbation of the Method I had employed in the resolution of the foregoing equations, though not of the Method I had used to determine the numbers of figures

figures that were exact in the values of the roots which had been obtained, which he thought rather tedious. Speaking of the papers containing the foregoing Tract, he ſays, " They are very clear and methodical; and the Method " is, I think, *the beſt general one* that can be applied to the numerical Solution " of all equations above a quadratick, and is that which I myſelf always uſe: " for I think it not worth while either to remember, or to turn to, any of Raph- " ſon's particular rules, or to exterminate either the ſecond, or any other terms. " And I think the approximation by ſimple equations much better than by qua- " draticks. As to your Methods of verifying the number of figures found in " the root, I think them too tedious; nor can I think of any ſo ſhort as that of " ſubſtituting the number found in [the firſt, or left-hand, ſide of] the equation, " and then [ſubſtituting in it in the ſame manner] the ſame number a little " increaſed, &c *; nor ſhould I uſe any other Method myſelf, if I were perform- " ing ſuch an operation."

I agree with Dr. Hutton in thinking the Methods uſed in the foregoing Tract to verify the number of figures found in the root, to be too tedious, and I therefore would adviſe my readers to have recourſe rather to Mr. Ivory's Method above-deſcribed in pages 353, &c - - - 359, for that purpoſe, or to the other Method uſed above in pages 376 and 390, which is the ſame in ſubſtance and principle as the Method employed in this laſt Tract, but more conciſely expreſſed, and rendered eaſier and fitter for practice by the omiſſion of ſome unneceſſary computations.

I will further obſerve on the foregoing Tract, that all the calculations contained in Art. 26, 27, 28, &c, - - - - 38, (by which we obtain the number 12.74 for a near value of the ſecond root of the equation $14{,}937\,x - 1998\,xx + 80\,x^3 - x^4 = 5000$ by means of the knowledge we have already obtained of the value of the firſt, or leaſt, root of it, which has been found to be = 0.350, 987,045,866,03,) might have been ſpared, and the ſame near value of the ſaid ſecond root, to wit, the number 12.74, or a ſtill nearer value of it, might have been obtained much more eaſily in the Method recommended by *Stevinus* and *Kerſey*, that is, by two or three conjectures and trials, as, for inſtance,

* An example of the Method of verifying the number of figures that are exact in the value of the root that has been obtained by one of theſe approximations, that is here alluded to by Dr. Hutton, may be ſeen in my octavo volume of *Tracts on the Reſolution of Equations by the Methods of Approximation*, publiſhed in the year 1800, in pages 317, 318, 319, and 320.

by

by ſuppoſing this ſecond root to be equal, firſt, to 10, and then to 12, and trying the effects of theſe conjectures. For the labour of making theſe conjectures and trials would have been much leſs than that of reducing the biquadratick equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$ to the cubick equation $x^3 - 79.6491 \times xx + 1970.051{,}130{,}81 \times x = 14{,}245.709{,}068{,}198{,}771$, (as is done in Art. 27,) and afterwards putting $p = 79.6491$, and $q = 1970.051{,}130{,}81$, and $r = 14{,}245.709{,}068{,}198{,}771$, and computing the values of pp, $\frac{pp}{3}$, $\frac{pp}{4}$, $3q$, $pp - 3q$, $2pp - 6q$, $\surd pp - 3q$, $\overline{2pp - 6q} \times \surd \overline{pp - 3q}$, $\frac{\overline{2pp - 6q} \times \surd \overline{pp - 3q}}{27}$, pq, $\frac{pq}{3}$, $\frac{p}{3}$, $\frac{pp}{9}$, $\frac{p^3}{27}$, $\frac{2p^3}{27}$, $\frac{pq}{3} - \frac{2p^3}{27}$, and $\frac{pq}{3} - \frac{2p^3}{27} - \frac{\overline{2pp - 6q} \times \surd \overline{pp - 3q}}{27}$, (as is done in Art. 28,) and comparing theſe quantities with each other in order to diſcover that the ſaid cubick equation $x^3 - 79.6491 \times xx + 1970.051{,}130{,}81 \times x = 14{,}245.709{,}068{,}198{,}771$ has three roots, (that is, real and poſitive roots,) of which one is leſs, and the other two are greater, than $-\frac{p}{3}$, or 26.5497, (as is done in Art. 29,) and then reducing the trinomial cubick equation $x^3 - 79.6491 \times xx + 1970.051{,}130{,}81 = 14{,}245.709{,}068{,}198{,}771$ to the binomial cubick equation $y^3 - 144.608{,}579{,}46 \times y = 246.170{,}166{,}297{,}206$, by ſubſtituting $\frac{p}{3} - y$ inſtead of x in the former quation, (as is done in Art. 30,) and determining whether this binomial cubick equation comes under Cardan's ſecond rule, or whether it may be reſolved by the triſection of a circular arc, (as is done in Art. 31,) and then reſolving the ſaid binomial cubick equation by the triſection of a circular arc (as is done in Art. 32,) and performing ſome further computations in Art. 33, 34, 35, 36, 37, and 38. Yet, though theſe computations do not lead us by the ſhorteſt and eaſieſt road to the number 12.74, which is a pretty good near value of the ſaid ſecond root of the equation $14{,}937x - 1998xx + 80x^3 - x^4 = 5000$, which we were then inveſtigating, I cannot conſider them as uſeleſs; becauſe it is fit that Students of Algebra, who are learning the Methods of reſolving Algebräick equations, ſhould know the connexions which different roots of the ſame equation have with each other, and how the knowledge of one of them may be made inſtrumental to the diſcovery of the others. When they are well acquainted with theſe connexions, they will be able to uſe their own diſcretion and

judgement

judgement to determine, in each particular equation, that they may have occaſion to reſolve (when ſuch equation has more than one root,) whether it will be moſt convenient, when one of the roots of the equation is known, to make uſe of that root as an inſtrument to aſſiſt them in diſcovering the other roots, or to drop all further conſideration of the root already known, and ſeek for the others by the ſame methods which they would have reſorted to, if the former, or known, root had not been known to them. I believe it will be found that the former Method of proceeding will be moſt convenient in ſome caſes, and the latter Method of proceeding in ſome others. And the diſtinction of theſe caſes from each other cannot well be pointed-out by any general rules, but muſt be left to the ſagacity and experience of the calculator in every particular equation of that kind that he has occaſion to reſolve.

Inner Temple, July 17, 1804.

FRANCIS MASERES.

A COMPUTATION

OF THE

LENGTH OF THE SINE OF A CIRCULAR ARC OF ONE MINUTE OF A DEGREE,

BY MR. RAPHSON'S METHOD OF RESOLVING HIGH EQUATIONS BY APPROXIMATION;

DERIVED FROM THE

CHORD OF AN ARCH OF SIXTY DEGREES,

(WHICH IS EQUAL TO THE RADIUS OF THE CIRCLE,)

BY ONE QUINQUISECTION, ONE TRISECTION, TWO BISECTIONS, A SECOND QUINQUISECTION, A SECOND TRISECTION, AND A THIRD BISECTION, AND BY TAKING THE HALF OF THE CHORD LAST-FOUND.

Performed in the Year 1777,

BY MR. CHARLES HUTTON, (now Dr. HUTTON,)

PROFESSOR OF MATHEMATICKS IN THE ROYAL MILITARY ACADEMY AT WOOLWICH, IN KENT.

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A ſhort Account of the following Computation.

IT is well known to Mathematicians that the Computation of the length of the Sine of an Arc of one Minute is a Problem of fundamental importance in Trigonometry towards forming a Table of the Sines and Tangents and Secants of circular arches to every minute of a Degree throughout the whole Quadrant. And the uſual Method of ſolving this Problem in the middle of the ſeventeenth century, (or before Sir Iſaac Newton had found-out and pub-

liſhed to the world the infinite Series $a - \frac{a^3}{2.3.rr} + \frac{a^5}{2.3.4.5.r^4} - \frac{a^7}{2.3.4.5.6.7.r^6}$ $+ \frac{a^9}{2.3.4.5.6.7.8.9.r^8} - \frac{a^{11}}{2.3.4.5.6.7.8.9.10.11.r^{10}}$ + &c for expreſſing the ſine s of any arc called a in powers of the ſaid arc and the radius r,) was to biſect the arc of 30 degrees, (of which the ſine is $= \frac{r}{2}$) continually, or to inveſtigate the ſines of the halves of the ſeveral arches of which the ſines ſucceſſively become known, till the calculator had obtained the value of the ſine of an arch that was leſs than an arch of one minute. This required no fewer than eleven ſucceſſive biſections of the arch of 30 degrees, by the laſt of which we ſhould obtain the length of the ſine of an arch of $52''$, $44'''$, $3''''$, $45'''''$ *. Theſe numerous biſections require a great deal of laborious calculation; and, when they have been performed with due care, they do not give us the value of the ſine of one minute, which was ſought, but only that of an arch that is leſs than one minute; from which we are afterwards obliged to derive the value of the ſine of one minute by means of the rule of Proportion. This made me conjecture that it might be more convenient to employ quinquiſections and triſections of an arch as well as biſections, in this Inveſtigation, becauſe then we ſhould arrive at laſt at the very quantity which we were ſeeking, to wit, the length of the ſine of one minute. And, when I mentioned this matter to Mr. Hutton, in the year 1777, he agreed with me in this opinion, and, at my deſire, undertook and performed, with great ſkill and ſucceſs, the following Computation of the length of the Sine of an Arch of one Minute in the manner that had been ſuggeſted. He ſuppoſes the radius of the circle to be called 1, and conſequently the chord of an arch of 60 degrees (which is equal to the radius of the circle,) to be equal to 1 likewiſe. And he then begins his calculation by quinquiſecting the arch of 60 degrees, or finding the chord of it's fifth part, or of an arch of 12 degrees, by reſolving, by Mr. Raphſon's Method of Approximation, the equation $a^5 - 5a^3 + 5a = b$, which expreſſes the relation between thoſe two chords, upon a ſuppoſition that b denotes the chord of the greater arch, of which the chord is given, to wit, in the preſent caſe, the chord of 60 degrees, and that the leſſer of the two values of a denotes the chord of

* The intermediate arches are as follow; to wit, an arch of 15° ; of 7°, 30′ ; of 3°, 45′ ; of 1°, 52, 30″ ; of 56′, 15″ ; of 28′, 7″, 30‴ ; of 14′, 3″, 45‴ ; of 7′, 1″, 52‴, 30⁗ ; of 3′, 30′, 56″, 15‴ ; and of 1′, 45″, 28‴, 7⁗, 30⁗⁗.

it's

it's fifth part, or of an arch of 12 degrees. And he finds this latter chord to be = 0.209,056,926,535,307; and he afterwards proves this number to be true in all it's fifteen figures.

He then trifects the arch of 12 degrees, or finds the chord of it's third part, or of an arch of 4 degrees, by refolving, by Mr. Raphfon's Method of Approximation, the equation that expreffes the relation between the faid two chords, to wit, the equation $3a - a^3 = c$, in which c denotes the chord of the greater arch, to wit, 12 degrees, and is therefore = 0.209,056,926,535,307, and the leffer of it's two roots, or the leffer value of a, denotes the chord of the third part of the faid greater arch, or the chord of an arch of 4 degrees. And he finds this latter chord to be = 0.069,798,993,405,002; and he afterwards proves this number to be true in all it's fifteen figures.

The next operation is to bifect an arch of 4 degrees, or to find the chord of it's half, or of an arch of 2 degrees, by refolving the equation that expreffes the relation between the chords of thofe arches, to wit, the equation $4a^2 - a^4 = c^2$, in which c denotes the chord of the greater arch, to wit, 4 degrees, and is therefore = 0.069,798,993,405,002, and the leffer of the two values of a denotes the chord of it's third part, or of an arch of 2 degrees. And he finds this latter chord to be = 0.034,904,812,874,569; and he afterwards proves this number to be true in the firft 14 places of figures, to wit, 0.034,904,812,874,56, and very nearly true alfo in the laft figure 9, which muft be either an 8 or a 9.

The next operation is to bifect an arch of 2 degrees, or to find the chord of it's half, or of an arch of 1 degree, by a calculation fimilar to that employed in the bifection of an arch of 4 degrees. And Dr. Hutton finds the chord of the latter arch, or of an arch of 1 degree, to be = 0.017,453,070,996,753; and he afterwards proves this number to be true in all it's fifteen figures.

Having thus found the chord of an arch of 1 degree, or 60 minutes, to be = 0.017,453,070,996,753, Dr. Hutton then proceeds to quinquifect this arc, or to find the chord of it's fifth part, or of an arch of 12 minutes, by refolving, by Mr. Raphfon's Method of Approximation, the equation $a^5 - 5a^3 + 5a = b$, which expreffes the relation between thofe two chords, upon a fuppofition that b denotes the chord of the greater arch, of which the chord is given,

given, or known, to wit, in the preſent caſe, the chord of an arch of 1 degree, or 60 minutes, (which is = 0.017,453,070,996,753,) and that the leſſer of the two values of a denotes the chord of it's fifth part, or of an arch of 12 minutes. And he finds this latter chord to be = 0.003,490,656,731,798; and he afterwards proves this number to be true in the firſt 14 places of figures, to wit, in the figures 0.003,490,656,731,79, and very nearly true in the 15th figure 8.

He then triſects the arch of 12 minutes, or finds the chord of it's third part, or of an arch of 4 minutes, by reſolving, by Mr. Raphſon's Method of Approximation, the equation that expreſſes the relation between thoſe two chords, to wit, the equation $3a - a^3 = c$, in which c denotes the chord of the greater arch, to wit, 12 minutes, and is therefore = 0.003,490,656,731,798, and the leſſer of the two values of a denotes the chord of it's third part, or of an arch of 4 minutes. And he finds this latter chord to be = 0.001,163,552,769,027; and he afterwards proves this number to be true in the firſt 14 places of figures, to wit, 0.001,163,552,769,02, and very nearly true in the laſt figure 7.

He then biſects the arch of 4 minutes, or finds the chord of an arch of 2 minutes, by reſolving the equation that expreſſes the relation between the chords of thoſe two arches, to wit, the equation $4a^2 - a^4 = c^2$, in which c denotes the chord of the greater arch, to wit, 4 minutes, and is therefore = 0.001,163,552,769,027, and the leſſer of the two values of a denotes the chord of it's half, or of an arch of 2 minutes. And he finds this latter chord to be = 0.000,581,776,409,127; and afterwards proves that this number is true in the firſt 14 places of figures, to wit, 0.000,581,776,409,12, and that it is very nearly ſo in the laſt figure 7.

He then divides this number 0.000,581,776,409,127, (which is equal to the chord of an arch of 2 minutes,) by 2, and the Quotient 0.000,290,888,204,563,5 will be equal to the ſine of half the ſaid arch of 2 minutes, or to the ſine of an arch of 1 minute, which is the quantity that was to be found by the calculation. And he afterwards proves that this value of the ſaid ſine, to wit, the number 0.000,290,888,204,563,5 is true in the firſt 15 places of decimal fractions, to wit, in the figures 0.000,290,888,204,563, and that it's laſt figure, 5, in the 16th place of decimals, differs from the truth by only an unit.

unit *. And thus the length of the ſine of an arch of only 1 minute is, by this valuable and well-conducted computation, found exactly to 15 places of decimal figures, without the help of Sir Iſaac Newton's Series for expreſſing the ſine of a given arch of a circle in terms involving the powers of the arch and radius, which is obtained by firſt inveſtigating, by means of Sir Iſaac's Method of Fluxions, the infinite ſeries for expreſſing the arch of a circle in terms involving the powers of it's ſine and the radius, and afterwards reverting the ſaid ſeries according to the known methods of reverſion, which are very abſtruſe and difficult to underſtand clearly, and likewiſe often very laborious in the practical application of them to particular ſerieſes.

Dr. Hutton's opinion reſpecting this Method of computing the ſine of an arch of one minute is thus expreſſed in a Letter to me, dated at Woolwich, on the 13th of March 1777.

Sir,

Having ſome leiſure time at the receipt of your Letter deſiring the Computation of the Sine of 1 Minute to 12, or 14, places of figures, I have made that computation myſelf in the manner you wiſhed to have it done, and I have herein incloſed you the whole proceſs at large. Each equation is carried to 15 places of decimals, the laſt figure being always that which is neareſt to the truth in the 15th place of decimals.—You will perceive that my manner of performing the Diviſions, Multiplications, and Extractions of roots (which are ſtrictly true,) ſaves a great deal of trouble in ſuch operations.—I was always of opinion that Raphſon's was an eaſier and readier Method of reſolving high equations than Halley's, &c, as it is both clearer and requires leſs time. But this Method of obtaining the Sine of 1′ is far inferiour, in point of expedition, to that of Infinite Serieſes. However, it is both quicker and more ſatisfactory than the old way, by the continual biſections of the arc of 30 degrees, as you very properly remark.

I remain, &c.

C. HUTTON.

* The more exact value of the ſine of an arch of 1 minute, as derived from Sir Iſaac Newton's Series $s = a - \frac{a^3}{6r^2} + \frac{a^5}{120r^4} - \frac{a^7}{5040r^6} +$ &c, is 0.000,290,888,204,563,424, which is true in all it's figures. See the third volume of this Collection of Tracts, intitled, *Scriptores Logarithmici*, pages 407, 408.

In

In another Letter from Dr. Hutton, dated from Woolwich, on the 25th of Auguſt of the ſame year 1777, I find the following obſervations on this Computation of the Sine of an Arch of 1 Minute.

Sir,

You herewith receive again the Computation of the Sine of one Minute, with the proof of each of the ſeven extractions, or ſections; which indeed took-up more time than I at firſt expected, the Labour being hereby nearly doubled. I have had a fair copy taken, in order to place each Proof on the right-hand page facing it's own proper ſection, as you will perceive.—You will perceive that I have availed myſelf of the Method you mentioned, of raiſing one power of a large number from [the ſame power of] another differing from it by an unit in the laſt place of figures; and in theſe powers I have omitted ſuch terms as I perceived would be too ſmall to take into the general ſum.

I am, &c.

CHARLES HUTTON.

The Method, alluded-to in this Letter, of raiſing one power of a large number from the ſame power of another number differing from the former number by an unit in the laſt place of figures, is illuſtrated by an example in my Tract, intitled, "*Obſervations on Mr. Raphſon's Method of reſolving Equations by Approximation*," in the octavo volume of Tracts on the Reſolution of Affected Algebräick Equations by various Methods of Approximation, that was publiſhed in the year 1800, and ſold by Mr. John White, Bookſeller in Fleet-Street, in pages 317, 318, 319, and 320 of that volume.

After theſe preliminary obſervations on Dr. Hutton's valuable Computation of the Sine of a circular Arch of 1 Minute in a Circle of which the radius is called 1, I ſhall preſent my readers with the Computation itſelf, as it is drawn-up and exhibited in the fair copy of it mentioned in the foregoing Letter.

Dr. HUTTON'S

DR. HUTTON'S COMPUTATION OF THE LENGTH OF THE SINE OF A CIRCULAR ARC OF ONE MINUTE OF A DEGREE.

PROBLEM I.

To quinquisect an Arc of 60 *Degrees.*

THE Equation for the quinquisection of any Arc, is $a^5 - 5a^3 + 5a = b$, where b is the chord of the given arc, and a the chord of it's fifth part. And Raphson's Theorem is $x = \frac{b + 5g^3 - g^5 - 5g}{5 + 5g^4 - 15g^2}$, or $\frac{b + 5g^3 - g^5 - 5g}{5 \times 1 + g^4 - 3g^2}$

Here $b = 1$ the chord of 60°

Let $g = .2$;

$.2 = g$	$1. = b$	1
$.04 = g^2$	$.04 = 5g^3$	$0.0016 = g^4$
$.008 = g^3$	$+1.04$	$+1.0016$
$.0016 = g^4$	$.00032 = g^5$	$-0.12 = 3g^2$
$.00032 = g^5$	$1.0 = 5g$	$+ .8816$
	-1.00032	$\times 5$
	$+.03968$	$+4.4080$

1. Then $b + 5g^3 - g^5 - 5g = +0.3968$
& $5 + 5g^4 - 15g^2 = +4.408$

Therefore $+4.408) +.03968 ($ $+.009$ *
$1 = x$

$.2$
$+.009$
Hence $g = .209$

$.209 = g$	$1. = b$	
$.209$	$0.045,646,65 = 5g^3$	
1881	$+1.045,646,65$	
418	$1.045 = 5g$	
$.043,681 = g^2$	$.000,398,78 = g^5$	
$.209$	$-1.045,398,78$	
393129	$+.000,247,87$	
87362		
$.009129329 = g^3$	$+1.001,908,03 = 1 + g^4$	
902 inverted	$-0.131,043 = 3g^2$	
1825866	$+ .870,865,03$	
82164	$\times 5$	
$.001908030 = g^4$	$+4.354,325,15$	
902		
381606		
17172		
$.000398778 = g^5$		

2. Again, $b + 5g^3 - g^5 - 5g = +.000,247,87$
$5 + 5g^4 - 15g^2 = +4.354,325,15.$
$4.354,325,15).000,247,87 ($ $+.000,056,92 = x$
3015
402
10

$.209$
$.000,056,92$
$g = .209,056,92$

* Raphson's Division of this is wrong.

3. Again, $\frac{b+5g^3-g^5-5g}{5\times 1+g^4-3g^2} = +.000,000,028,454,586 \div +4.353,978,608,881,255.$

4.353,978,608,881,255).000,000,028.454,586(.000,000,006,
2330714 [535,307 = x
153725
23106
1336
30

+.20905692
+.0000000006535307

Therefore .209,056,926,535,307 = the chord of 12°.

.209,056,92 = g
.209,056,92
41811384
188151228
125434152
104528460
188151228
41811384

.043,704,795,799,886,4 = g^2
29650902

8740959159977
393343162199
2185239790
262228775
39334316
874096

.009,136,789,999,153 = g^3
29650902

182735799831
82231109992
456839500
54820740
8223111
182736

.001,910,109,175,910 = g^4
29650902

38202183518 2
1719098258 3
95505459
11460655
1719098
38202

.000,399,321,541,179 = g^5

1 - - - - - - - - - - = b.
0.045,683,949,995,765 = 5g^3
+1.045,683,949,995,765

1.045,284,60 - - - - = 5g
.000,399,321,541,179 = g^5
−1.045,683,921,541,179
+.000,000,028,454,586

+1.001,910,109,175,910 = 1 + g^4
−0.131,114,387,399,659 = $3g^2$
0.870,795,721,776,251
5

4.353,978,608,881,255

Proof

N.B. From this process, it may be observed that Mr. Raphson did not carry his approximations out so far as he might have done, perhaps owing to his mistake of putting an 8. for a 9 in the first quotient.

Proof of the foregoing Quinquisection.

.209,056,926,535,307 = a

a inverted 703535629650902

418113853070614 0
188151233881771
10452846326 77
1254341559 21
188151233 88
41811385 3
12543415 6
1045284 6
62717 1
10452 8
627 2
146

.043,704,798,532,388,69 = a^2

703535629650902

87409597064777
39334318679 15
2185239926 6
262228791 2
39334318 7
874096 0
262228 8
21852 4
1311 1
218 5
131
3

.009,136,790,856,025,9 = a^3

·703535629650902

1827358171205 2
822311177042
456839542 8
54820745 1
8223111 8
182735 8
54820 7
4568 4
274 1
45 7
2 7
1

For the next or b, take $a - .1^{15} = a - \frac{1}{10^{15}}$, which is one lefs in the 15th place of figures.

Then $b^5 = a^5 - 5a^4 \times .1^{15}$

$-5b^3 = \quad -5a^3 + 15a^2 \times .1^{15} -$ &c.

$+5b = \quad +5a - 5 \times .1^{15}$

$b = a^5 - 5a^4 \times .1^{15} - 5a^3 + 15a^2 \times .1^{15} + 5a - 5 \times .1^{15}$

But the
first b was $= a^5 \quad -5a^3 \quad +5a$

Take the upper
from the under,
and the difference is $\frac{5a^4}{10^{15}} - \frac{15a^2}{10^{15}} + \frac{5}{10^{15}}$,
or $5 \times \frac{a^4 - 3a^2 + 1}{10^{15}}$.

Now $a^4 + 1 = \quad 1.001,910,109,414,756,6$
$\quad -3a = \quad -0.627,170,779,605,921,$

Divide by 10^{15}). $.374,739,329,808,835,6 = a^4 - 3a + 1$

Then $\frac{a^4 - 3a^2 + 1}{10^{15}} = \quad .000,000,000,000,000,374,74$

and $5 \times \frac{a^4 - 3a^2 + 1}{10^{15}} = .000,000,000,000,001,873,70,$

and this number is the difference between the firſt or greater value of b, and the next leſs, or that which reſults by taking a leſs by 1 in the 15th place of decimals. Conſequently

From - - 1.000,000,000,000,000,8
Take - - 0.000,000,000,000,001,87

Leaves $b = \quad .999,999,999,999,998,93,$

which is ſomething too little, and is rather more be-

.001,910,109,414,756,6 $= a^4$
.703535629650902

3820218829513
171909847328
955054707
114606565
17190985
382022
114607
9551
573
96
6

.000,369,321,603,595,3 $= a^5$
1.045,284,632,676,535, $= 5a$

1.045,683,954,280,130,3 $= a^5 + 5a$
Subt. .045,683,954,280,129,5 $= 5a^3$

1.000,000,000,000,000,8 $= b$,
but ſomething too great.

low (1) the true value than the other is above it. Conſequently the true value of a lies between 7 and 6 in the 15th figure, but nearer to 7 than to 6. And the number a is rightly found.

PROBLEM II.

To triſect an Arc of 12 *Degrees.*

The Equation for the triſection is $3a - a^3 = c$.

And Raphſon's Theorem is $x = \frac{c+g^3-3g}{3-3g^2} = \frac{c+g^3-3g}{3\times 1-g^2}$:

Here $c = .209,056,926,535,307$.
Let $g = .07$
1. Then $c+g^3-3g = .000,600$,
and $3-3g^2 = +2.9853$
Therefore $+2.9853)-.000,600(-.0002 = x$
$+.07$
$-.0002$
$g = +.0698$

$.07 = g$
$.07$
$.0049 = g^2$
07
$+.000,343 = g^3$
$+.209,057 = c$
$+.209,400$
$-.21 \quad = 3g$
$-000,600$

3.0
$0.0147 = 3g^2$
2.9853

2. Again,

2. Again $c + g^3 - 3g = -.000,003,00$
& $3 - 3g^2 = 2.985,383,88$
$\therefore$ 2.985,383,88) — .000,003,00(—.000,001,01 $= x$
2

+.0698
—.000,001,01
$g = +.069,798,99$

.0698 $= g$
.0698
5584
6282
4188
.004,872,04 $= g^2$
8960 inverted
29232
4385
390
.000,340,07 $= g^3$
.209,056,93 $= c$
.209,397,00
.2094 $= 3g$
—.000,003,00

3.0
0.01461612
2.98538388

---

3. Again $c + g^3 - 3g = +.000,000,010,165,239$
& $3 - 3g^2 = 2.985,384,302,984,940$
$\therefore$ 2.985,384,3) +.000,000,010,165,239(+.000,000,003,405,002 $= x$
1209086
14932
5

+.069,798,99
+.000,000,003,405,002,
+.069,798,993,405,002 = the chord of 4°.

.069,798,99 $= g$
.069,798,99
62819091
62819091
55839192
62819091
48859293
62819091
41879394
.004,871,899,005,020,1 $= g^2$
99897960
292313940301
43847091045
3410329304
438470910
38975192
4384709
438471
.000,340,053,629,932, $= g^3$
.209,056,926,535,307, $= c$
.209,396,980,165,239,
.209,396,97 $= 3g$
.000,000,010,165,239,
+3.0
—0.014,615,697,015,060, $= 3g^2$
+2.985,384,302,984,940,

Proof

Proof of the Trisection of 12 *Degrees.*

Here $3a - a^3$ muſt be $= c$, where a is $= .069,798,993,405,002$, &c $= .209,056,926,535,307$.

$a = .069,798,993,405,002,$
200504399897960

418793960430001
62819094064500
4885929538350
628190940650
55839194720
6281909410
628190940
20939700
2791960
34900
100

$a^2 = .004,871,899,480,351,5$
200504399897960

29231396882110
4384709532320
341032963620
43847095320
3897519580
438470950
43847100
1461570
194880
2440

$-a^3 = .000,340,053,679,698,9$
$3a = .209,396,980,215,006,$

$3a - a^3 = .209,056,926,535,307,1$
$c = .209,056,926,535,307,0$

Too great in the 16th place by 1.

For the next or b, take $a - \frac{1}{10^{15}} = a - \cdot 1^{15}$.

Then $3b = 3a - \frac{3}{10^{15}}$

and $-b^3 = -a^3 + \frac{3a^2}{10^{15}}$ &c.

$\therefore 3b - b^3 = 3a - a^3 - \frac{3}{10^{15}} + \frac{3a^2}{10^{15}}$ &c.

which taken from $3a - a^3$ the other value,

leaves $3 \times \frac{1-a^2}{10^{15}}$ for the difference between the two values.

Therefore from 1 000 &c.
take $0\,004,871,899,480,351,5 = a^2$
leaves $.995,128,100,5$ &c. $= 1 - a^2$

$10^{15})2.985,384,301,5$ &c. $= 3 \times \overline{1-a^2}$

$.000,000,000,000,002,985,$&c $= 3 \times \frac{1-a^2}{10^{15}}$

This difference taken from the greater value,
or $.209,056,926,535,307,1$, leaves, for the leſs value, $.209,056,926,535,304,115$,
which is much more below the true value of c than the other is above it, and therefore the true value lies between 2 and 1 in the 15th place, but much nearer to the former than to the latter.

PROBLEM

PROBLEM III.

To bisect the Arc of 4 Degrees.

The Equation for the biſection is $4a^2 - a^4 = c^2$, where c is the chord of the whole arc, and a is that of the half-arc.

By reſolving this quadratick equation we find $a = \sqrt{2 - \sqrt{4 - c^2}}$. And by uſing ·069,798,993,405,002 for c, this Theorem will give the value of $a =$ .034,904,812,874,569, as below.

.069,798,993,405,002, $= c$
200504399897960
418793960043001
628190940645O
48859295383 5
62819094065
5583919472
628190941
62819094
2093970
279196
3490
1

$-.004,871,899,480,351,5 = c^2$
$+4.00$

3·995,128,100,519,648,5 (1·998,781,654,038,191,4 $= \sqrt{4 - c^2}$
1 2.0

29)299
389)3851
3988)35028
39967)312410
399748)3264105
3997561)6612119
39975620)261455864
399756325)2160210885
399756330)161429260(4038191 4
····· 1526728
327459
7654
3656
58
18
2

.001,218,345,961,808,6(·034,904,812,874,569, = the chord of 2 degrees.
9
64)318
689)6234
69804)335961
698088)5674580
6980961 8987686
6980962)2006725(2874569
····· 610533
52056
3189
397
48
6

N. B. This laſt extraction might have been omitted, as the ſquare is the Number uſed in the next operation over the leaf.

Proof

Proof of the Bisection of 4 Degrees.

Here $4a^2 - a^4$ muſt $= c^2$, where $a = .034,904,812,874,569,\&c = .069,798,993,405,002$, or $c^2 = .004,871,899,480,351,5$, as was found by the laſt proving.

$a = .034,904,812,874,569$
$96547821840943 0$

10471443862371
1396192514982
314143315871
139619251 5
27923850 3
3490481
698096
279239
24433
1396
175
21
3

$a^2 = .001,218,345,961,808,6$
the ſame inverted $68081695438121 00$

12183459618
2436691924
121834596
97467677
3655038
4873 8
60917
10965
731
12
10

$-a^4 = -.000,001,484,366,882,6$
$+4a^2 = +.004\ 873,383,847,234 4$

$4a^2 - a^4 = .004,871,899,480,351,8$
$c^2 = .004,871,899,480,351,5$

$4a^2 - a^4 - c^2 = .000,000,000,000,000,3$

Therefore the value of a is above the truth.

Take therefore the next value $= 1$ leſs in the laſt or 15th place, ſo that b ſhall be $= a - \frac{1}{10^{15}}$.

Then $4b^2 = 4a^2 - \frac{8a}{10^{15}} +$ &c.

and $b^4 = a^4 - \frac{4a^3}{10^{15}} +$ &c.

Therefore $4b^2 - b^4 = 4a^2 - a^4 - \frac{8a - 4a^3}{10^{15}}$ &c. ſubtracted from $4a^2 - a^4$

leaves $4a \times \frac{2 - a^2}{10^{15}} =$ the difference between the two values.

2.000
$a^2 = 0.001,218,345,961,808,6$

$2 - a^2 = 1.998,781,654,038,191,4$
$4a$ inverted is $67289415291693 1$

199878165403819 1
59963449621145 7
17989034886343 7
11992689924229
1998781654 04
179890348863
39975633 08
9993908 27
1998781 7
799512 7
179890 3
15990 3
399 8
139 9
12 0

$10^{15}).279,068,398,445,298,3 = 4a \times \overline{2 - a^2}$

$.000,000,000,000,000,3 =$ the difference between the two values of c, and which being taken from the laſt or former number, there remains the exact value of c, and therefore the 15th figure is either an 8 or a 9.

PROBLEM

PROBLEM IV.

To bisect an Arc of 2 Degrees.

Using the laſt theorem here again, viz. $a = \sqrt{2 - \sqrt{4 - c^2}}$, when c is now = .034,904,812,874,569, or c^2 = .001,218,345,961,808,6, the value of a will be found as below.

$+40$

$-c^2 = -0.001,218,345,961,808,6$

$3.998,781,654,038,191,4(1.999,695,390,312,782,5 = \sqrt{4-c^2}$

1

29)2.99

389)3887

3989)38681

39986)278065

399920)3814940

3999385)21557938

39993903)156101319

399939069)3611961014

399939078)·12509393(03127825

·····

511221

111282

31294

3298

099

19

$+2.0$

$-\sqrt{4-c^2} = -1.999,695,390,312,782,5$

$.000,304,609,687,217,5(.017,453,070,996,753$ = the chord of 1 degree.

1

27)204

344)1560

3485)18496

34903)107187

3490607).24782175

3490614)·347926(0996753

·····

33771

2356

262

18

1

Proof of the Bisection of 2 Degrees.

Here again $4a^2 - a^4$ muſt $= c^2$, where $a =$ .017,453,070,996,753, and $c^2 =$ .001, 218,345,961,808,6.

$a =$ .017,453,070,996,753
357699070354710

1745307099675
1221714969773
69812283987
8726535498
523592130
12217147
157078
15708
1047
122
8

$a^2 =$ .000,304,609,687,217,5
inverted 5712786906403000

913829062
12184387
1827658
27415
1828
243
21
1

$a^4 =$ .000,000,092,787,061,5
$4a^2 =$ .001,218,438,748,870,0

$4a^2 - a^4 =$ .001,218,345,961,808,5
$c^2 =$ .001,218,345,961,808,6

$c^2 - 4a^2 - a^4 =$ .000,000,000,000,000,1.

Therefore $4a^2 - a^4$ is too little by 1 in the 16th figure.

Like as in the laſt extraction, the difference of the values will here be $4a \times \frac{2-a^2}{10^{15}}$.

2.00
$a^2 =$ 0.000,304,6 &c.

10^{15}) 1.999,695,39 &c. $= 2 - a^2$

.000,000,000,000,002 $= \frac{2-a^2}{10^{15}}$
.069,812,283,987,012 $= 4a$

.000,000,000,000,000,14 = the difference.
.001,218,345,961,808,50

.001,218,345,961,808,64
$c^2 =$ .001,218,345,961,808,60

.000,000,000,000,000,04. Therefore the next value is too great by 4 in the 17th figure.

And the root is right.

PROBLEM

PROBLEM V.

To quinquisect an Arc of 1 *Degree.*

The equation is $a^5 - 5a^3 + 5a = b = .017,453,070,996,753$.

and the theorem is $x = \frac{b+5g^3-g^5-5g}{5\times 1+g^4-3g^2}$.

Take $g = .0$

Then $b + 5g^3 - g^5 - 5g = +.017,453$, &c.

and $5 \times 1 + g^4 - 3g^2 = +5$.

Therefore $+5) + .017,453$,&c $(+.003,490,6 = x$

$$\begin{array}{r} 0.0 \\ +0.003,490,6 \\ \hline g = 0.003,490,6 \end{array}$$

2. Again $b + 5g^3 - g^5 - 5g = +.000,000,283,648,578$,

and $5 \times 1 + g^4 - 3g^2 = +4.999,817,236,416,885$.

$\therefore 4.999,817,236$,&c.$) + .000,000,283,648,621 (+.000,000,056,731,798$

$$\begin{array}{r} 33657769 \\ 3658856 \\ 158984 \\ 8990 \\ 3990 \\ 490 \\ 40 \\ 1 \end{array}$$

$$\begin{array}{r} +.003,490,6 \\ +.000,000,056,731,798 \\ \hline +.003,490,656,731,798 = \text{the chord of 12 minutes.} \end{array}$$

$$\begin{array}{r} .003,490,6 = g \\ .003,490,6 \\ \hline 209436 \\ 314154 \\ 139624 \\ 104718 \\ \hline .000,012,184,288,36 = g^2 \\ 6094300 \text{ inverted} \\ \hline 36552865 \\ 4873715 \\ 1096586 \\ 7311 \\ \hline .000,000,042,530,477 = g^3 \\ 6094300 \\ \hline 127591 \\ 17012 \\ 3828 \\ 26 \\ \hline .000,000,000,148,457 = g^4 \\ 6094300 \end{array}$$

445
59
13

.000,000,000,000,517 $= g^5$

.000,000,212,652,385 $= 5g^3$
.017,453,070,996,753 $- b$

+.017 453,283,649,138 $= b + 5g^3$

$-.017,453,000,000,517 = -g^5 - 5g$

+.000.000,283,648,621

+1.000,000,000,148,457 $= 1 + g^4$
$-$0.000 036,552,865,080 $= -3g^2$

.999,963,447,283,377
$\times 5$

+ .999,817,236,416,885

Proof of the Quinquisection of 1 *Degree.*

a = .003,490,656,731,798
inverted 8971376560 94200

104719701954
13962626927
3141591058
20943940
1745328
209439
24434
1047
35
24
3

a^2 = .000,012,184,684,418,9
8971376560 94300

365540532
48738738
10966216
73108
6092
731
85
4

a^3 = .000,000,042,532,550,6

Now by the Proof of the 1st Extraction it appears that the difference between the two values of $a^5 - 5a^3 + 5a$ and $b^5 - 5b^3 + 5b$, is $5 \times \frac{a^4 - 3a^2 + 1}{10^{15}}$, when the difference between a and b is $\frac{1}{10^{15}}$.

Now $a^4 + 1$ = 1.000,000,000,148,466,5
and $3a^2$ = 0.000,036,554,053,256,7

0.999,963,446,095,209,8 $= a^4 - 3a^2 + 1$

$\frac{5}{10^{15}}$)4.999,817 230,476.049,0

.000,000,000,000,005 the difference,

$a^5 - 5a^3 + 5a$ = .017,453,070,996,755
$b^5 - 5b^3 + 5b$ = .017,453,070,996,750
b = .017.453,070,996,753

.000,000,000,000,003. Error is here 3 in defect. Therefore the 15th figure is between 8 and 7.

a^3 =

$a^3 = .000,000,042,532,550,6$
$\quad 897137656094300$
$\quad 1275977$
$\quad 170130$
$\quad 38279$
$\quad 255$
$\quad 21$
$\quad 3$

$a^4 = .000,000,000,148,466,5$
$\quad \&c\ 6094300$
$\quad 4454$
$\quad 594$
$\quad 133$
$\quad 1$

$a^5 = .000,000,000,000,518,2$
$5a = .017,453,283,658,990$

$a^5 + 5a = .017,453,283,659,508,2$
$5a^3 = .000,000,212,662,753,0$

$a^5 - 5a^3 + 5a = .017,453,070,996,755,2$
$b = .017,453,070,996,753$

Error $.000,000,000,000,002.$

The error is here in excefs by 2 in the 15th place.

PROBLEM

PROBLEM VI.

To trisect an Arc of 12 *Minutes.*

The equation is $3a - a^3 = c$. And the theorem $x = \frac{c+g^3-3g}{3-3g^2}$; where c is = .003,490,656,731,798.

Take $g = 0$,

1. Then $\frac{c+g^3-3g}{3-3g^2} = \frac{c}{3} = .001,163,55\&c. = x$.

Hence 0

$$g = \frac{0.001,163,55}{.001,163,55}$$

2. Again $c + g^3 - 3g = +\ .000,000,008,307,061$
and $3 - 3g^2 = +\ 2.999,995,938,454,192$.

∴2.999,995,938,&c.).000,000,008,307,070(+.000,000,002,769,027
2307078
207081
27081
0081
021
0

.001,163,55
+.000,000,002,769,027
.001,163,552,769,027 = the chord of 4 minutes.

.001,163,55 = g
.001,163,55
581775
581775
349065
698130
116355
116355
.000,001,353,848,602,5 = g^2
55361100 inverted
1353849
135385
81231
4062
677
68
.000,000,001,575,272 = g^3
+.003,490,656,731,798 = c
.003,490,658,307,070 = $c+g^3$
−.003,490,65 = $-3g$
+.000,000,008,307,070 = $c+g^3-3g$
+3.
−0.000,004,061,545,808 = $-3g^2$
2.999,995,938,454,192 = $3-3g^2$

Proof

Proof of the Trisection of 12 *Minutes.*

a = .001,163,552,769,027
inverted 720967255361100

	1163552769
	698131661
	34906583
	5817764
	581776
	23271
	8145
	698
	105

a^2 = .000,001,353,855,046,2
720967255361100

	1353855
	812313
	40616
	6769
	677
	27
	9
	1

a^3 = .000,000,001,575,281,7
$3a$ = .003,490,658,307,081
$3a - a^3$ = .003,490,656,731,799,3
c = .003,490,656,731,798,
1

The error is in the 15th place a 1 in excefs.

By the proof of the third Extraction, it is found that the difference between the two values is $3 \times \frac{1-a^2}{10^{15}}$.

Now from 1.00
take a^2 = 0.000,001,353,855,046,2
10^{15}) .999,998,646,144,953,8
.000,000,000,000,001
3
The difference .000,000,000,000,003
taken from $3a - a^3$ = .003,490,656,731,799
leaves next lefs value = .003,490,656,731,796
But c = .003,490,656,731,798

The difference is in defect in the 15th place a 2. And therefore the true value is between 7 and 6 in the 15th place.

PROBLEM

PROBLEM VII.

To bisect an Arc of 4 Minutes.

The equation is $4a^2 - a^4 = c^2$, the root of which is $a = \sqrt{2 - \sqrt{4 - c^2}}$.

where c^2 is $= .001,163,552,769,027^2 = .000,001,353,855,046,309,$

720967255361100
1163552769027
1163552769031
69813166141
3490658307
581776385
58177638
2327105
814487
79813
10472
23
8

$$- .000,001,353,855,046,309 = -c^2$$

4.0

3.999,998,646,144,953,691(1.999,999,661,536,209,783 $= \sqrt{4 - c^2}$

29)299
389)3899
3989)39898
39989)399764
399989)3986361
3999989)38646044
39999986)264614395
399999926)2461447936
3999999321) 6144838091
3999999322) 2144838770(536209783
144839109
24839129
839133
39133
3133
333
13

2.0

2.0
1.999,999,661,536,209,783
.000,000,338,463,790,217(.000,581,776,409,127 = the chord of 2′.
108)884
1161)2063
11627)90279
116347)889002
1163546)7457317
1163552)476041(409127
10620
148
32
9

2).000,581,776,409,127 = the chord of 2 minutes

.000,295,888,204,563,5, = the fine of 1 minute, within an unit in the 16th place of decimals.

Proof of the Bisection of 4 Minutes.

$a = .000,581,776,409,127$
$\cdot 7219046771850 00$

290888204 5635
46542112 7302
5817764091
407243486 4
40724348 6
3490658 4
232710 6
52360
582
116
41

$a^2 = .000,000,338,463,790,216,7$
$209736483300 0000$

1015391
101539
27077
1354
203
10
2

$a^4 = .000,000,000,000,114,557,6$
$4a^2 = .000,001,353,855,160,866,8$

$4a^2 - a^4 = .000,001,353,855,046,309,2$
$c^2 = .000,001,353,855,046,309$

Error in excefs in the 19th place by 2 only.

By the proof of the third Extraction, the difference of the two values is $4a \times \frac{2-a^2}{10^{15}}$.

Now, from 2.0
Take $a^2 =$ 0.000,000,338,463,790,3
10^{15})1.999,999,661,536,209,7
.000,000,000,000,002
$4a = .002,327,105,6$ &c. mult. by
The difference .000,000,000,000,000,005
taken from
$4a^2 - 4a^4 = .000,001,353,855,046,309,2$
leaves the next
value $= .000,001,353,855,046,304,2$
But c^2 is $= .000,001,353,855,046,309$

The error is therefore in defect, and amounts only to 5 in the 18th place of decimal fractions.

Therefore the true value lies between 7 and 6 in the 15th place, but much nearer to 7 than to 6.

End of Dr. HUTTON's *Computation of the Length of the Sine of an Arch of one Minute of a Degree in a Circle of which the radius is called* 1.

MIRIFICI

Logarithmorum

Canonis descriptio;

Ejúsque usûs in utrâque

Trigonometriâ, ut etiam in

omni Logisticâ Mathematicâ,

Amplissimi, Facillimi, &
expeditissimi, explicatio.

Authore ac Inventore,

IOANNE NEPERO,

Barone Merchistonii,

&c. Scoto.

EDINBURGI,

Ex officinâ ANDREÆ HART

Bibliopôlæ, clɔ. dc. xiv.

EX OFFICINA VANDENHOECK
ET RUPRECHT.

IOANNE NEPERO
Vicecomite de Merchiston
Logarithmorum
Canonis descriptio:
Eiusdem de usu
Logarithmorum
Liber

Illuſtriſſimo, & optimæ ſpei

PRINCIPI CAROLO,

Potentiſſimi, & Invictiſſimi,

IACOBI D.G.

magnæ Britanniæ, Franciæ, &

Hiberniæ Regis, filio unico, Walliæ Prin-

cipi, Duci Eboraci, & Rotheſaiæ, magno Scotiæ

Seneſcallo, ac Inſularum Domino, &c.

D. D. D.

QUUM *nullum ſit ſtudium, vel doctrinæ genus, (Illuſtriſſime Princeps) quod generoſa ac heroïca ingenia, ad præclara quæque & ſublimia magìs acuat, contráque tarda & inſulſa pectora magìs obtundat, quàm matheſis: non mirandum eſt eruditos & magnanimos principes eam magnoperè præteritis omnibus ſæculis in deliciis habuiſſe, imperitos verò & ignavos homines eandem, velùt ignorantiæ ſuæ & ignaviæ hoſtem, ſemper odio acerrimo proſequutos eſſe. Cur non igitùr novitium hoc noſtrum inventum, cùm obtuſa ingenia & humi repentia refugiat, ad ſublime Celſitudinis tuæ ingenium & patrocinium confugiet & tranſvolabit? Præſertìm cùm nova hæc Logarithmorum methodus, omnem illam priſtinæ Matheſeos in calculo difficultatem (quæ alioquì generoſam tuam indolem offendere poſſet) penitùs è medio tollat: & ad ſublevandam memoriæ imbecillitatem ita ſe accommodet, ut illius adminiculo facile ſit, plures quæſtiones mathematicas unius horæ ſpatio, quàm priſtinâ & communitèr receptâ formâ ſinuum, tangentium, & ſecantium, vel integro die abſolvere. Ideóque tuæ Celſitudini tanto gratius hoc inventum fore ſperamus, quanto facilem magìs & expeditam reddit Logiſticam: quid enim jucundius, & in omni diſciplinarum genere præſtantius eſſe poſſit, quàm præclara quæque & ſublimia, exactè, ex tempore, facili negotio, nullóque vel temporis vel laboris diſpendio expedire? Rogamus igitùr (ſereniſſime Princeps) ut munuſculum hoc, licèt exiguum, & longè infrà meritorum*

&

& dignitatis tuæ faſtigium, certiſſimi tamen obſequii pignus & ſymbolum, pro humanitate tuâ boni conſulas. Quod ſi te feciſſe intellexero, vel hâc ſolâ ratione animos mihi, jam morbis penè confecto, addideris, ad alia propedièm, his fortaſſe majora & tanto principe magè digna, moliendum. Interìm illuſtriſſimos tuos parentes, magna magnæ BRITANNIÆ *luminaria, téque præclarum tam præclaræ ſtirpis ramum, & futuræ noſtræ tranquillitatis ſpem, diù nobis incolumes ſervet & protegat ſummus ille Rex Regum, & Dominus dominantium, cui omnis honor & gloria in æternum tribuatur.*

Sereniſſimæ tuæ Celſitudinis obſe-

quio addictiſſimus

IOANNES NEPERVS.

IN

IN MIRIFICVM

Logarithmorum Canonem

Præfatio.

QUUM nihil ſit (chariſſimi mathematum cultores) mathematicæ praxi tàm moleſtum, quódq; Logiſtas magìs remoretur, ac retardet, quàm magnorum numerorum multiplicationes, partitiones, quadrataéque ac cubicæ extractiones. quæ, præter prolixitatis tædium, lubricis etiàm erroribus plurimùm ſunt obnoxiæ: Cœpi igitùr animo revolvere, quâ arte certâ & expeditâ, poſſem dicta impedimenta amoliri. Multis ſubinde in hunc finem perpenſis, nonnulla tandem inveni præclara compendia, alibi fortaſſe tractanda: verùm inter omnia nullum hoc utilius, quod unà cum multiplicationibus, partitionibus, & radicum extractionibus arduis & prolixis, ipſos etiam numeros multiplicandos, dividendos, & in radices reſolvendos ab opere rejicit, & eorum loco alios ſubſtituit numeros, qui illorum munere fungantur per ſolas additiones, ſubtractiones, bipartitiones, & tripartitiones. Quod quidem arcanum, cùm (ut cætera bona) ſit, quo communius, eo melius: in publicum mathematicorum uſum propalare libuit. Eo itaque liberè fruamini (matheſeos ſtudioſi) & quâ à me profectum eſt benevolentiâ, accipite. Valete.

Ad Lectorem Trigonometriæ *ſtudioſum.*

QVi Cœli, atque ſoli, & ſinuoſos æquoris arcus
Metiri, & curvos vis numerare gradus;
Quíque per expanſum tenſas cognoſcere rectas,
Menſurat radio quas GEODÆTA ſuo:
Macte animo, inſta operi, LOGARITHMIS utere, quos hîc
Rara Caledonii, dat Baro, gemma ſoli.
Fruſtrà erit hinc multis tabulas extendere chartis:
Fruſtrà erit & calamo crebra litura tuo.
Quæ niſi multiplici nunquam potuêre priores,
Actu uno hîc facili tu numerare queas.
Ad SOPHIAM quid ferto novi, quod priſtina vincat,
Quiſquis ab ingenio nomen habere cupis.

Patricius Sandæus.

In Logarithmos D. I. NEPERI.

REGIOMONTANVS fertur per aphæreſin, atque
Proſtheſin ignotos elicuiſſe ſinus:
Atque aliquos aliquot ſinuum quæſita Logiſtas
Conſimili legimus notificâſſe modo:
Nemo tamen cunctos poterat ſic ſolvere nodos,
Aut certâ rectam lege docere viam.
Muſarum NEPERVS honos, & gloria gentis
Scotigenæ, parvo præſtat utrumque libro.
Nomine ſic NEPAR, PARILI ſit & omine NON PAR,
Quum non hac habeat NEPAR in arte PAREM.

Aliud.

BVchanane tibi Nepervm adſciſce ſodalem,
Floreat & noſtris *Scotia* noſtra viris:
Nam velut ad ſummum culmen perducta Poëſis
In te ſtat, nec quò progrediatur habet:
Sic etiàm ad ſummum eſt culmen perducta Matheſis,
Ínque hoc ſtat, nec quò progrediatur habet.

Ad Lectorem.

HIc liber eſt minimus, ſi ſpectes verba; ſed uſum
Si ſpectes, Lector, maximus hic liber eſt.
Diſce; ſcies parvo tantum debere libello
Te, quantum magnis mille voluminibus.

Andreas Ivnivs
Philoſophiæ Profeſſor in
Academiâ Edinburgenâ.

IN LOGARITHMOS.

QVæ tibi cunque ſinus, tangentes, atque ſecantes,
** Prolixo præſtant, atque † labore gravi:*
Abſque labore gravi, & ſubitò, tibi, Candide Lector,
Hæc Logarithmorum parva tabella dabit.

* potiùs *Prolixè.*
† potiùs *cúm que.*

MIRI-

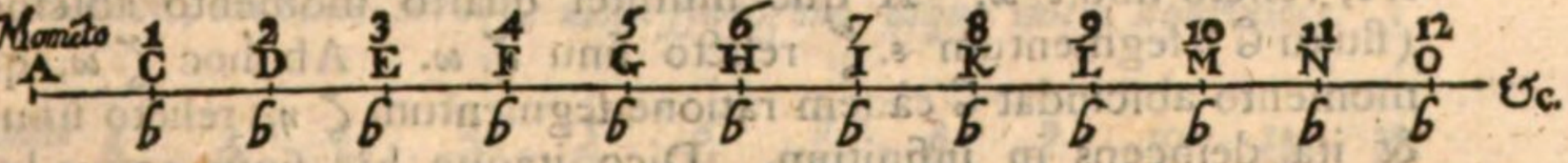
MIRIFICI
Logarithmorum canonis descri-
ptio, ejúsque usûs in utrâque Trigonome-
triâ, ut etiàm in omni Logisticâ mathematicâ,
amplissimi, facillimi, & expe-
ditissimi, explicatio.

LIBER I.

CAPVT. I.

De Definitionibus.

1. Def. *Linea* æqualitèr crescere *dicitur, quum punctus eam describens, æqualibus momentis per æqualia intervalla progreditur.*



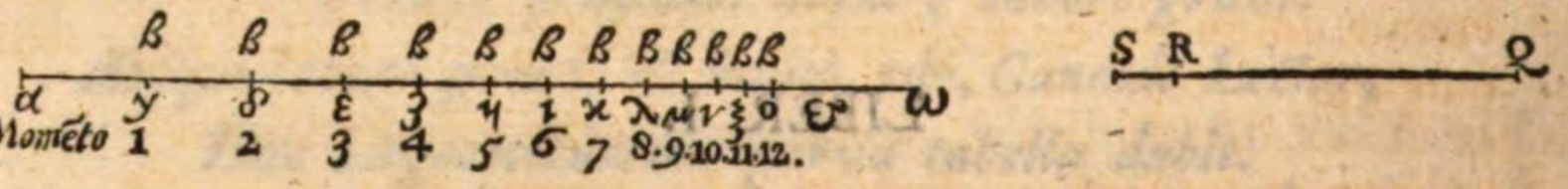
Sit punctus A, à quo ducenda sit linea fluxu alterius puncti qui sit B. Fluat ergò primo momento B ab A in C, secundo momento à C in D, tertio momento à D in E; atque ita deinceps in infinitum, describendo lineam A C D E F, &c. intervallis A C, C D, D E,

D E, E F, & cæteris deinceps æqualibus, et momentis æqualibus descriptis. Dicetur hæc linea, per definitionem superiùs traditam, *æqualitèr crescere.*

Corollarium. *Unde hoc incremento quantitates æqui-differentes temporibus æqui-differentibus produci est necesse.*

Ut in superiori schemate unico momento B ab A in C, & tribus momentis ab A in E progressus est, sic sex momentis ab A in H, & octò momentis ab A in K. Sunt autèm illorum momentorum unius & trium, & horum sex & octò, differentiæ æquales, scilicet, duorum. Sic etiàm erunt quantitatum illarum A C, & A E, & harum A H, & A K differentiæ C E, & H K, æquales. Æqui-differentes ergò, ut suprà.

2.Def. *Linea* proportionalitèr *in breviorem* decrescere *dicitur, quum punctus eam transcurrens æqualibus momentis, segmenta abscindit ejusdem continuò rationis ad lineas à quibus abscinduntur.*



Exempli gratiâ. Sit linea sinûs totius α. ω proportionalitèr minuenda. Sit punctus transcursu suo eam minuens β. Sit denique ratio segmentorum singulorum ad lineas à quibus abscinduntur, ut Q. R. ad Q. S. Quâ ergò ratione secatur Q. S. in R. eâdem ratione (per 10. 6. Eucl.) secetur α. ω in γ. atque hic ϛ. transcurrens ab α in γ primo momento ab α. ω abscindat α. γ, relictâ lineâ, seu sinu, γ. ω. Ab hâc autem γ. ω procedens β secundo momento abscindat simile segmentum quale est Q. R ad Q. S, quod sit γ. δ, relicto sinu δ. ω. A quo proinde tertio momento abscindat β simili ratione segmentum δ. ε, relicto sinu ε. ω. A quo similitèr quarto momento abscindatur (fluxu ϛ) segmentum ε. ζ, relicto sinu ζ. ω. Ab hoc ζ. ω, quinto momento abscindat ϛ eâdem ratione segmentum ζ. η, relicto sinu η. ω. & ità deinceps in infinitum. Dico itaque hîc sinûs totius lineam α. ω (ex præmissâ definitione) *proportionalitèr decrescere in sinum* η. ω. aut in alium quemvis ultimum in quo sistit ϛ; & sic in aliis.

Cor.

Cor. *Unde hoc æqualibus momentis decremento, ejuſdem etiàm rationis proportionales lineas relinqui eſt neceſſe.*

Quæ enim ſuperiùs eſt continua proportio ſinuum minuendorum, $\alpha\omega, \gamma\omega, \delta\omega, \varepsilon\omega, \zeta\omega, \eta\omega, \iota\omega$, & $\varkappa\omega$, &c. atque ſegmentorum ab eis abſciſſorum $\alpha\gamma, \gamma\delta, \delta\varepsilon, \varepsilon\zeta, \zeta\eta, \eta\iota, \iota\varkappa$, & $\varkappa\lambda$. Eadem erit neceſſariò etiàm ſinuum relictorum proportio, ſcilicet, $\gamma\omega, \delta\omega, \varepsilon\omega, \zeta\omega, \eta\omega, \iota\omega, \varkappa\omega$, & $\lambda\omega$, ut ex 19. prop. 5. & 11. prop. 7. Eucl. patet.

3. Def. *Quantitates ſurdæ, ſeu numero inexplicabiles, numeris quàm proximè definiri dicuntur, quum numeris majuſculis, qui à veris ſurdarum valoribus unitate non differant, definiuntur.*

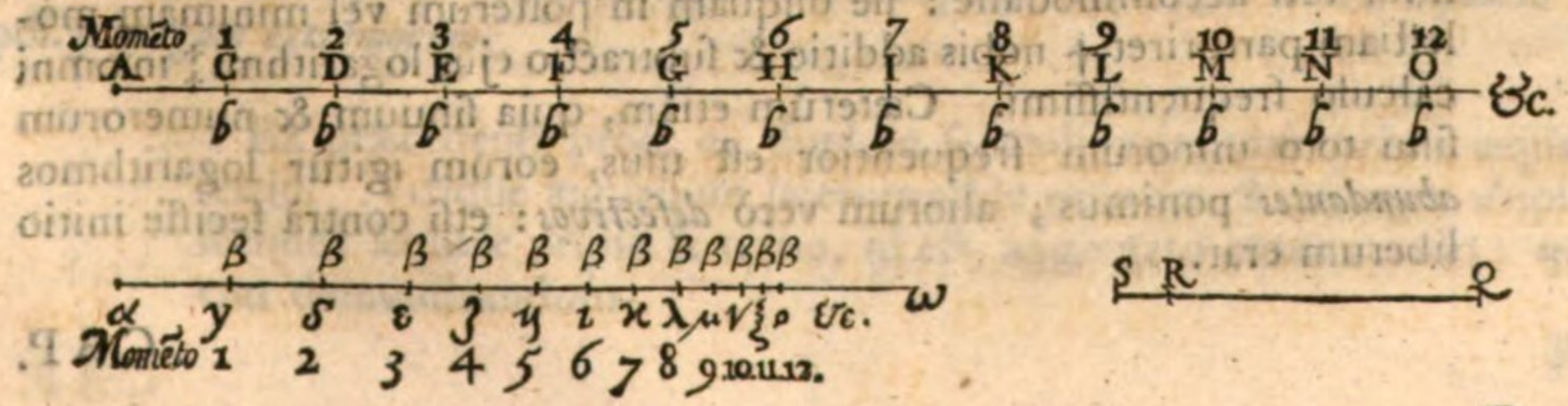
Vt ſit ſemi-diameter, ſeu ſinus totus, rationalis numerus 10,000,000, erit ſinus, 45. graduum radix quadrata 200,000,000,000,000, quæ ſurda, ſeu irrationalis, & numero inexplicabilis eſt, atque inter terminos 7,071,067, minorem, et 7,071,068, majorem, includitur. Ab horum itaque utrovis non differt unitate. Surdus igitùr ſinus ille 45 graduum quàm proximè dicitur definiri et explicari, quum per numeros integros 7,071,067 vel 7,071,068. neglectis fractionibus, definitur. In magnis etenim numeris ex fragmentis unitatis ſpretis nullus error ſenſibilis emergit.

4. Def. Synchroni *motus ſunt, qui ſimùl & eodem tempore fiunt.*

Vt in ſuperioribus, eſto quod B. moveatur ab A in C. eodem tempore quo G movetur ab α in γ. dicentur rectæ A C, & $\alpha\gamma$ *ſynchrono motuo* deſcribi.

5. Def. & poſtul. *Quum quolibet motu & tardior & velocior dari poſſit, ſequetur neceſſariò cuique motui æqui velocem (quem nec tardiorem, nec velociorem definimus) dari poſſe.*

6. Def. Logarithmus *ergò cujuſque ſinûs, eſt numerus quam proximè definiens lineam, quæ æqualitèr crevit intereà dum ſinûs totius linea proportionalitèr in ſinum illum decrevit, exiſtente utroque motu ſynchrono, atque initio æquiveloce.*



Exempli

Exempli gratiâ. Repetantur ambo ſuperiora ſchemata, & moveatur B. ſemper & ubique eâdem ſeu æquali velocitate quâ cœpit moveri 𝔊 initio quum eſt in α. deinde primo momento procedat B. ab A in C, & eodem momento procedat 𝔊 ab α in γ. proportionalitèr: erit numerus definiens A C logarithmus lineæ, ſeu ſinûs, $\gamma\omega$. Tum ſecundo momento promoveatur B à C in D, & eodem momento promoveatur proportionalitèr 𝔊 à γ in ; erit numerus definiens A D logarithmus ſinûs $\delta\omega$. Sic tertio momento procedat æqualitèr B à D in E, & eodem momento promoveatur proportionalitèr 𝔊 à δ in ε; erit numerus definiens A E logarithmus ipſius ſinûs $\varepsilon\omega$. Item quarto momento procedat B in F, & β in ζ; erit numerus A F Logarithmus ſinûs $\zeta\omega$. Atque eodem continuò ſervato ordine erit (ex definitione ſuperiùs traditâ) numerus A G logarithmus ſinûs $\eta\omega$; A H logarithmus ſinûs $\iota\omega$; A I log. ſinûs K ω; A K log. ſinûs $\lambda\omega$; & ita in infinitum.

Cor. *Unde ſinûs totius* 10,000,000 *nullum ſeu* 0, *eſt logarithmus: & per conſequens, numerorum majorum ſinu toto logarithmi ſunt nihilo minores.*

Quum enim ex definitione pateat, quod à ſinu toto decreſcentibus ſinibus, à nihilo accreſcant logarithmi, ideò contrà, creſcentibus numeris (quos adhuc ſinus vocamus) in ſinum totum, ſcilicèt, in 10,000,000. decreſcant, in 0, ſeu nihilum, logarithmi eſt neceſſe. Et per conſequens numerorum creſcentium ultrà ſinum totum 10,000,000. (quos *ſecantes* aut *tangentes*, & non amplius *ſinus* vocamus) logarithmi erunt minores nihilo.

Itaque logarithmos ſinuum, qui ſemper majores nihilo ſunt, abundantes *vocamus, et hoc ſigno* +, *aut nullo, prænotamus. Logarithmos autem minores nihilo* defectivos *vocamus, prænotantes* * *eis hoc ſignum* —.

* Potiùs *præponentes.*

Admonitio.

ERat quidem initio liberum cuilibet ſinui, aut quantitati nullum ſeu 0, pro logarithmo attribuiſſe: ſed præſtat id præ cæteris ſinui toti accommodaſſe: ne unquam in poſterum vel minimam moleſtiam parturiret + nobis additio & ſubtractio ejus logarithmi ‡ in omni calculo frequentiſſimi. Cæterùm etiàm, quia ſinuum & numerorum ſinu toto minorum frequentior eſt uſus, eorum igitùr logarithmos *abundantes* ponimus; aliorum verò *defectivos*: etſi contrà feciſſe initio liberum erat.

+ Potiùs *pareret.*

‡ *Seu* qui in omni calculo occurrit frequentiſſimè.

CAP. II.

De Logarithm. propositionibus.

Pro- pos.1. *PRoportionalium numerorum, aut quantitatum, æqui-differentes sunt Logarithmi.*

Vt proportionalium sinuum, scilicet, $\gamma\omega$, qui se habet ad $\varepsilon\omega$. ut $\iota\omega$. ad $\lambda\omega$, Logarithmi respectivè sunt numeri definientes A C, A E, A H, & A K, (ut per def. 6. patet.) differunt autèm A C, & A E differentiâ C E: atque A H, & A K differentiâ H K. Sunt autèm (ex 1. def. & suo corollario) C E & H K æquales. Æqui-differentes igitùr sunt Logarithmi præfatorum sinuum proportionalium. Et ita in omnibus proportionalibus.

Nam quas affectiones & symptomata Logarithmi ab ortu & genesi suâ acquisiverint, eas in posterum retineant est necesse. Ab ortu autèm & in genesi suâ imbuuntur hâc affectione, & hæc lex illis præscribitur, ut sint æqui-differentes, quum eorum sinus seu quantitates sint proportionales (prout ex def. Logarithmi, & utriusque motûs patet, & in constructione logarithmorum ampliùs aliquando patebit.) Proportionalium ergò quantitatum æqui-differentes sunt Logarithmi.

Pro- pos.2. *Ex trium proportionalium Logarithmis, duplum secundi, seu medii, minutum primo, æquatur tertio.*

Quum, per prop. 1. differentia logarithmorum primi & secundi æquetur differentiæ logarithmorum secundi & tertii, id est, secundus minutus primo æquetur tertio minùs secundo: Ideò, addito secundo ad utrumque æquationis latus, proveniet bis secundus, seu duplum secundi, minutum primo, æquale tertio: quod erat probandum.

Pro- pos.3. *Ex trium proportionalium logarithmis, duplum secundi, seu medii, æquatur aggregato extremorum.*

Ex præcedente prop. 2. duplum secundi minutum primo æquatur tertio. Vtrique æqualium laterum adde primum, & exurget duplum secundi æquale primo & tertio, id est, aggregato extremorum: quod erat demonstrandum.

Pro-

Propoſ.4. *Ex quatuor proportionalium logarithmis, aggregatum ſecundi et tertii minutum primo æquatur quarto.*

Quum per 1. prop. ex quatuor proportionalium logarithmis, ſecundus minutus primo, æquetur quarto minus tertio, utrique æqualitatis lateri, adde tertium, & fient ſecundus & tertius minuti primo æquales quarto: quod erat propoſitum.

Propoſ.5. *Ex quatuor proportionalium logarithmis aggregatum mediorum (ſecundi, ſcilicet, & tertii) æquatur aggregato extremorum, primi, videlicet, & quarti.*

Per prop. 4. præcedentem, ſecundus & tertius minuti primo erant æquales quarto: utrique æqualitatis lateri adde primum, & fiet ſecundus, plus tertio æqualis quarto, plus primo: quod demonſtrandum erat.

Propoſ.6. *Ex quatuor continuè proportionalium logarithmis triplum alterutrius mediorum æquatur aggregato extremi remoti, et dupli vicini.*

Per ſecundam prop. duplum ſecundi, ſeu medii, minutum primo eſt æquale tertio: & per tertiam prop. duplum hujus, quod eſt, quadruplum ſecundi minutum duplo primi, æquabitur aggregato ſuorum extremorum, videlicèt quarto plus ſecundo. Iam ſi ab utroque æqualitatis latere ſubduxeris ſecundum, fiet triplum ſecundi minutum duplo primi æquale quarto. Hujus rurſùs æqualitatis lateribus adde duplum primi, & exurget triplum ſecundi æquale quarto plus primi duplo: quod probandum ſuſcepimus.

Admonitio.

HVc uſque logarithmorum geneſin & ſymptomata explicavimus: quo verò calculo, quâve logiſticæ methodo, habeantur, hoc loco explicandum foret. Sed, quia ipſum canonem integrum, ejúſque logarithmos omnes cum ſuis finibus, ad ſingulas quadrantis minutias primas, exhibemus, ideò in tempus magìs idoneum doctrinam conſtructionis logarithmorum tranſilientes *, ad eorum uſum properamus, ut, prælibatis priùs uſu & rei utilitate, cætera aut magìs placeant poſthac edenda, aut minùs ſaltèm diſpliceant ſilentio ſepulta. Præſtolor enim eruditorum de his judicium & cenſuram, priuſquam cætera, in lucem temerè prolata, lividorum detrectationi exponantur.

* Potiùs *tranſferentes*, ſeu *differentes*.

CAP.

CAP. III.

Descriptionem complectens tabulæ logarithmorum, & septem ejus columnarum.

Sectio. 1. *PRima columna est expressè arcuum ab* 0. *in* 45. *Gradus crescentium: et subintelligitur esse etiàm suorum* * *ad semicirculum reliquorum* †.

Sectio. 2. *Septima autem columna est arcuum à quadrante in* 45. *gradum decrescentium: et subintelligitur esse etiàm suorum ad semicirculum reliquorum.*

Sectio. 3. *Unde alterius columnæ arcus, sunt arcuum alterius è regione respondentium complementa.*

4. *Atque in primâ exprimitur omnis trianguli rectilinei rectanguli angulus acutus minor.*

5. *In septimâ autem ei è regione collocatur ejusdem* [*trianguli*] *rectanguli angulus acutus major.*

6. *In secundâ columnâ sunt sinus arcuum primæ columnæ.*

7. *Súntque hi crus minus subtendens minorem angulum* [*trianguli*] *rectanguli, cujus basis, seu hypotenusa, est sinus totus.*

8. *In sextâ columnâ sunt sinus arcuum septimæ columnæ.*

9. *Súntque hi crus majus subtendens majorem angulum ejusdem* [*trianguli*] *rectanguli, cujus, scilicet, hypotenusa est sinus totus.*

10. *Unde omni triangulo rectilineo rectangulo fit æquiangulum et simile ex sinu toto, et sinu secundæ columnæ, et sinu sextæ ei è regione respondente.*

11. *Tertia columna continet Logarithmos arcuum, et sinuum sinistrorum.*

12. *Qui sunt Logarithmi proportionis cruris minoris rectanguli ad ejusdem hypotenusam.*

13. *Itémque hi sunt arcuum, et sinuum dextrorum Logarithmi complementorum, quos* antilogarithmos *appellamus.*

14. *Quinta columna continet Logarithmos arcuum, et sinuum dextrorum.*

15. *Qui sunt Logarithmi proportionis cruris majoris* [*trianguli*] *rectanguli ad ejusdem hypotenusam.*

16. *Itémque hi sunt arcuum et sinuum sinistrorum antilogarithmi, seu Logarithmi complementorum.*

17. *Quarta denique, seu media, columna continet differentias inter logarithmos tertiæ columnæ, et quintæ. Unde duplex est hæc columna, Abundans et defectiva.*

18. *Abundantes, sunt differentiæ quæ oriuntur ex subtractione logarithmorum quintæ à logarithmis tertiæ.*

19. *Defectivæ verò, sunt differentiæ ortæ ex subductione logarithmorum tertiæ à logarithmis quintæ: quæ ideò sunt minores nihilo.*

* Potiùs eorundem arcuum.

† Id est, complementorum.

20. *Differentiæ abundantes dicuntur* differentiales numeri arcuum ſiniſtrorum.
21. *Sùntque logarithmi proportionis minoris cruris, [trianguli] rectanguli ad ejuſdem crus majus.*
22. *Itémque ſunt Logarithmi fœcundorum, ſive tangentium arcuum ſiniſtrorum.*
23. *Differentiæ autèm defectivæ dicuntur* numeri differentiales arcuum dextrorum.
24. *Súntque Logarithmi proportionis majoris cruris [trianguli] rectanguli ad ejuſdem crus minus.*
25. *Itémque ſunt Logarithmi fœcundorum, ſive tangentium arcuum dextrorum.*
26. *Omnis etiàm arcus ſiniſter, ejúſque ad ſemicirculum reliquus, dicitur arcus complementi arcuum, ſinuum, et Logarithmorum dextrorum, atque differentialium defectivorum.*
27. *Et contrà, omnis arcus dexter, ejúſque ad ſemicirculum reliquus, dicitur arcus complementi arcuum, ſinuum, & Logarithmorum ſiniſtrorum, atque differentialium abundantium.*

Admonitiones.

28. *Hìc notandum eſt, ſi Logarithmos tertiæ columnæ defectivos feceris (præpoſito, ſcilicet, — ſigno,) fient Logarithmi hypotenuſarum, ſive ſecantium arcuum dextrorum ſeptimæ columnæ.*
29. *Et hi etiàm fient Logarithmi proportionis hypotenuſæ [trianguli] rectanguli ad ejuſdem crus minus.*
30. *Et ſi Logarithmos quintæ columnæ defectivos feceris, fient Logarithmi hypotenuſaram, ſive ſecantium arcuum ſiniſtrorum primæ columnæ.*
31. *Fient etiam hi Logarithmi proportionis hypotenuſæ [trianguli] rectanguli ad ejuſdem crus majus. Verùm quia ad [triangulorum] rectilineorum ſcientiam comparandam, ſoli ſinus, eorúmque arcus, & logarithmi cum differentialibus: ad Sphæricorum autèm, ſtretis ſinibus, ſoli arcus, & eorum logarithmi, & differentiales ſufficiunt: ideò hypotenuſas & fœcundos tabulâ excluſimus: ſicuti & ſinus ipſos in ſphæricis negligi volumus. Oſtendemus tamèn obitèr te poſſe (ſi libuerit) eis omnibus ſatìs expeditè in [triangulis] rectilineis uti, in ſphæricis verò minimè.*

CAP. IV.

De uſu tabulæ, & numerorum ejus.

Sectio. 1. *Sinuum, tangentium, & ſecantium præciſè in tabulis ſuis repertorum, Logarithmos non minùs præciſè dare.*

Per ſect. 11. & 14. cap. 3. reperto ſinu dato in ſecundâ, aut ſeptimâ, columnâ noſtræ tabulæ, reperietur ejus Logarithmus in ejuſdem lineæ tertiâ vel quintâ columnâ. Habentur igitùr ſic exactè ſinuum tabulatorum logarithmi. Tangentium autem & ſecantium numeris in ſuis tabulis repertis habentur arcus. Ex arcubus verò cognitis noſtra tabula exhibet tangentium logarithmos ſeu differentiales cum ſignis ſuis in mediâ columnâ per ſect. 22. & 25. Et [exhibet] ſecantium logarithmos reciprocè in tertiâ & quintâ columnâ; præpoſito tamèn his — ſigno per ſect. 28. & 30. Habentur igitùr ſinuum, tangentium, & ſecantium tabulatorum logarithmi.

Exempla ſinuum.

SInûs 6946584 Logarithmum quæro. Sinum illum præciſè reperio in ſecundâ columnâ reſpondentem arcui 44. Gr. 0. m. & in eâdem lineâ tertiæ columnæ adſtat illi 3643349 ſuus logarithmus, quem quæſivi.

Itèm ſinûs 7213574 quæratur logarithmus. Sinus hic invenietur reſpondens arcui 46. Gr. 10. m. & ei vicinus 3266204 logarithmus ejus quæſitus.

Exempla tangentium.

QVæratur tangentis 2186448. logarithmus. Huic tangenti in ſuâ tabulâ reſpondet arcus 12. Gr. 20. m. & huic arcui in mediâ columnâ tabulæ noſtræ reſpondet logarithmus, ſeu differentialis abundans 15203064. quæſitus. Itèm, ſi tangentis 45736291 logarithmum quæſiveris, offendes in tabulâ tangentium ejus arcum 77. Gr. 40. m. hujúſque arcûs in tabulâ noſtrâ differentialem eandem, defectivam tamen, ſcilicet, — 15203064.

Exempla secantium.

SEcanti 18118009 respondet in tabulâ secantium arcus 56. gr. 30. m. & huic arcui in tabulâ nostrâ convenit reciprocè defectivus — 5943212, logarithmus secantis 18118009 suprascripti. Sic secantis 13118337 invenies logarithmum — 2714255; & secantis 13960592 offendes logarithmum — 3336533.

2. *Numerorum datorum, & in tabulis sinuum, tangentium, & secantium non repertorum, logarithmos æstimare.*

Numerum dato simillimum, sive is fuerit dati decuplus, centuplus, millecuplus, 10,000^{plus} 100,000^{plus} aut 1000,000^{plus}, quære in secundâ, aut sextâ columnâ tabulæ nostræ, aut, si mavis, in tabulis tangentium, aut secantium: & hujus arcum nota. Ejus enim logarithmus è tabulâ nostrâ elicitus, est quem quæris: mente tamen reservando, aut, memoriæ gratiâ, notis exprimendo, numerum locorum, seu figurarum multiplicitatis. Vt, si quæratur logarithmus numeri 137, in tabulis non reperti: reperies inter sinus numeros 14544, 136714, & 1371564; & inter tangentes 13705046; inter secantes verò numerum 13703048; qui est omnium dato simillimus, dummodò ejus ultimæ vel dextimæ quinque figuræ deleri subintelligantur. Hujus ergò secantis 13703048, & sui arcûs 43. gr. 8. m. logarithmus (per præced. aut per sect. 28. & 30. cap. 3.) quæratur; & invenietur — 3150332, qui pro logarithmo dati numeri 137 etiàm habetur: recordando tamen ultimas quinque figuras abscindendas esse, aut, memoriæ gratiâ, expressè hoc modo signandas — 3150332. — 00000. Similitèr, si per tangentem 13705046, superiùs expressum, quæsiveris logarithmum numeri 137; ex tangentis illius arcu 53. gr. 53. m. invenietur (per sect. 25.) in mediâ columnâ — 3151790, logarithmus illius tangentis 13705046: qui quia excedit 137 datum quinque locis seu figuris, ideò — 3151790 — 00000 erit logarithmus numeri dati 137. Tanto tamen minùs exactus est hic logarithmus, quanto magìs 13705046. est dissimilis numero 13700000. seu centies millecuplo dati: sed hic error partes $\frac{5046}{100,000}$ unitatis non exuperat. Si tandem per sinum suprascriptum 1371564 quæsiveris logarithmum dati 137, is (per hanc, & 11. sect. cap. 3.) deprehendetur esse 19866327 — 0000. Nec secùs operandum erit signo + quando numerus figurarum datæ quantitatis excedit numerum figurarum sinûs ei simillimi; quod rarò contingit. Ut, si quæratur numeri, seu discretæ quantitatis, 232702.

logarithmus,

logarithmus, invenies in tabulâ ſinum 23271 ei omnium ſimillimum; ſed unica deeſt huic figura. Hujus ergò logarithmo tabulato (per ſect. 11. cap. 3.) reperto, qui eſt 60631284, adjiciatur unica cyphra ſigno + interpoſito; & fiet 60631284 + o pro logarithmo numeri 232702 quæſito. Sed modus logarithmos æſtimandi omnium optimus eſt, quo primò creati ſunt: de quo alibi.

3. *Vnde, ut ſuperiore primâ ſectione logarithmi ſimplices, et puri exhibentur: ita hâc præcedente, appoſitis cyphris, impuri emergunt.*

4. *Similium ſignorum logarithmos addere, eſt aggregatum utriuſque cum ſigno communi exhibere.*

Vt ex additione — 56312. ad — 73495. provenient — 129807. Itémque addito 4216. ad + 5392. producuntur 9608. Sic 3219 — oo. ad 4360 — ooo. faciunt 7579 — ooooo.

5. *Diſſimilium ſignorum logarithmos addere, eſt differentiam eorum cum ſigno majoris numeri exhibere.*

Vt ex additione — 210. ad 332. producitur + 122. Itèm ex additione — 210. ad 192. producitur — 18. Sic — 210. + ooo. ad 332 — oo. ſunt 122 + o. Item — 210 — ooo. ad 192 + oo. ſunt — 18 — o.

6. *Duorum logarithmorum, hic illius defectivus, ille autem hujus abundans propriè dicitur: cum & numerum, & cyphras communes ſeu eoſdem: ſigna verò omnia + & — penitùs contraria habeant.*

Vt abundantis 56312 defectivus eſt — 56312. Itèm abundantis 56312 — oo defectivus eſt — 56312 + oo. Sic abundantis 56312 + oo defectivus eſt — 56312 — oo.

7. *Abundantem ſubtrahere, eſt ejus defectivum addere.*

Vt ſubtrahere abundantem 56312. ex — 73495. idem erit, quod addere illius defectivum, qui (per 6.) eſt — 56312. ad eundem — 73495. fiéntque (per 4. præmiſſam) — 129807. Sic ſubtrahere 56312 + oo ex — 73495 — ooo eſt idem quod addere — 56312 + oo ad — 73495 — ooo; fiúntque (per 4. & 5. præcedentes) — 129807 — ooooo.

8. *Defectivum*

8. *Defectivum ſubtrahere eſt ejus abundantem addere.*

Vt ſubtrahere defectivum — 4216 ex + 5392 eſt idem quod addere 4216 ad 5392 & (per 4.) producere 9608. Sic idem eſt ſubtrahere — 4216 + 00 ex 5392 + 0 quod addere 4216 — 00 ad 5392 + 0 & producere 9608 — 0.

9. *Logarithmum numero-tenùs augere, vel minuere, ſalvo valore priſtino, eſt ad illum addere, aut ab eo ſubtrahere, quemvis ex logarithmis ſequentibus, ſcilicet,* 23025842 + 0, *vel* 46051682 + 00, *vel* 69077527 + 000, *vel* 92103369 + 0000, *vel* 115129211 + 00000, *nihil prorſùs ſignificantibus.*

Vt ſit Logarithmus 39156 — 0; cui ſi addideris illorum quemvis, ut, exempli gratiâ, 23025842 + 0, fiet inde 23064998, major numero, valore autem prorſùs idem qui 39156 — 0. Námque hujus 39156 — 0. logarithmi, quantitas ſeu valor numeralis (per 12. & 13. ſect. ſeq. hujus) eſt 9 9 6 0 9 2 0. à quibus deme unicam figuram ultimam, prout — 0 notat, & fiet 996092. Illius autem Logarithmi 23064998. valor numeralis (per ſeq. ſect. 12. & 13. hujus) eſt etiàm 996092, idem qui priùs.

Exemplum minutionis.

SIt Logarithmus 25451769. minuendus, à quo ſi ſubduxeris 23025842 + 0. relinquitur 2425927 — 0. ejuſdem valoris, cujus prior hic 25451769. Nam ſimplicis & puri Logarithmi 2425927 valor eſt decuplus valoris utriuſvis eorum. Sunt ergò eorum valores invicèm æquales. Nihil enim aliud ſignificat additio Logarithmi 23025842 + 0, quam quod valor numeri cui additur, ſit decupartiendus, & huic decimæ parti cyphra unica ſit adjicienda: ſubtractio verò ejuſdem ſignificat valorem logarithmi à quo ſubtrahitur decuplari, & ab hoc decuplo cyphram unicam abjici, remanet itaque in utrâque priſtinus valor. Sic 46051684 + 00 additus ſignificat ad centeſimam partem valoris duas cyphras adjici: & ſubtractus, quod à centuplo duæ cyphræ rejiciantur: & ſic de reliquis ſuprà expreſſis.

10. *Si itaque ad logarithmum minutum aliquot cyphris addideris, aut à logarithmo auctо cyphris ſubtraxeris aliquem ex logarithmis ſupraſcriptis totidem cyphrarum, producetur ex impuro logarithmus purus ejuſdem valoris.*

Vt

Vt in ſuperiore primo exemplo ſit logarithmus impurus 39156 — 0. purgandus à cyphrâ ſuâ & — ſigno. Adde ergò illi 23025842 + 0; fiet indè, ut ſuprà, 23064998, logarithmus purus priſtini valoris. Sic à logarithmo 63584468 + 00, impuro, ſi ſubtraxeris 46051684 + 00 totidem, ſcilicet, cyphrarum, relinquetur logarithmus 17532784 purus, & ejuſdem valoris, cujus prior ille impurus.

11. *Si ad logarithmum numero defectivum addideris, aliquem ex ſupradictis logarithmis nonæ ſectionis numero majorem, proveniet logarithmus ejuſdem valoris numero abundans.*

Vt ad logarithmum — 28595270 — 0000. adde quemvis ex numeris nonæ ſectionis numero majorem. v. g. 46051684 + 00. & fiet inde 17456414 — 00. ejuſdem valoris, & numero abundans.

12. *Logarithmorum in tabulâ noſtrâ numero-tenùs inventorum ſinus, tangentes, ſecantes, ſeu numerales valores quoſcunque, exhibere poteris, per cap. 3. ſect. 11. 14. 22. 25. 28. 30. ſive ſint puri, ſive impuri.*

Vt logarithmo 36 graduum &. 40 minutorum, qui eſt 5155724 in tertiâ columnâ, reſpondet ſuus ſinus 5971586 in ſecundâ: & ejus defectivo — 5155724 reſpondet in tabulâ ſecantium 16745970 ſecans 53 gr. 20 m. Itèm logarithmo differentiali 2950794, in quartâ columnâ, reſpondet tangens (in ſuâ tabulâ) 7444724. & ejus defectivo — 2950794 reſpondet tangens 13432331, graduum, ſcilicet, 53. & 20. minut. Sic logarithmi 2204930 in quintâ columnâ numeralis valor eſt in ſextâ columnâ 8021232, ſinus, ſcilicet, gr. 53. & 20. m; & ejuſdem defectivi, ſcilicet, — 2204930, numeralis valor eſt ſecans 12466913, conveniens gradibus 36 & 40 min.

Exemplum impurorum.

SIt logarithmi impuri 97796 — 0 inquirendus valor. Huic numero-tenùs reſpondet in tabulâ noſtrâ ſinus 9902681; à quo aufer dextimam figuram (prout — 0 indicat) & fient 990268, valor logarithmi 97796 — 0 quæſitus. Sic logarithmi 25451769 + 00 valor eſt 78459100, quia logarithmo 25451769 puro reſpondet in tabulâ noſtrâ ſinus 784591. Itèm logarithmi — 349136 — 00 in quartâ columnâ apud gradum 46 reperti, valor erit 103553; quia tangens 46 graduum eſt 10355302. Sic logarithmi — 6350305 — 00, in tertiâ columnâ apud gradum 32 reperti, valor eſt 188708, quia

quia ſecans complementi 32 graduum, ſcilicet, 58 graduum, eſt 18870800; cujus duæ ultimæ & dextimæ figuræ o o delendæ ſunt propter — o o annexa logarithmo.

13. *Logarithmorum datorum, in tabulâ noſtrâ non repertorum, numerales valores æſtimare.*

Ad vulgares Geodeſias ſufficit plerumque, logarithmi tabulati propinquioris dato, numeralem valorem pro dati accipere. Verùm, ſi propiùs ad metam accedere deſideras, logarithmum datum per nonam hujus numero-tenùs auge, vel minue, ſalvo valore priſtino, donec aut in tabulâ reperiatur, aut alicui tabulato ſatìs ſimilis devenerit: & hujus logarithmi valor per præmiſſam inventus, eſt quem quæris. Ut, exempli gratiâ, quæratur valor hujus logarithmi 23149721 + o; cui in tabulâ non reperitur ſimilis vel ſatìs propinquus. Verùm, ſi ab illo ſubduxeris 23025842 + o, relinquetur 123879; cui ſub 81 gradu reperietur ſatìs propinquus & ſimilis 123881; cujus ſinus 9876883, per præmiſſam inventus, eſt valor oblati logarithmi 23149721 + o quæſitus.

Admonitio.

PRo hâc ſectione, & ſecundâ hujus, monitum volumus, numerorum datorum logarithmos, &, contrà, logarithmorum datorum numerales valores (ubi non reperiuntur in tabulâ) omnium accuratiſſimè exhiberi per modum ipſum quo creantur, aut reſolvuntur, logarithmi, qui eſt, ut à ſinu dato per media Geometricè proportionalia deſcendas, donec in proximè minorem ſinum tabulatum perveneris: ſimilitèr ab hujus logarithmo tabulato deſcendas etiàm per totidem media Arithmetica congrua; & horum ultimus erit illorum primi logarithmus: & contrà per reſolutionem, ut à logarithmo dato per media Arithmetica in Logarithmum tabulatum proximè minorem deſcendas, & ab hujus valore tabulato ſimilitèr etiam deſcendas per totidem media Geometrica & congrua; & horum ultimus erit numeralis valor illorum Logarithmorum primi. Verùm, "quæ æqui-differentia Arithmetica cuique continuatæ proportioni Geo-"metricæ conveniat & ſit congrua," exquirere non eſt mediocris ingenii. Quarè de his (Deo aſpirante) ubi de Logarithmis condendis & creandis agetur, ampliùs aliquando differemus.

CAP. V.

De ampliſſimo Logarithmorum uſu, & expeditâ per eos praxi.

Proble-ma 1. *EX trium proportionalium Logarithmis, dato logarithmo medio et altero extremo, reliquum extremum, ejuſve proportionalem, vel arcum, per unicam duplationem et ſubtractionem dare.*

Quum (per ſecundam prop. cap. 2.) duplum medii (ſcilicèt, logarithmi) minutum altero extremorum, æquetur reliquo, ideò à duplo medii logarithmi dati aufer logarithmum extremi datum, et relinquetur logarithmus extremi quæſiti: cui in tertiâ, quartâ, aut quintâ columnâ tabulæ inventæ reſpondet arcus in primâ & ſeptimâ: ſinus autèm in ſecundâ, aut ſextâ: & ſui ſecantes aut tangentes in tabulis ſuis per cap. 3. ſect. 1. 2. 6. 8. 11. 14. 22. 25. 28. 30. pro extremo quæſito habentur.

Exemplum.

DEntur 10,000,000, primum proportionale, & 7,071,068 ſecundum; quæratur tertium. Id vulgò exquiritur medium in ſe quadratè multiplicando, & hoc quadratum per primum dividendo. Sed nos faciliùs, medii logarithmum 3,465,735 duplando, & ab hoc duplo, (quod eſt 6,931,470) logarithmum primi (qui eſt 0) auferendo: & ita reſtat 6,931,470. logarithmus quæſitus: cujus arcum 30. graduum, & ſinum 5,000,000, (ſcilicèt, proportionale quæſitum) juxtà eum invenies. Sunt ergò 10,000,000, 7,071,068, 5,000,000, tria proportionalia, quorum ultimum ſolâ duplatione & ſubtractione acquiſivimus; quod polliciti ſumus.

Itèm duo proportionalia, 10,562,556 primum, & 7,660,445 ſecundum, aut ſaltèm eorum logarithmi, — 547302 & 2,665,149, dentur: Tertium ſic habebis. Ab hujus duplo, 5,330,298, aufer — 547,302, & (per 8. ſect. cap. 4.) producitur logarithmus 5,877,600, 33. graduum, & 45. min. cujus ſinus, 5,555,702, eſt tertium proportionale quæſitum.

Prob. 2. *Ex trium proportionalium logarithmis, datis logarithmis extremis, medium, ejuſque proportionale, et arcum, per unicam additionem et bipartitionem dare.*

Quum (per ſect. 3. cap. 2.) duplum Logarithmi medii æquetur aggregato extremorum, ideò extremorum Logarithmos adde [inter ſe]: productum [ſeu ſummam] bipartire; & emerget Logarithmus medii: atque inde medium, & medii arcus, innoteſcit in columnis, & per ſectiones, ut ſuprà.

Exempli gratiá.

Dentur extrema 10,000,000 & 5,000,000; quæratur medium. Id vulgò acquiritur multiplicando data illa invicèm, & producti radicem quadratam extrahendo. Verùm nos ſic faciliùs. Datos extremorum Logarithmos, 0 primi, & 6,931,470 ultimi, addimus, & aggregatum 6,931,470 bipartimur, fiétque 3,465,735 optatus medii Logarithmus. Vnde ipſum medium 7,071,068, & ejus arcus 45. gr. ratione ſupradictâ habentur.

Itèm, ſint extrema data 10,562,556 &5,555,702; eorum Logarithmi —547,302 & 5,877,600. Horum additorum ſumma eſt 5,330,298 per ſect. 5. cap. 4; quam bipartimur, & fit 2,665,149 Logarithmus, & ejus arcus 50 graduum: Et ſinus, ſeu medium proportionale quæſitum, eſt 7,660,445, ſolâ additione & bipartitione inventum.

Prob. 3. *Ex quatuor proportionalium Logarithmis, datis tribus, eorúmve arcubus, invenire quartum Logarithmum, ejúſque ſinum, et arcum, per unicam additionem, et ſubtractionem.*

In hoc problemate quæſitum ſemper pro quarto ſtatuimus, ita ut datorum primum ſe habeat ad ſecundum, ut tertium ad quæſitum. Quúmque ita conſtitutorum aggregatum ex Logarithmis ſecundi et tertii, minutum Logarithmo primi, æquetur quarti Logarithmo per 4. ſect. cap. 2; Ideò Logarithmos ſecundi & tertii adde [inter ſe], & hinc aufer Logarithmum primi, & proveniet Logarithmus quarti quæſiti: & inde ipſum quartum, et ejus arcus.

Exempli gratiá.

Sit, ut 7,660,445 ad 9,848,078, ita 5,000,000 ad quartum, quod quærimus. Hoc vulgus acquirit ducendo ſecundum in tertium, et dividendo per primum. Tu autem ſic faciliùs: Logarithmos ſecundi 153,088, & tertii 6,931,469, addes; fiet 7,084,557: à quo auferes Logarithmum primi, qui eſt 2,665,149, & relinquetur 4,419,408, Logarithmus quarti: cujus ſinus 6,427,876 eſt ipſum quartum deſideratum, & ejus arcus eſt 40. graduum. Idem proveniret ſi (ſpretis ſinibus) ſolùm darentur tres ſui arcus 50. gra. 80. gr. & 30. gr. Namque

ex Logarithmis arcuum 80. gr. & 30. gr. ablato Logarithmo 50. gr. remanebit Logarithmus 40. gr. Et ita ipſe arcus 40. gr. innoteſcet abſque finibus, eorúmve multiplicatione aut diviſione; proùt initio polliciti ſumus.

Aliud exemplum.

SIt ut tangens, ſeu fœcundus numerus, 43 gr. ad ſinum 57 gr., ita fœcundus, ſeu tangens 35 gr. ad ſinum quartum tacitum, cujus arcum, neglectis et ſpretis tàm finibus quàm tangentibus, ſic inveniemus. Logarithmum differentialem 35 gr. ſcilicèt, 3,563,784, in mediâ columnâ inventum, ad Logarithmum 57 gr., videlicet 1,759,372, in quintâ columnâ locatum, addimus: à producto [ſeu ſummâ] videlicet, 5,323,156, differentialem 43 gr. (qui eſt 698,698,) ſubducimus, et relinquitur 4,624,458, Logarithmus quarti (ſinûs, ſcilicèt,) quo in tertiâ columnâ per 11. ſect. cap. 3. reperto, reperies juxtà eum in primâ columnâ 3 9. grad. 2. minut. ferè; qui eſt arcus quæſitus quarti proportionalis, ſeu ſinûs ſpreti.

Hâc ratione proportionalium arcus, abſque eorum ſinibus, tangentibus, ſecantibus, aut proportionalibus quibuſcunque, acquiruntur.

Quod certè compendium ad triangulorum planorum angulos dimetiendos, et ad univerſam ſphæricorum Trigonometriam, conducit plurimum: ut ſuo loco patebit.

Prob. 4. *Quatuor continuè proportionalium datis extremis eorúmve arcubus, mediorum quodvis, eorúmve arcuum quemvis, invenire, inductâ ſimplici tripartitione pro arduâ cubicæ radicis extractione.*

Quum in horum Logarithmis, triplum cujuſque medii æquetur aggregato extremi remoti et dupli vicini, per prop. 6. cap. 2; Ideò duplum Logarithmi extremi alterutrius ad Logarithmum extremi reliqui adde, et productum tripartire; et proveniet Logarithmus medii priori extremo proximi: et eodem modo [invenietur] alterum medium. Vt, exempli gratiâ: Sint extrema, primum 4,029,246, ultimum verò 10,562,556. Quæruntur media; quæ abſque extractione radicis cubicæ ſic invenies. Datorum Logarithmi ſunt 9,090,051 & — 547,302: ad illius duplum 18,180,102 adde hunc; & fiet 17,632,800; qui, tripartitus, producit 5,877,600 Logarithmum, cujus ſinus, 5,555,702, eſt prius medium quæſitum. Item ſimili modo ad hujus, — 547,302, duplum, quod eſt — 1,094,604, adde illum 9,090,051; et producetur 7,995,447; qui, tripartitus, producit 2,665,149, Logarithmum, cujus ſinus 7,660,445 eſt poſterius medium etiàm quæſitum. Quatuor itaque proportionalia continua ſunt 4,029,246, 5,555,702, 7,660,445, et 10,562,556.

Aliud exemplum.

SInt extrema data 14,142,135 et 5,000,000. Illius, in tabulâ ſecantium inventi, Logarithmus in tabulâ noſtrâ eſt — 3,465,735; hujus verò, 5,000,000, Logarithmus eſt 6,931,470; cujus duplo 13,862,940, adde — 3,465,735; fiet 10,397,205; quem tripartire, et fit + 3,465,735, logarithmus medii proportionalis minori extremo, 5,000,000, proximi, quod eſt 7,071,068. Sic duplo — 3,465,735 (quod eſt — 6,931,470) adde + 6,931,470; et fiet inde 0, ſeu nihil, quod, tripartitum, etiàm reddit 0, cujus ſinus et valor eſt 10,000,000, pro reliquo et majore medio. Quatuor itaq. hæc continuè proportionalia ſunt 14,142,135, 10,000,000, 7,071,068, 5,000,000.

CONCLVSIO.

EX his prælibatis judicent eruditi quantum emolumenti adferent illis logarithmi: quandoquidem per eorum additionem multiplicatio, per ſubtractionem diviſio, per bipartitionem extractio quadrata, per tripartitionem [extractio] *cubica, et per alias faciles proſtaphæreſes* omnia graviora calculi opera evitantur: cujus rei ſpecimen generale hoc priore libro exhibuimus. Sequente autem de eorundem proprio et particulari uſu in nobili illa Geometriæ ſpecie, quæ Trigonometria dicitur, tractaturi ſumus.*

* Id eſt, proſtheſes et aphæreſes ſeu additiones et ablationes, ſeu ſubtractiones.

Finis prioris libri.

LIBER

LIBER SECUNDVS.

De canonis mirifici Logarithmorum præclaro uſu in Trigonometriâ.

CAP. I.

QVum Geometria ſit ars benè metiendi; Dimenſio ſit magnitudinum propoſitarum; Magnitudines figuram (potentiâ saltèm) conſtituant; Figura ſit triangulum, aut triangulatum: Triangulatum verò compoſitum ſit ex triangulis, quibus ſuíſque partibus menſuratis, menſurabitur et illud, illiúſque partes omnes: Certum igitùr eſt ex triangulorum doctrinâ omnis Geometricæ quæſtionis Solutionem Logiſticam pendere.

Triangulum aut rectilineum eſt, aut Sphæricum.

De [triangulis] rectilineis, prop. 1.

Prop. 1. *Rectilinei* [trianguli] *tres anguli æquantur duobus rectis.*

Vnde, duobus datis, aufer eorum aggregatum ex 180. gradibus, et proveniet tertius. Itèm unico ex 180. gradibus ablato, reſtat reliquorum duorum aggregatum.

Rectilineum [triangulum] aut rectangulum eſt, aut obliquangulum.

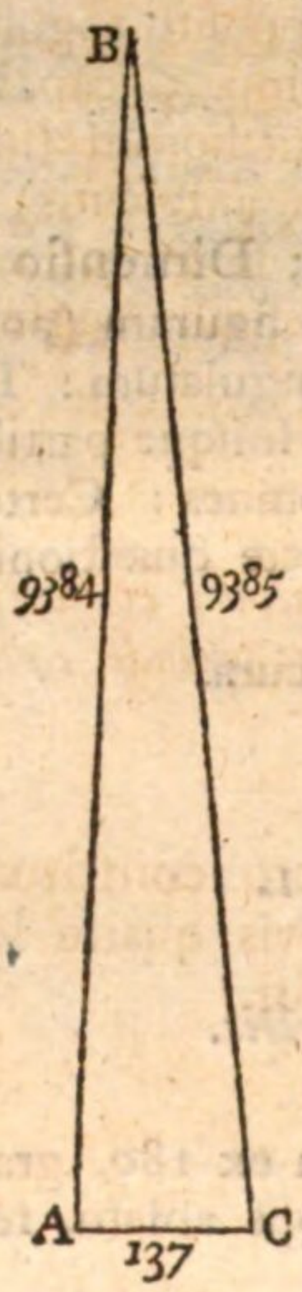
In

In rectangulis *crura* vocamus, [lineas] quæ rectum angulum ambiunt: *hypotenuſam*, [lineam] quæ [eum] ſubtendit.

Prop. 2. *In* [triangulo] *rectangulo Logarithmus cruris æquatur aggregato ex Logarithmo anguli ei oppoſiti, et Logarithmo hypotenuſæ.*

Quum ex Trigonometriæ principiis pateat, alterutrumvis crus ſe habere ad ſinum anguli ei oppoſiti, ut hypotenuſa ad ſinum totum: et (per prop. 5. cap. 2. lib. 1.) horum quatuor proportionalium logarithmi ſecundi et tertii, æquentur logarithmis primi et quarti: quarti autem Logarithmus ſit 0, ſeu nihil (per corollarium 6. def. cap. 1. lib. 1.) Ideò (ut ſuprà) Logarithmus cruris æquatur aggregato ex Logarithmo anguli quem ſubtendit, et Logarithmo hypotenuſæ.

Corol. *Vude hypotenuſæ, cruris, et anguli quem ſubtendit, duobus quibuſcunque datis, tertium, atque inde reliquæ omnes* [trianguli] *rectanguli partes innoteſcent.*



Quia enim hæc tria cum ſinu toto conſtituunt quatuor proportionalia, certum eſt eorum quodvis quarto loco poſſe conſtitui, et per 3. probl. cap. 5. lib. 1. acquiri.

Vt trianguli oblati A B C in A rectanguli, detur hypotenuſa B C 9385, cum crure A B 9384. Quæruntur anguli obliqui C. & B. Ex Logarithmo igitùr A B 635,870 — 000, aufer Logarithmum B C 634,799 — 000. Superſunt 1071, Logarithmus anguli C, cui in tabulâ reſpondent 89 g. 9′¼ pro angulo C, et, ex adverſo, 0 g. 50′¼ pro ejus complemento, angulo, ſcilicèt, B.

Vice verſâ, ſi detur angulus C, cum crure recti anguli A B, & quæratur hypotenuſa B C.

Ex

Ex Logarithmo A B, 6 3 5, 8 7 0 — o o o aufer Logarithmum anguli C 1 o 7 1, et provenient 6 3 4, 7 9 9 — o o o, Logarithmus B C, 9 3 8 5 hypotenuſæ quæſitæ.

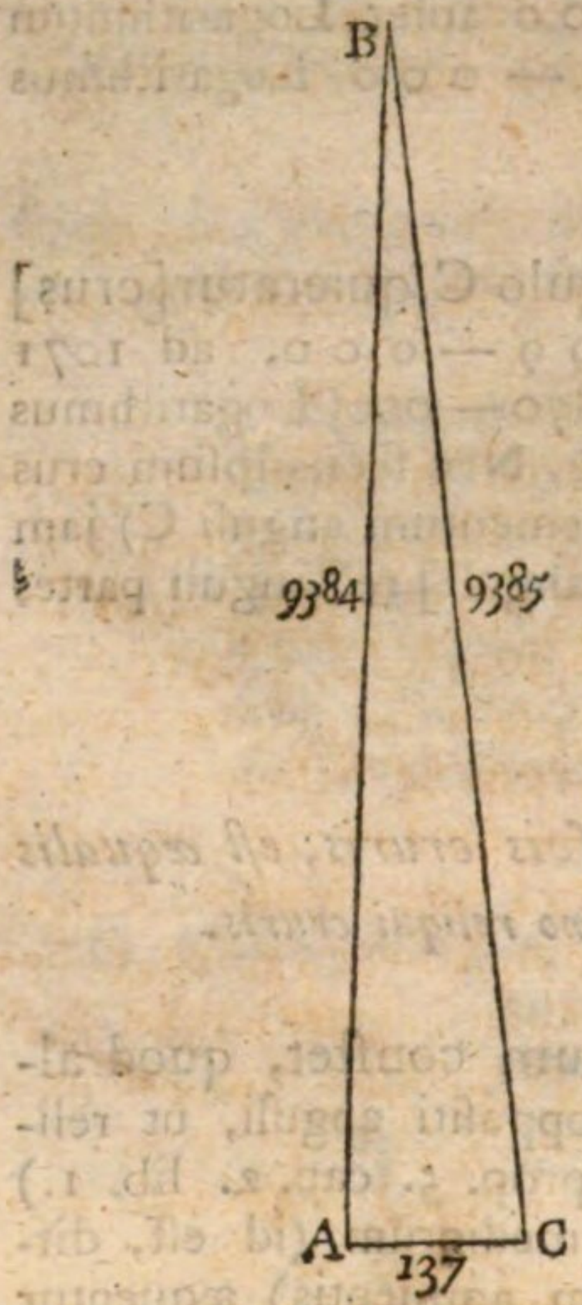
Tertiò, ſi, datis [hypotenuſâ] BC, & angulo C, quæratur [crus] A B: adde Logarithmum B C, 6 3 4, 7 9 9 — o o o, ad 1071 Logarithmum anguli C, et producentur 635,870 — ooo, Logarithmus numeri 9384 cruri A B quæſito reſpondentis. Nec ſecùs ipſum crus reliquum A C ex angulo B (qui eſt complementum anguli C) jam cognito habetur. Atque ita omnes hujus [trianguli] rectanguli partes innoteſcunt.

Prop. 3. *In* [triangulo] *rectangulo Logarithmus cujuſvis cruris, eſt æqualis aggregato ex differentiali oppoſiti anguli, et Logarithmo reliqui cruris.*

Quum ex vulgari doctrinâ triangulorum conſtet, quod alterutrum crus ſe habeat ad tangentem ſibi oppoſiti anguli, ut reliquum crus ad ſinum totum: et quum (per prop. 5. cap. 2. lib. 1.) ex his quatuor proportionalibus Logarithmi mediorum (id eſt, differentialis anguli, et Logarithmus cruris eum ambientis) æquentur Logarithmis cruris eundem ſubtendentis, et ſinûs totius (qui eſt nihil, ſeu o.) ideò Logarithmus cruris eſt æqualis aggregato, &c. ut ſuprà.

Corol. *Vnde ex cruribus* [anguli] *recti, & angulo alteri eorum oppoſito, duobus quibuſcunque datis, tertium (perhanc,) atque proinde cæteræ omnes* [trianguli] *rectanguli partes (per præced.) innoteſcent.*

Quandoquidem hæc tria cum ſinu toto conſtituant quatuor proportionalia, certum eſt eorum quodvis quarto loco poſſe collocari, & per 3. probl. cap. 5. lib. 1. acquiri.



Vt præcedentis trianguli A B C, in A rectanguli, datis cruribus A B, 9384 & A C, 137. Quæratur angulus B. Ex Logarithmo A C, 42,924,534 — 000. Aufer 635,870 — 000, Logarithmum A B, & provenient 42,288,664, differentialis anguli B, 0 gr. 50 m. ii, quæsiti. Verùm, si dentur crus A C, 137, & angulus B, 0 gr. 50′. ii, habebitur crus A B auferendo 42,288,664, differentialem anguli B, à Logarithmo A C; qui est 42,924,534 — 000. Inde enim proveniens 635,870 — 000, est Logarithmus numeri 9384, qui crus est A B quæsitum. Tertiò, datis crure A B, 9384, & angulo B, 0 gr. 50 m. ii. ut habeatur crus A C. adde 635,870 — 000, Logarithmum cruris A B, ad 42,288,664, differentialem anguli B, & provenient 42,924,534 — 000, Logarithmus 137, cruris A C, quæsiti. Hypotenusa autèm B C per præced. prop. habetur. Angulus etiàm C, patet, quum sit complementum anguli B, jam cogniti. Et ita per hanc, & præmissam, ex latere quovis, & parte aliâ quâvis [trianguli] rectanguli, datis reliquæ omnes ejus partes innotescent.

Completam ergò habes [triangulorum] rectangulorum rectilineorum scientiam. Sequitur obliquangulorum.

CAP. II.

De triangulis rectilineis, præsertìm obliquangulis.

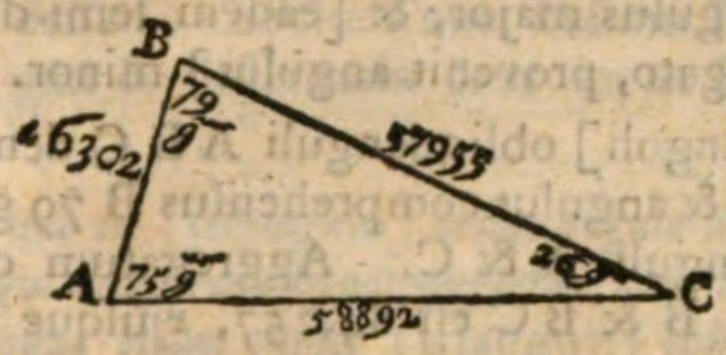
Prop. 4. *IN omni triangulo, aggregatum ex Logarithmis anguli cujusvis, & lateris eum ambientis, æquatur aggregato ex Logarithmis lateris, & anguli eis oppositorum.*

Quia omnium laterum ad oppositorum angulorum sinus eadem est ratio: et ita factum ex anguli cujusvis sinu recto, & latere quovis eum ambiente, æquatur facto ex latere subtendente priorem angulum, &

& ſinu anguli ſubtenſi à priore latere. Ideò (per prop. 5. cap. 2. lib. 1.) aggregatum ex Logarithmis &c. æquatur. ut ſuprà.

Corol. *Vnde ex duobus angulis quibuſcunque datæ ſpeciei, & ſuis ſubtendentibus, ſi tria dantur, quartum quodcunque, atque inde cæteræ omnes trianguli partes, innoteſcent.*

Horum enim quatuor proportionalium quodvis quæſitum poteſt quarto loco conſtitui, & per 3. probl. cap. 5. lib. 1. inveniri.



Vt [trianguli] obliquanguli A B C detur A B 26,302, & B C 57,955, & angulus C 26 graduum: Quæratúrque angulus A, qui ſic habetur. Adde 5,454,707 — 00 Logarithmum B C ad 8,246,889 Logarithmum, ſcilicet, C 26 graduum, & fient 13,701,596 — 00. Hinc aufer Logarithmum A B, qui eſt 13,354,921 — 00; reſtant 346,675, Logarithmus 75 graduum, & paulò pluris, anguli, ſcilicet, A quæſiti, ſi A prædicatur acutus: alioqui 105 gr. (per 1. & 2. ſect. cap. 3. lib. 1.) ſi pronuncietur obtuſus.

Vice verſâ, ſi detur angulus A jam 75 graduum, atque angulus C, & latus B C, ut ſuprà: & quæratur A B. Adde 5,454,707 — 00 Logarithmum B C, ad 8,246,889, Logarithmum anguli C; fient, ut ſuprà, 13,701,596 — 00; à quibus aufer 346,675, logarithmum anguli A; provenient 13,354,921 — 00, Logarithmus lateris A B, et numeri ejus 26,302 quæſiti. Habitis jam angulis A. 75 gr. & C. 26 gr. erit angulus B. 79 gr. per 1. hujus. Ex quo jam habito, non ſecùs acquiritur latus ei oppoſitum A C 58,892, quàm nuperrimè ex angulo C innotuit latus ei oppoſitum A B. Itaque jam patent omnes hujus [trianguli] obliquanguli partes.

In [triangulis] *obliquangulis* crura *vocamus*, [lineas] *quæ angulum quemvis ambiunt:* baſim [lineam] *quæ* [eum] *ſubtendit.*

Prop. 5. *In* [triangulis] *obliquangulis, Logarithmus aggregati crurum ſubductus à ſummâ factâ ex Logarithmo differentiæ crurum, & differentiali ſemi-aggregati eorum oppoſitorum angulorum, relinquit differentialem ſemi-differentiæ eorundem.*

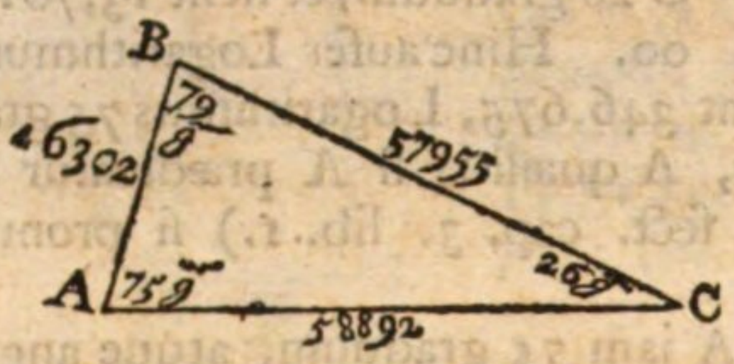
Quia, ut aggregatum crurum ad differentiam crurum, ita tangens ſemi-aggregati ſuorum oppoſitorum angulorum, ſe habet ad tangentem ſemi-differentiæ eorundem. Vnde analoga ſunt, & (per prop.

1. cap. 2. lib. 1.) eorundem differentiæ, ſeu exceſſus, ſunt æquales. Neceſſariò igitùr (per prop. 4. cap. 2. lib. 1.) concludimus ut ſuprà.

Corol. *Vnde ex duobus cruribus, & angulo comprehenſo, innoteſcunt (per hanc) anguli reliqui oppoſiti: atque inde (per præmiſſam) reliquum latus.*

Nam, ſubducto Logarithmo aggregati crurum à ſummâ factâ ex logarithmo differentiæ crurum & differentiali ſemi-aggregati oppoſitotum angulorum additis, proveniet differentialis ſemi-differentiæ eorundem angulorum: quâ ſemi-differentiâ additâ ad ſemi-aggregatum dictum, proveniet angulus major, & [eâdem ſemi-differentiâ] ſubtractâ [à dicto ſemi-aggregato, provenit angulus] minor.

Vt repetiti ſuperioris [trianguli] obliquanguli A B C dentur crura, A B 26,302, & B C 57,955, & angulus comprehenſus B 79 graduum: Quærantur autem reliqui anguli A & C. Aggregatum crurum A B & B C eſt 84,257, ejúſque Logarithmus eſt 24,738,819 — 0; differentia autèm eorundem A B & B C eſt 31,653, ejúſque Logarithmus eſt 34,529,210 — 0. Quúmque B angulus detur 79 g. erit (per 1. hujus) aggregatum angulorum A & C, graduum 101, ſemi-aggregatum verò 50 gr. 30 m. cujus differentialis eſt — 1,931,766; quo ad 34,529,210 — 0 addito, fient 32,597,444 — 0; hinc ablatis 24,738,819 — 0, provenient + 7,858,625 differentialis graduum 24. 30 m. qui ſunt ſemi-differentia angulorum A & C quæſitorum. Hanc ergò ſemi-differentiam 24 g. 30 m. adde ad ſemi-aggregatum 50 g. 30 m.; fient 75 gradus, pro angulo A, quæſitorum majore; & ſubtrahe eoſdem 24½ gradus ab eiſdem 50½ gradibus, & relinquentur 26 gradus pro angulo B, quæſitorum minore.



Defi- *In* [triangulis] *obliquangulis vera baſis ſemper eſt vel aggregatum caſuum: &*
nitio. *tunc differentia caſuum* baſis alterna *vocatur: vel vera baſis eſt differentia caſuum: et tumaggregatum caſuum vocamus* alternam.



Vt trianguli A B C, caſus minor eſt A D: caſus major eſt D C. Caſum aggregatum A C eſt baſis vera. Et in hoc triangulo aufer caſum minorem A D, ſeu ei æqualem D E à caſu majore D C, relinquetur differentia caſuum E C, quam *baſim alternam* vocamus. Contrà verò in

triangulo E B C casus minor est D E (cui æquatur D A.) Casus major est D C, & casuum differentia E C est basis vera. Casuum autèm aggregatum, scilicèt, A C, *basim alternam* vocamus.

Prop. 6. *In* [triangulis] *obliquangulis summa Logarithmorum aggregati et differentiæ crurum, est æqualis summæ Logarithmorum basium, veræ, et alternæ.*

Quia basis vera se habet ad aggregatum crurum, ut differentia crurum ad basim alternam; Ideò (per prop. 5. cap. 2. lib. 1.) necessariò concludimus, basium Logarithmos æquari Logarithmis aggregati & differentiæ crurum, ut suprà.

Corol. *Vnde ex* [triangulo] *obliquangulo datorum laterum, fiunt duo* [triangula] *rectangula notarum hypotenusarum cum altero cujusque* crure, quæ (per 2. hujus) reliquas etiàm omnes* [trianguli] *obliquanguli partes notas reddunt.*

* potiùs, *utriusque.*

Nam, addito Logarithmo aggregati crurum ad Logarithmum differentiæ crurum, & hinc ablato Logarithmo basis veræ, proveniet Logarithmus basis alternæ, per prop. 4. cap. 2. & probl. 3. cap. 5. lib. 1. Harum itaque basium semi-aggregatum est casus major: semi-differentia verò casus minor. Vt, [exempli gratiâ] superioris trianguli ABC dentur latera, videlicet, crus A B 26,302, & crus B C 57,955, & basis A C 58,892, & quærantur cætera. Aggregatum crurum est 84,257, ejúsque Logarithmus est 24,738,819— 0. Differentia crurum est 31,653, ejúsque Logarithmus est 34,529,210 — 0. Hos Logarithmos adde; fient inde 59,268,029 — 00; à quibus aufer 5,293,461 — 00 Logarithmum basis A C; restant 53,974,568 Logarithmus numeri 45,286 basis alternæ: quam ad veram adde; fient inde 104,178, quorum dimidium est 52,089, D C, casus major. Eandem ab eâdem aufer; fient inde 13,606, quorum dimidium est 6803, A D, casus minor.

Rectanguli itaque [trianguli] A D B, habitis jam, hypotenusâ AB, & crure altero A D; atque rectanguli [trianguli] B D C habitis, hypotenusâ B C, & crure D C, innotescunt (per 2. hujus) anguli [triangulorum] rectangulorum apud A & B & C, &, per consequens, omnes etiàm [trianguli] obliquanguli oblati partes ex præmissis propalantur.

Nec secùs agendum foret si darentur latera trianguli E B C, & cæteræ partes quærerentur. Ex cruribus enim & basi verâ E C, innotescit basis alterna A C, atque ex his uterque casus, & cætera, ut suprà.

CONCLVSIO.

PErfectam igitùr et completam jam habes omnium triangulorum rectilineorum doctrinam, quæ ſi aliquantulum operoſa in Logarithmis rectarum variabilium inveniendis videatur: In motibus tamen planetarum computandis (in quibus, ſcilicèt, eccentricitates orbium, elongationes Augium et apogæorum, epicyclorum diametri, et aliæ rectæ, eædem et invariabiles permanent) eorum Logarithmi, exactè ſemèl notati, ſempèr in poſterum, ſine ullâ mutatione ſubſervient, mirandâ certè facilitate, et certitudine.

Sequuntur jam Sphærica triangula, omnium difficillima, ut vulgò ab aliis traduntur; per Logarithmos tamèn noſtros, omnium facillima.

CAP. III.

De Triangulis Sphæricis.

Sententia 1. *IN Triangulis Sphæricis angulus omnium quadranti quantitate proximus, et latus eum ſubtendens dubia ſunt, An ejuſdem, an diverſæ, ſint ſpeciei, niſi id aut computus, aut hypotheſis, prodat.*

2. *Duorum verò obliquiorum angulorum quilibet eſt ejuſdem ſpeciei, cujus eſt latus eum ſubtendens. Vnde, alterius* [ſpecie] *datâ, reliqui patet ſpecies.*

3. *Si trianguli angulus aliquis propinquior ſit quadranti, quàm latus eum ſubtendens, erunt duo ejus latera ejuſdem ſpeciei, et tertium quadrante minus.*

4. *Si verò trianguli latus aliquod propinquius ſit quadranti, quàm eo ſubtenſus angulus: erunt duo ejus anguli ejuſdem ſpeciei, et tertius quadrante major.*

5. *Triangulum Sphæricum aut eſt quadrantale, aut non.*

6. *Quadrantale eſt cujus aut latus, aut angulus, æquatur quadranti.*

Vnde, [trianguli]non-rectanguli quadrantalis ſcientiam æquè facilè ac [trianguli] rectanguli comparari poſſe, docemus.

7. *Quadrantale triangulum aut eſt multiplex, aut ſimplex.*

8. *Multiplex quadrantale aut eſt trirectangulum, aut birectangulum.*

9. *Trirectangulum eſt cujus ſingulæ partes quadranti æquantur.*

10. *Vnde omne triangulum, cujus trium partium non oppoſitarum ſingulæ quadranti æquantur, Trirectangulum eſt.*

11. [Trian-

11. [Triangulum] *Birectangulum eſt, cujus duo tantùm anguli, et ſua ſubtendentia altera, ſigillatìm quadranti æquantur.*

12. *In omni* [triangulo] *birectangulo angulus obliquus æquatur ſuo ſubtendenti lateri.*

13. *Omne Triangulum cujus pars aliqua æquatur quadranti, & angulus aliquis obliquus æquatur ſuo ſubtendenti, Birectangulum eſt.*

14. *Omne triangulum habens duas quaſcunque partes ſigillatim quadranti æquales, et tertiam inæqualem, Birectangulum eſt.*

15. *Cætera* quadrantalia ſimplicia *dicuntur.*

CAP. IV.

De [triangulis] ſimplicibus Quadrantalibus.

1. [Triangulum] *Quadrantale ſimplex eſt, cujus unica tantùm pars quadranti æquatur, cæteræ autèm quinque partes ſunt non-quadrantes.*

2. *Harum quinque partium non quadrantium, Tres quæ à recto angulo, ſeu quadrante latere, ſitu remotiores ſunt, in ſua complementa convertimus, et retento priſtino ordine omnes quinque in circularem, ſeu pentagonalem, ſitum ſtatuimus, et* circulares *vocamus.*

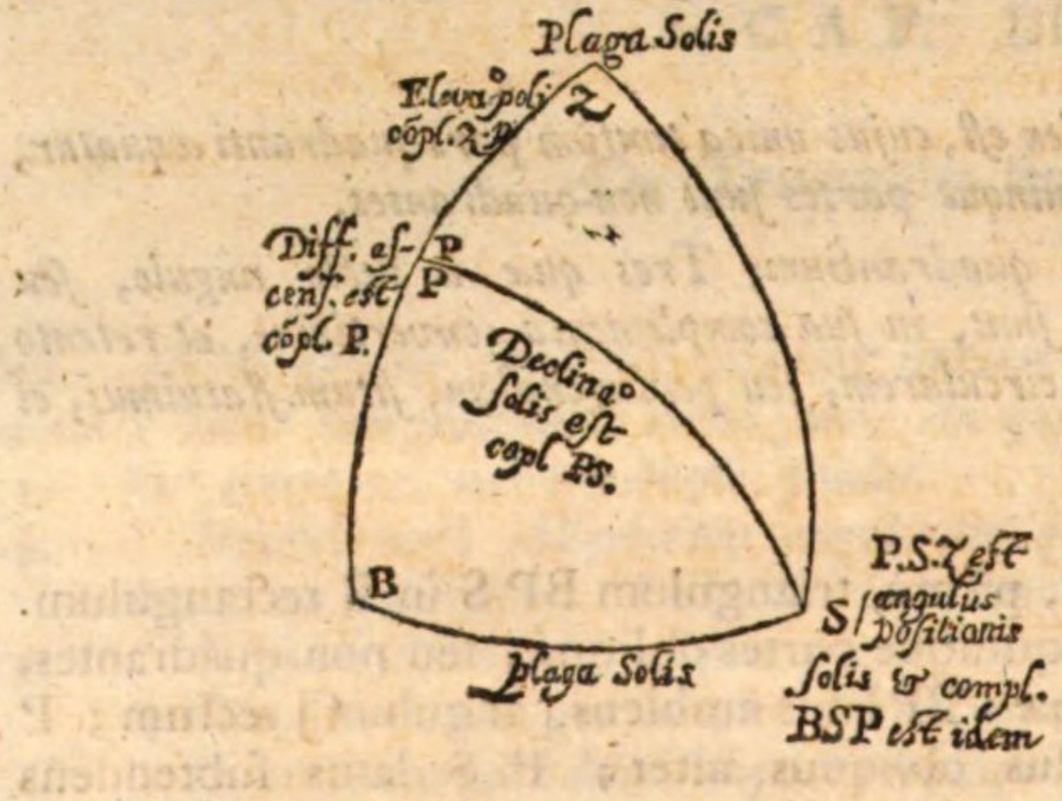


Sit, primò, triangulum BP S in B rectangulum. Ejus quinque partes obliquæ, ſeu non-quadrantes, ſunt hæ: BP latus ambiens [angulum] rectum; P angulus obliquus alter; P S latus ſubtendens rectum; S angulus reliquus obliquus; S B reliquum latus ambiens [angulum] rectum. Pro quibus nos, facilioris calculi gratiâ, aſſumimus latus B P ipſum; complementum anguli P; Complementum lateris P S; complementum anguli S; atque ipſum latus S B: &, ſervato naturali ſitu, has quinque partes ordine ſtatuimus, ut à margine, & *circulares* vocamus.

Similitèr ſit, ſecundò, triangulum quadrantale ſimplex, non-rectangulum (ex centris ſolis orientis, poli, & zenith factum) S P Z, in latere



latere Z S quadrantale. Ejus quinque partes non-quadrantes priſtinæ ſunt: Z, angulus alter ambitus à latere quadrante; Latus P Z, diſtantia poli à zenith; P, angulus ſubtenſus à quadrante; Latus P S, diſtantia poli à Sole; & angulus denique S, alter angulorum quos quadrans ambit. Pro quibus nos, ad faciliorem computum noſtrum, aſſumimus ipſum angulum Z, ſeu P Z S, qui eſt arcus plagæ Solis à ſeptentrione; Complementum P Z, quod eſt ipſa elevatio poli; Complementum anguli P, ſeu anguli Z P S, quod eſt differentia aſcenſionalis, id eſt, differentia temporis ortûs vel occaſûs Solis ab horâ ſextâ; Complementum lateris P S, quod eſt Solis declinatio: & angulum ipſum S ſeu P S Z, quem *poſitionis Solis* (reſpectu, ſcilicet, poli & zenith) vocamus. Has quinque partes etiàm circulari vel pentagono ſitu ſtatuimus, ut à margine, & *circulares* vocamus. Nec aliæ fient circulares partes ſuperioris trianguli rectanguli B P S, ſi P polum, S ſolem, & B cardinem borealem, ſeu ſeptentrionalem, poſueris. Fient enim latus B P elevatio poli, complementum P differentia aſcenſionalis, Complementum P S declinatio ſolis, complementum S angulus poſitionis ſolis: ac denique B S plaga ſolis. Quæ ſunt eædem prorſùs circulares partes, quæ ſuprà, & eodem ſitu lævorſùm, quo illæ dextrorſùm, diſpoſitæ. Et ita in omnibus [triangulis] quadrantalibus, tàm rectangulis quàm non-rectangulis.



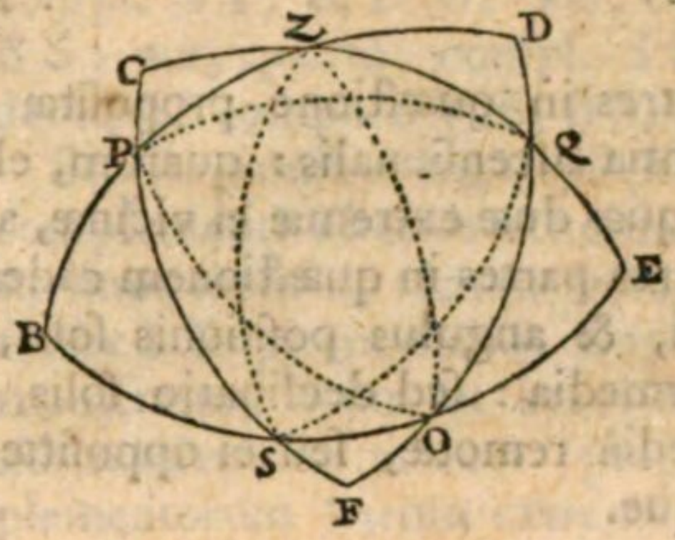
Corol. *Hinc fit quod plurima ſint triangula in partibus ſuis naturalibus haud cõnformia, quæ in partibus his circularibus prorſùs conveniunt, & hæc noſtrâ circularium methodo reſolvuntur.*

Vt ſatìs lucidè apparet in duobus ſuperioribus triangulis B P S, & P Z S conjunctis. In quibus omnes naturales partes (præter P S & B S hujus, & P S & P Z S illius) prorſùs differunt: circulares verò partes omnes (ut ſuprà dictum eſt) conveniunt.

4. *Hæc*

4. *Hæc circularium partium uniformitas manifeſtiſſimè patet in rectangulis factis, in ſuperficie globi ex quinque circulis magnis, quorum primus ſecet ſecundum, ſecundus tertium, tertius quartum, quartus quintum, quintus deniq. primum ad rectos angulos : reliquæ verò ſectiones omnes ad angulos obliquos fient.*

Exempli gratiâ : Meridianus regionis D B, ſecat horizontem B E in puncto B. Horizon B E ſecat circulum E C, qui ſolem ambit (id eſt, qui circà ſolem tanquam polum ducitur) in puncto E. Circulus E C, qui ſolem ambit, ſecat meridianum ſolis C F in puncto C. Meridianus ſolis C F æquatorem F D in puncto F : & tandem æquator F D ſecat meridianum regionis D B in puncto D. Et omnes hæ quinque ſectiones in punctis B. E. C. F. D orthogonalitèr & ad rectos angulos fiunt : factis cæteris ſectionibus in punctis Z. P. S. O. Q. ad angulos obliquos. Fiéntque ex his ſectionibus [triangula] rectangula quinque P B S, S F O, O E Q, Q D Z, & Z C P, quorum quamvis partes naturales differant, & in ſingulis triangulis varientur, circulares tamen quinque partes eædem ſunt, quæ ſuprà, abſque ullo diſcrimine.



5. *Eadem circularium partium uniformitas patet etiàm in* [triangulis] *quadrantalibus non-rectangulis, factis in ſuperficie globi ex quinque punctis, quorum primus diſtet à ſecundo, ſecundus à tertio, tertius à quarto, quartus à quinto, & quintus à primo diſtantiis & arcubus æqualibus quadranti, aliæ verò punctorum diſtantiæ inæquales ſint quadranti.*

Vt in eodem præcedente ſchemate puncta, P à Q, Q ab S, S ab Z, Z ab O, atque O à P, diſtant ſpatiis quadranti æqualibus : at verò P ab Z, Z à Q, Q ab O, O ab S, & S à P, diſtant ab invicèm arcubus non-quadrantibus. Et fient ex his diſtantiis [triangula] quadrantalia non-rectangula quinque, P Z Q, Z Q O, Q O S, O S P, & S P Z : quorum quamvis naturales partes differant, partes tamen circulares eædem & immutabiles hîc permanent, quæ ſuprà : Scilicèt, elevatio poli, differentia aſcenſionalis, declinatio ſolis, angulus poſitionis ſolis, & plaga ſolis : quæ omnibus ſuperioribus triangulis ex æquo conveniunt, nec his duntaxàt ſolis, verumetiàm omnibus triangulis quæ oriuntur ex interſectionibus cæteris horum decèm arcuum ad integros circulos productorum : quæ, quia plurima & confuſa ſunt, miſſa hic facimus.

facimus. Hâc epitome ſatìs eſt monuiſſe omnem confuſionem naturalium partium, & ſuarum regularum, his paucis circularibus partibus & ſuâ regulâ unicâ evitari ac tolli.

6. *Quinque circularium partium, tres ſemper in quæſtionem cadunt, quarum duæ dantur, tertia quæritur.*

7. *Atque harum trium una eſt intermedia, & duæ ſunt extremæ, quæ, ſcilicet, intermediæ aut circumponuntur, aut opponuntur.*

Verbi gratiâ, Sint partes tres in quæſtione propoſitæ hæ, plaga ſolis, elevatio poli, & differentia aſcenſionalis: quarum, elevatio poli pars intermedia dicitur, & reliquæ duæ extremæ ei vicinæ, aut circumpoſitæ vocantur. Verùm, ſi tres partes in quæſtionem cadentes forent, declinatio ſolis, elevatio poli, & angulus poſitionis ſolis, vocabitur (ut priùs) elevatio poli intermedia: ſed declinatio ſolis, & angulus poſitionis ſolis, extremæ à mediâ remotæ, ſeu ei oppoſitæ, dicentur. Par ratio eſt in reliquis quinque.

8. *Logarithmus intermediæ æquatur differentialibus circumpoſitarum extremarum, ſeu antilogarithmis oppoſitarum extremarum.*

Hoc theorema probatur inductione omnium trium partium ſeu triplicitatum, quæ ex quinque circularibus partibus [trianguli] quadrantalis prioris B P S rectanguli, conſtitui poſſunt, & in quæſtionem cadere. Poſterioris autèm [trianguli] non-rectanguli P Z S triplicitates omittimus, quia ejus omnes partes circulares (ex 18, & 19, & 20 præmiſſis) eædem prorſùs ſunt quantitate, quæ prioris. Quinque ergò partium circularium [trianguli] rectanguli B P S, (quæ ſunt B S, ſeu plaga ſolis orientis; complementum B S P, ſeu angulus poſitionis ſolis; complementum S P, ſeu declinatio ſolis; complementum S P B, ſeu differentia aſcenſionalis; & P B, ſeu elevatio poli;) tres illæ quæ in quæſtionem extremarum circumpoſitarum cadunt, ſunt, aut, primò, B S, complementum B S p, & compl. S p: aut, ſecundò, complem. B S P, complem. S P, & compl. S P B: aut, tertiò, comp. S P, compl. S P B, & P B: aut, quartò, compl. S P B, P B, & B S: aut, quintò, ſunt P, B, B S, & complem. B S P.

Verùm, quia in omnibus his triplicitatibus, Tangens alterius extremæ eſt ad ſinum rectum intermediæ, ut ſinus totus ad tangentem reliquæ extremæ (proùt ex vulgaribus demonſtrationibus Trigonometriæ patet,); Ideò (per noſtras demonſtrationes prop. 5. cap. 2. lib. 1.) Logarithmi mediarum (qui ſunt Logarithmus ſolius intermediæ

per

per corol. 6. def. cap. 1. lib. 1.) æquantur logarithmis tangentium utriuſque extremæ. Sed Logarithmi tangentium harum extremarum ſunt differentiales earundem (ex ſect. 22. & 25. cap. 3. lib. 1.) Logarithmus igitùr ſolius intermediæ æquatur differentialibus circum poſitarum extremarum, ut in priore parte Theorematis aſſeruimus. Sequitur poſterioris partis confirmatio.

Earundem ergò quinque partium circularium, tres illæ quæ in quæſtionem extremarum intermediæ oppoſitarum cadunt, ſunt aut, primò, P B, comp. B S P, & compl. S P B: aut, ſecundò, B S, compl. S P, & P B: aut, tertiò, compl. B S P, compl. S P B, & B S: aut, quartò, compl. S P, P B, & compl. B S P: aut, quintò denique, compl. S P B, B S, & compl. S P.

Sed in omnibus his triplicitatibus ſeu quinque caſibus, ſinus rectus complementi alterius extremæ ſe habet ad ſinum rectum intermediæ, ut ſinus totus ad ſinum rectum complementi reliquæ extremæ (quod fuſiùs à Regiomontano, Copernico, Lanſbergio, Pitiſco, & aliis demonſtratur, quàm ut brevi hâc epitome repetendum ſit.) Ideò per noſtras demonſtrationes (prop. 5. cap. 2. lib. 1.) Logarithmi complementorum harum extremarum æquantur Logarithmis mediarum, id eſt, (ut dictum eſt) Logarithmo ſolius intermediæ. At Logarithmi complementorum harum extremarum oppoſitarum ſunt earundem ipſarum partium antilogarithmi (ex def. ſect. 13. & 16. cap. 3. lib. 1.) Sequitur ergò in his caſibus, quod logarithmus ſolius intermediæ æquetur antilogarithmis ſuarum extremarum oppoſitarum, ut aſſerit poſterior theorematis pars. Totum itaque theorema conſtat. Præter hanc probationem per inductionem omnium caſuum, qui occurrere poſſunt, poteſt idem theorema lucidè perſpici ex 19ª. & 20ª. præcedentibus, in quorum ſchemate, homologa circularium partium conſtitutio earundem analogiæ ſimilitudinem arguit: ita ut quod de unâ intermediâ & ſuis * extremis circumpoſitis, aut oppoſitis, verè enuntiatur, de cæteris quatuor intermediis & ſuis † extremis reſpectivè cirmcumpoſitis, aut oppoſitis, negari non poſſit.

* Potiùs, *ejus.*
† Potiùs, *earum.*

Porisma generale.

9. *Hinc ſequitur in* [triangulis] *quadrantalibus ſimplicibus, quod ex duabus partibus quibuſcunque datis tertia quævis innoteſcet. Semper enim aut intermedia quæritur, & ejus logarithmus habetur addendo differentiales circumpoſitarum extremarum datarum: aut altera extremarum quæritur, & ejus differentialis emergit ex ſubtractione differentialis reliquæ extremæ datæ à Logarithmo intermediæ notæ: ut in quinque prioribus triplicitatibus* [trianguli] *rectanguli præcedentis theorematis, et totidem*

dem [trianguli] *non-rectanguli: aut intermedia quæritur, et ejus Logarithmus provenit addendo antilogarithmos oppositarum extremarum datarum: aut denique altera extremarum oppositarum quæritur: et ejus antilogarithmus ex subductione antilogarithmi reliquæ extremæ oppositæ datæ ex Logarithmo intermediæ notæ habetur. Vt in quinque posterioribus casibus* [trianguli] *rectanguli præcedentis theorematis, et totidem* [trianguli] *non-rectanguli. Horum autem Logarithmorum, antilogarithmorum, et differentialium jam inventorum cuilibet respondent duo arcus diversarum specierum. Ex specie igitur quæsiti arcûs, per secundam, tertiam, quartam, hujus, aut per hypothesim, notâ, ipse arcus verus innotescet.*

Vt in priore exemplo septimæ, Tres quæstionis partes circulares sunt, plaga solis, elevatio poli, & differentia ascensionalis, id est, In [triangulo] rectangulo B P S, partes BS, P B, et complem. S P B: vel in [triangulo] non-rectangulo, quadrantali, P Z S, partes P Z S, compl. PZ & compl. S P Z: quarum trium dentur extremæ circumpositæ, scilicet, plaga solis orientis B S, vel P Z S, 70 gr. & differentia ascensionalis compl. S P B, vel compl. S P Z, 16 gr. 24′. 27″: & quæratur intermedia pars P B, vel compl. P Z, quæ est elevatio poli. Addatur ergò differentialis 70. gr., viz. — 10,106,827, ad differentialem 16 gr. 24′. 27″. videlicet, ad 12,226,180, & provenient 2,119,353, Logarithmus 54 graduum pro elevatione poli quæsitâ.

Admontio.

PRæter elevationem poli hoc modo inventam, habetur etiàm, secundò, eâdem praxi plaga solis ex elevatione poli, & angulo positionis solis. Itèm, tertiò, angulus positionis solis ex plagâ solis, & ejusdem declinatione, datis. Quartò, declinatio solis ex angulo positionis solis, & differentiâ ascensionali. Quintò, differentia ascensionalis ex declinatione solis, & elevatione poli.

Secundum exemplum.

DEtur plaga solis orientis B S, seu P Z S, 70. graduum: & elevatio poli 54. graduum, quæ est P B, aut compl. P Z. Quæratur autèm differentia ascensionalis, scilicet, compl. S P B, vel compl. S P Z. Et, quia hîc similitèr extremæ partes circumponuntur intermediæ, ergò aufer differentialem plagæ solis seu 70 graduum, qui est — 10,106,827. ex Logarithmo elevationis poli 54 graduum, scilicèt, ex 2,119,353; & provenient inde 12,226,180, differentialis

ferentialis graduum 16. 24′ 27″ ferè, arcus differentiæ aſcenſionalis quæſitæ.

Admonitio.

AD hujus exempli imitationem habetur, ſecundò, declinatio ſolis ex differentiâ aſcenſionali, et elevatione poli, datis. Itèm, tertiò, angulus poſitionis ſolis ex declinatione ſolis, & differentiâ aſcenſionali. Quartò, plaga ſolis ex angulo poſitionis ſolis, & declinatione ejuſdem. Quintò, elevatio poli habetur ex plagâ ſolis, & angulo poſitionis ſolis. Itèm contrà habetur, Sextò, differentia aſcenſionalis ex declinatione ſolis, & angulo poſitionis ſolis datis: Septimò, declinatio ſolis ex angulo poſitionis ſolis, & plagâ ejus: Octavò, angulus poſitionis ſolis habetur ex plagâ ſolis & elevatione poli, datis: Nonò, plaga ſolis ex elevatione poli, & differentiâ aſcenſionali: Decimò, tandem, elevatio poli habetur ex differentiâ aſcenſionali, & declinatione ſolis, datis.

Tertium exemplum.

IN poſteriore exemplo ejuſdem ſeptimæ tres quæſtionis partes circulares proponuntur hæ, declinatio ſolis, elevatio poli, & angulus poſitionis ſolis. Eæ ſunt in [triangulo] rectangulo B P S compl. P S, B P & compl. B S P, & in [triangulo] non-rectangulo, quadrantali, P Z S eæ ſunt, compl. P S, compl. Z P, & Z S P. Quarum trium dentur extremæ oppoſitæ, ſcilicet, declinatio ſolis, quæ eſt compl. P S 11 gr. 35′. 51″, & angulus poſitionis ſolis, qui eſt compl. B S P, ſeu Z S P, 34 gr. 19′. 21″ ferè. Et quæratur intermedia pars B P, ſeu compl. Z P, quæ eſt elevatio poli. Addatur ergò antilogarithmus 11 gr. 35′. 51″, qui eſt 206,271, ad antilogarithmum 34 gr. 19.′ 21″, qui eſt 1,913,082; provenient 2,119,353, Logarithmus 54 graduum pro elevatione poli quæſitâ.

Admonitio.

PRæter elevationem poli hoc jam modo inventam, poteris, ſecundò, per eandem praxim habere plagam ſolis ex ejuſdem declinatione, & differentiâ aſcenſionali, datis. Tertiò, angulum poſitionis ſolis ex differentiâ aſcenſionali & elevatione poli. Quartò, declinationem

ſolis ex elevatione poli & plagâ ſolis. Et quintò, invenies differentiam aſcenſionalem ex plagâ ſolis & angulo poſitionis ſolis datis.

Quartum exemplum.

DEtur declinatio ſolis, compl. S P. 11 gr. 35′. 51″, & elevatio poli B P, ſeu compl. P Z graduum 54. Quæratur autem angulus poſitionis ſolis, compl. B S P, ſeu P S Z. Et, quia hic ſimiliter extremæ partes intermediæ opponuntur, igitùr auferendus erit antilogarithmus 11 graduum 35′. 51″. qui eſt 206,271, ex logarithmo 54 graduum, qui eſt 2,119,353; & ſupererunt 1,913,082, antilogarithmus 34 graduum 19′. 21″. ferè, qui ſunt angulus poſitionis ſolis quæſitus.

Admonitio.

PRæter angulum poſitionis ſolis hâc primâ praxi acquiſitum, habetur, ſecundò, eâdem praxi declinatio ſolis ex datis differentiâ aſcenſionali & plagâ ſolis. Tertiò, habetur differentia aſcenſionalis ex datis elevatione poli & angulo poſitionis ſolis. Quartò, elevatio poli invenitur ex plagâ ſolis & ejuſdem declinatione datis. Quintò, plaga ſolis acquiritur ex angulo poſitionis ſolis & differentiâ aſcenſionali. Sextò, (contrario ordine) angulus poſitionis ſolis invenitur ex plagâ ſolis & differentiâ aſcenſionali datis. Septimò, declinatio ſolis habetur ex angulo poſitionis ſolis, & elevatione poli, datis. Octavò, differentia aſcenſionalis ex declinatione ſolis, & ejuſdem plagâ invenitur. Nonò, elevatio poli habetur ex datâ differentiâ aſcenſionali, & angulo poſitionis ſolis. Decimò, tandem, acquiritur plaga ſolis, ex elevatione poli, & declinatione ſolis, datis.

Atque ita ad imitationem horum quatuor exemplorum, triginta variæ ſolvuntur quæſtiones circularium partium, hoc eſt, triginta variæ quæſtiones in [triangulo] quadrantali, rectangulo, & totidem in [triangulo] non-rectangulo ſolvuntur hoc poriſmate, beneficio unius tantummodò additionis vel ſubtractionis. Cæterùm ad intelligentiam poſterioris partis hujus poriſmatis, de arcuum ſpeciebus, vide exempla, (tertium quartum, quintum, & ſextum) ſequentia.

CAP.

CAP. V.

De [triangulis] non-quadrantalibus mixtis.

HActenùs [triangulorum] *quadrantalium: ſequitur triangulorum Sphæricorum non-quadrantalium doctrina.*

1. *Non-quadrantale eſt triangulum Sphæricum, cujus nec latus, nec angulus quadrans eſt.*
2. *Non-quadrantale reducitur ad bina quadrantalia, ſi à vertice ad ejus baſim (proùt opus fuerit extenſam) dimittatur perpendicularis, aut quadrans arcus.*
3. *Perpendicularis cadit intrà triangulum, ſi anguli apud baſim ſint ejuſdem ſpeciei: extrà verò, ſi diverſæ: & contrà.*
4. *Quadrans arcus cadit extrà triangulum, ſi crura ſint ejuſdem ſpeciei: Intrà verò, ſi diverſæ: & contrà.*
5. *Ex non-quadrantalis ſex partibus, tres datæ ſolùm ſufficiunt ad reliquarum ſcientiam comparandam: niſi forſàn trium datarum, quarum una alteri opponatur, tertia ſit propinquior quadranti, quam altera ejuſdem generis data. In hoc enim caſu requiritur etiàm dari ſpecies partis quæ tertiæ opponitur, ut reliquæ ſciantur.*

Hujus caſus exempla ſunt quartum & ſextum exemplum ſequentium.

6. *Partes tres datæ aut miſcellaneæ ſunt, aut puræ.*
7. *Miſcellaneæ ſunt, quarum una eſt diverſi generis à reliquis duabus.*

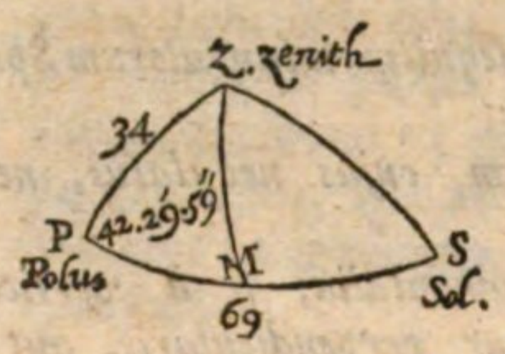
Vt cùm dantur duo latera, & angulus aliquis: aut duo anguli cum latere aliquo.

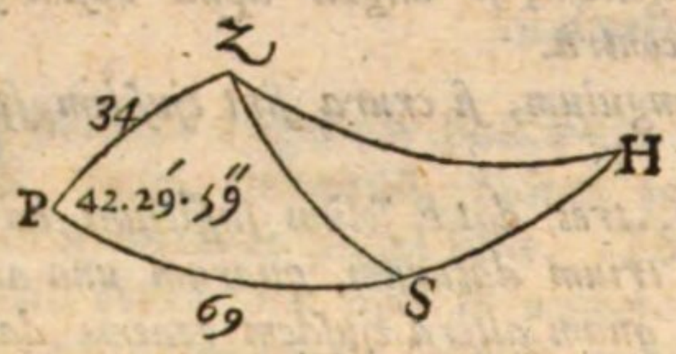
8. *In partibus miſcellaneis datis, ſi ab illo termino lateris dati in cujus reliquo termino ſit angulus datus, cadat ad baſim perpendicularis aut quadrans datum illum angulum ſubtendens, reducetur non-quadrantale ad bina quadrantalia per nonam ſect. cap. 4. hujus ſcibilia.*

Vnde & [triangoli] non quadrantalis partes (quia cum horum quadrantalium partibus, aut partium reliquiis, communes ſunt) facilè innoteſcunt; cognitis tamen priùs per 2. 3. & 4. ſect. cap. 3. hujus, aut ex hypotheſi, partium ſpeciebus.

Exemplum

Exemplum duorum laterum, & anguli interpoſiti datorum.





VT ſit (uſûs & exercitii gratiâ) Triangulum Sphæricum non-quadrantale in ſuperficie primi mobilis deſcriptum P Z S, polum, Zenith, & ſolem referens: cujus ſex partes ſunt, latus P Z, quod eſt interſtitium poli & Zenith, ſeu complementum elevationis poli; Latus Z S, interſtitium Zenith & ſolis, ſeu complementum altitudinis ſolis; Latus P S, interſtitium poli & ſolis, ſeu complementum declinationis ſolis ab æquatore; Angulus Z P S, hora diei, ſeu tempora horaria æquatoris; Angulus P Z S, quæ plaga eſt, ſeu azimuth ſolis à ſeptentrione; Angulus P S Z, qui angulus eſt ſitûs & poſitionis ſolis ad polum & zenith. Harum ſex partium dentur tres quæcunque miſtellaneæ. Verbi gratiâ, angulus horarius Z P S 42. 29′. 59″. (qui horam ſecundam 49′. 59″. 56‴ pomeridianam notat.) & latus P Z 34. complementum elevationis poli; Atque latus, P S, 69°, complementum declinationis ſolis. Ex quibus, ut acquirantur tres reliquæ partes, ab Z, termino lateris P Z dati, dimittatur perpendicularis Z M, aut, ſi mavis, quadrans Z H angulum datum Z P S ſubtendens, reducénſque [triangulum] non-quadrantale oblatum P Z S ad duo triangula in angulo M quadrantalia, quæ ſunt P M Z & Z M S, ut in primo ſchemate: vel (ſi varietate delecteris) ad duo triangula in latere Z H quadrantalia, quæ ſunt Z H P & Z H S, ut in ſecundo ſchemate: Quorum quadrantalium omnes partes per 9 ſect. cap. 4. hujus acquires. Nam ex datis P Z 34, atque Z P M, ſeu Z P S, 42. 29′. 59″, invenies perpendicularem Z M 22. 11′ 47″, & angulum P Z M. 52. 46′. 38″, & latus P M 26. 26′. 29″, quo ablato à P S 69, reſtat M S 42. 33′. 31″. quo & perpendiculari Z M jam cognitis, invenies, per 9. ſect. cap. 4, hujus, angulum M S Z, ſeu quæſitum P S Z, 31. 6′ 5″. & latus quæſitum S Z 47. atque angulum M Z S 67. 38′. 11″. quo ad P Z M 52. 46′. 38″. addito, fit angulus reliquus quæſitus P Z S 120. 24′. 49″. Tres itaque habes partes quæſitas officio perpendicularis Z M, primi ſchematis. Eaſdem quoque officio quadrantis Z H ſecundi ſchematis venari poteris. Ex datis enim, ut ſuprà, P Z 34.

&

& Z P S, ſeu Z P H, 42. 29′. 59″, invenies (per eandem 9^am. ſect. cap. 4. hujus,) angulum Z H P. 22. 11′. 47″, & angulum P Z H 142. 46′. 38″, & latus P H 116. 26′. 29″; ex quo aufer P S 69; reſtat S H 47. 26′. 29″; quo et, angulo apud H, 22. 11′. 47″. jam habitis, invenies per 9. cap. 4. hujus, angulum H S Z 148. 53′. 55″. ejúſque ad ſemicirculum reliquum, ſcilicèt, 31. 6′. 5″, angulum P S Z quæſitum: atque latus quæſitum S. Z. 47; Denique angulum H Z S 22. 21′. 49″; quo ex H Z P 142. 46′. 38″. ablato, reſtat angulus reliquus quæſitus P Z S 120. 24′. 49″; prorſùs ut ſuprà.

Admonitio.

HVjus exempli imitatione novêm variæ ſolvuntur hujus & cujuſque trianguli quæſtiones. Ex datis enim elevatione poli, horâ diei, & declinatione ſolis illius diei, habetur (ut ſuprà) primò azimuth ſeu plaga ſolis, ſecundò altitudo ſolis, Tertiò angulus poſitionis ſolis. Item datis declinatione ſolis, angulo poſitionis ſolis, & altitudine ſolis, habetur, quartò, plaga ſolis, quintò elevatio poli, ſextò hora, ſeu arcus horarius. Item datis altitudine ſolis, plagâ ſolis, & elevatione poli, habetur, ſeptimò, hora diei; octavò, declinatio ſolis; nonò, denique, angulus poſitionis ſolis.

Secundum exemplum duorum angulorum, & lateris interpoſiti, datorum.

PRæcedentium ſchematum datis angulis, horario, ſcilicet, Z P S 42. 29′. 59″. & plagæ ſolis P Z S 120, 24′. 49″. cum complemento elevationis poli, latere, ſcilicet, interpoſito P Z 34; Tres cæteræ partes exquiruntur. Nam habitis primò (ut ſuprà) Z M. 22. 11′. 47″. & P M. 26 gr. 26′. 29″. & angulo P Z M. 52. 46′. 38″. quo ex P Z S. 120. 24′. 49″. ablato, relictóque angulo M Z S. 67. 38′. 11″. ex hoc atque Z. M. jam notis, invenientur tandem latus quæſitum Z S. 47. & Z S M. ſive angulus quæſitus, Z S P, 31. 6′. 5″. atque M S 42. 33′. 31″. quo ad P M addito, fit reliquum latus quæſitum P S 69. Háſque beneficio perpendicularis primi ſchematis habes; nec ſecùs eaſdem officio quadrantis ſecundi ſchematis acquirere poteris: acquiruntur enim (per nonam quarti hujus) ex datis, angulis

angulis P H Z & P Z H, & ex hoc ſubducto P Z S. dato, reſtat S Z H, quo & angulo P H Z, jam notis propalantur cæteræ omnes partes.

Admonitio.

HVjus exempli imitatione novèm variæ ſolvuntur hujus, & cujuſque trianguli quæſtiones. Ex datis enim (ut ſuprà) horâ diei elevatione poli, & plagâ ſolis, habetur, primò, declinatio ſolis; Secundò, angulus poſitionis ſolis; tertiò altitudo ſolis. Item datis horâ diei, declinatione ſolis, & angulo poſitionis ſolis, habetur, quartò, altitudo ſolis; quintò, plaga ſolis; ſextò, elevatio poli. Item datis angulo poſitionis ſolis, altitudine ſolis, & plagâ ſolis, acquiritur ſeptimò, elevatio poli; octavò, hora diei; nonò, declinatio ſolis.

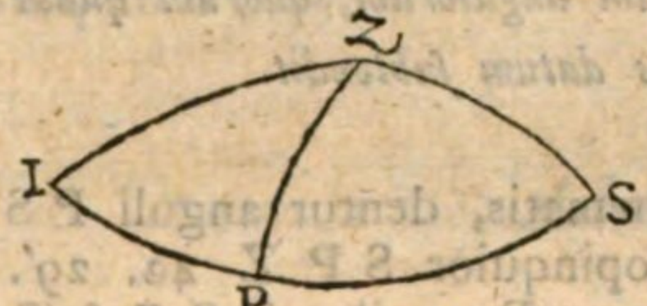
Tertium exemplum duorum datorum laterum, quorum quadranti propinquius ſubtendit angulum datum.

PRæcedentium ſchematum dentur latera P Z 34. & eo quadranti propinquius Z S 47. cum eo quem hoc ſubtendit angulo Z P S 42. 29′. 59″; acquirantur (per 9. ſect. cap 4. hujus) Z M. 22. 11′. 47″. & P Z M 52. 46′. 38″. & P M 26. 26′. 29″ & ſimili modo habebis Z S M. ſeu quæſitum angulum Z S P: certiſſimè enim ſcitur hic per 2. ſect. cap. 3. hujus, minor quadrante, ſcilicèt, eſſe 31. 6′. 5″. & non eſſe 148. 53′. 55″. Habebis etiàm angulum M Z S. 67. 38′. 11″; quo ad P Z M 52. 46′. 38″. addito, fit reliquus quæſitus angulus P Z S 120. 24′. 49″. Habebis denique & M S. 42. 33′. 31″; quo ad P M. 26. 26′. 29″ addito, fit quæſitum latus P S 69. Nec ſecùs eadem acquirere poteris (ſi libet) officio quadrantis Z H ſecundi ſchematis.

Quartum

Quartum exemplum duorum datorum laterum, quorum quadranti minùs propinquum ſubtendit angulum datum: magis autèm propinquum ſubtendit angulum datæ tantùm ſpeciei.

DEntur latera Z S 47, & eo quadranti minùs propinquum P Z 34, cum eo quem hoc ſubtendit angulo Z S P 31. 6′. 5″. detúrque quòd [angulus] quem Z S ſubtendit (angulus, ſcilicèt, S P Z.) ſit ſpecie minor quadrante. Dimiſſo itaque ab Z ad baſim P S perpendiculari Z M, (ut priùs) aut quadrante Z I (ut hîc) ſubtendente angulum datum Z S P. Acquirantur per 9. ſect. 4. hujus cæteræ partes, ut (exercitii, & varietatis gratiâ) ex hujus ſchematis quadrante Z I acquires angulum Z I S 22. 11′. 47″. & I Z S 157. 38′. 11″. & S 132. 33′. 31″. & ſimili modo habebis angulum I P Z, &, per conſequens, angulum quæſitum S P Z 42. 29′. 59″; Quia ex hypotheſi expreſſè quadrante minor declaratur: alioquìn, niſi ejus daretur ſpecies, foret (ex primâ, cap. 3. & quintâ ſect. hujus cap.) incertus: potuit enim alitèr fuiſſe 137. 30′. 1″. Habebis etiàm ſic angulum I Z P 37. 13′. 22″; quo ex I Z S 157. 38′. 11″. ablato, relinquitur reliquus quæſitus angulus P Z S 120. 24′. 49″. Habebis denique & I P 63. 33′. 31″; quo ex I S 132. 33′. 31″. ablato, remanet quæſitum latus P S 69.



Eaſdem etiàm metas attinges ſi, beneficio perpendicularis Z M primi ſchematis, partium logiſticen quæſiveris.

Admonitio.

PRæcedentis tertii & hujus quarti exemplorum imitatione octodecìm variæ ſolvuntur hujus & cujuſque trianguli quæſtiones. Ex datis enim (ut in tertio exemplo) elevatione poli, altitudine ſolis, & horâ diei, habetur, primò, plaga ſolis; ſecundò, angulus poſitionis ſolis; tertiò, declinatio ſolis. Itèm datis (ut in hoc quarto exemplo) elevatione poli, altitudine ſolis, & angulo poſitionis ſolis, habetur, quartò, plaga ſolis; quintò, hora diei; ſextò, declinatio ſolis. Itèm, datis

datis altitudine ſolis, declinatione ſolis, & horâ diei, habetur, ſeptimò, angulus poſitionis ſolis; octavò, plaga ſolis; nonò, elevatio poli. Itèm, datis altitudine ſolis, declinatione ſolis, & plagâ ſolis, habetur, decimò, angulus poſitionis ſolis; undecimò, hora diei; duodecimo, elevatio poli. Itèm, datis declinatione ſolis, elevatione poli, & angulo poſitionis ſolis; habetur, decimotertiò, plaga ſolis; decimoquartò, altitudo ſolis; decimoquintò, hora diei. Item, datis declinatione ſolis, elevatione poli, & plagâ ſolis, habetur, decimoſextò, hora diei; decimoſeptimò, angulus poſitionis ſolis; &, decimo-octavò, altitudo ſolis.

Quintum exemplum duorum datorum angulorum, quorum quadranti propinquiorem latus datum ſubtendit.

TRianguli P Z S primi ſchematis, dentur anguli P S Z 31. 6′. 5″. & eo quadranti propinquior S P Z 42. 29′. 59″. cum latere Z S 47. hunc ſubtendente. Ex quibus P S Z & S Z habetur (per nonam quarti hujus) perpendicularis Z M 22. 11′. 47″. & cæteræ partes [trianguli] quadrantalis S Z M, ſcilicèt, M Z S 67. 38′. 11″. & M S 42. 33′. 31″. Sicut & ex perpendiculari hoc cum dato Z P S, ſeu Z P M, angulo, habentur partes omnes [trianguli] quadrantalis Z M P. Scilicèt, primò, latus quæſitum P Z; certiſſimè enim ſcitur hoc (per 2. ſentent. cap. 3. hujus) [eſſe] minus quadrante, videlicet, eſſe 34. non autèm eſſe 146. Deinde habetur P Z M 52. 46′. 38″; quo ad S Z M. 67. 38′. 11″. addito, fit quæſitus angulus P Z S. 120. 24′. 49″. Vltimò, habetur P M 26. 26′. 29″; quo ad M S 42. 33′. 31″. addito, fit reliquum latus quæſitum P S 69. Has etiàm ipſas partes alitèr (ſi mavis) ex duobus proximè præcedentis Schematis [triangulis] quadrantalibus Z I S & Z I P acquirere poteris.

Sextum exemplum duorum datorum angulorum, quorum quadranti minùs propinquum ſubtendit latus datum, magìs autèm propinquum ſubtendit latus datæ tantùm ſpeciei.

TRianguli P Z S primi Schematis dentur anguli Z P S 42. 29′. 59″. & eo quadranti minùs propinquus Z S P 31. 6′. 5″. cum eum ſubtendente latere P Z 34. Detúrque quòd [Latus] angulum Z P S ſubtendens, (ſcilicèt, latus Z S) ſit ſpecie minus quadrante.

Ex

Ex his datis quæratur perpendicularis Z M 22. 11′. 47″. & cæteræ [trianguli] quadrantalis P Z M partes, ſcilicèt, P Z M 52. 46′. 38″. & P M 26. 26′. 29″. S cut &. ex perpendiculari hoc cum dato, Z S M, ſeu Z S P, 31. 6′. 5″. quærantur partes omnes [trianguli] quadrantalis ZMS; ſcilicet, primò, latus optatum ZS 47; quia ex hypotheſi expreſſè quadrante minus declaratur, alioquìn potuit fuiſſe 133. Nam (per 1. cap. 3. & quintam hujus) incertum eſt niſi ejus expreſſè detur ſpecies. Deinde angulus M Z S 67. 38′. 11″; quo ad M Z P. 52. 46′. 38″. addito, fit optatus angulus P Z S 120. 24′. 49″. Denique habetur S M 42. 33′. 31″.; Quo ad P M. 26. 26′. 29″. addito, fit optata baſis P S 69. Eaſdem etiàm partes ex duobus [triangulis] quadrantalibus P H Z & S H Z ſecundi ſchematis, quàm facillimè acquirere poteris.

Admonitio.

PRæcedentis quinti & hujus ſexti exemplorum imitatione, octodecìm variæ ſolvuntur hujus, & cujuſque trianguli, quæſtiones. Ex datis enim (ut in quinto exemplo) angulo poſitionis ſolis, horâ diei, & altitudine ſolis, habetur, primò, elevatio poli; ſecundò, plaga ſolis; tertiò, declinatio ſolis. Itèm, datis (ut in hoc ſexto exemplo) horâ diei, angulo poſitionis ſolis, & elevatione poli, habetur, quartò, altitudo ſolis; quintò plaga ſolis; ſextò, declinatio ſolis. Item, datis horâ diei, plagâ ſolis, & altitudine ſolis, habetur, ſeptimò, declinatio ſolis; octavò, angulus poſitionis ſolis; nonò, elevatio poli. Item, datis horâ diei, plagâ ſolis, & declinatione ſolis, habetur, decimò, altitudo ſolis; undecimò, angulus poſitionis ſolis; duodecimò, elevatio poli. Itèm, datis plagâ ſolis, angulo poſitionis ſolis, & declinatione ſolis, habetur, decimotertiò, elevatio poli; decimoquartò, hora diei; decimoquintò, altitudo ſolis. Itèm, datis plagâ ſolis, angulo poſitionis ſolis, & elevatione poli, habetur, decimoſextò, declinatio ſolis; decimoſeptimò, hora diei; decimo-octavò, altitudo ſolis. Atque ita hujus ſolius canonis methodo, quinquaginta quatuor variæ ſolvuntur quæſtiones ejuſdem trianguli non-quadrantalis. Cæteræ inferiùs ſolventur.

9. *Ex his itaque patet quod duorum angulorum & ſuorum ſubtendentium laterum tribus datis, quarti ſatèm Logarithmus innoteſcet, tacitâ etiàm* [triangulorum] *quadrantalium deſcriptione. Ab aggregato enim ex Logarithmis anguli & lateris ſibi adjacentis datorum, aufer Logarithmum tertii dati, et proveniet inde Logarithmus quarti quæſiti, ipſúmque quartum, niſi ſit incertæ ſpeciei, innoteſcet.*

Vt ex ſuperioribus, tertio, quarto, quinto, & ſexto, exemplis percipi poteſt. Angulorum enim baſis Z P S, & Z S P, & ſuorum ſubtendentium crurum Z S & Z P, dentur tria, quæ (verbi gratiâ) ſint crura Z S 47. ejúſque Logarithmus 3,128,580, & Z P 34. ejúſque Logarithmus 5,812,606, cum huic adjacente angulo Z P S 42. 29′. 59″; cujus Logarithmum 3,921,720 adde ad 5,812,606, [et] fit 9,734,326 (Logarithmus, ſcilicèt, taciti & ſuppreſſi perpendicularis Z M, vel anguli Z H S, ſeu Z I P,) à quo aufer 3,128,580; remanet 6,605,746 Logarithmus quarti Z S P, quæſiti. Ipſum itaque quartum Z S P, erit 31. 6′. 5″. Quoniam per 2. ſect. cap. 3. minus quadrante arguitur. Contrà autèm, datis Z P 34. ejúſque Logarithmo 5,812,606, & Z S 47. ejúſque Logarithmo 3,128,580, cum huic adjacente angulo Z S P 31. 6′. 5″.; ad cujus Logarithmum 6,605,746, adde 3,128,580, [et] fit aggregatum (ut ſuprà) 9,734,326; à quo aufer 5,812,606, [et] provenient 3,921,720 Logarithmus quarti quæſiti, ſcilicèt, Z P S, cujus arcus per 1 ſect. cap. 3. incertus eſt an ſit 42. 29′. 59″. an 137. 30′. 1″. niſi declaret hypotheſis majórne, an minor, ſit quadrante.

CAP. VI.

De [triangulis] non-quadrantalibus puris.

HActenùs de partibus miſcellaneis datis: Sequuntur puræ.

1. *Puræ ſunt tres partes ejuſdem generis datæ. Súntque aut tria latera data, et quæruntur anguli: aut tres anguli dati, et quæruntur latera.*

Admonitio.

Admonitio.

2. *PVræ, quamvis ſimplicitate priores, ob difficultatem tamen earundem, meritò hic poſteriorem ſortiuntur locum.*

3. *In triangulis Sphæricis, primò, ſumma ex Logarithmis crurum ſubducta à ſummâ ex Logarithmis aggregati & differentiæ ſemi-baſis et ſemi-differentiæ crurum, relinquit duplum Logarithmi dimidii anguli verticalis.*

Quia docent Regiomontanus libro 5. cap. 2. de triangulis, & alii, ut rectangulum comprehenſum ſub ſinibus rectis crurum, ſe habet ad quadratum ſinûs totius; Ita differentiam ſinuum verſorum baſis & differentiæ crurum ſe habere ad ſinum verſum anguli verticalis: quum autèm ut illa differentia ad hunc ſinum verſum, ita rectangulum factum ex ſinibus rectis aggregati & differentiæ ſemi-baſis & ſemi-differentiæ crurum, ſe habet ad quadratum ſinûs recti dimidii anguli verticalis (eſt enim noviſſimum hoc rectangulum ad illam differentiam ſinuum verſorum, & hoc ultimum quadratum ad illum ſinum verſum, in ratione 5,000,000 cuplâ, & intellige quinquies millies millecuplâ, exiſtente ſinu toto 10,000,000). Ideò ſequetur quòd, ut rectangulum ſub ſinibus rectis crurum ſe habet ad quadratum ſinûs totius, ita rectangulum factum ex ſinibus rectis aggregati et differentiæ ſemi-baſis & ſemi-differentiæ crurum, ſe habebit ad quadratum ſinûs recti dimidii anguli verticalis: & per conſequens (ex corol. def. 6. cap. 1. & prop. 4. cap. 2. & probl. 3. cap. 5. lib. 1.). Summa ex Logarithmis crurum, ſubducta ex Logarithmis aggregati & differentiæ ſemi-baſis & ſemi-differentiæ crurum, relinquit duplum Logarithmi dimidii anguli verticalis, ut ſuprà.

4. *Secundò, Summa ex Logarithmis crurum ſubducta à ſummâ ex Logarithmis aggregati et differentiæ ſemi-baſis et ſemi aggregati crurum, relinquit duplum antilogarithmi dimidii anguli verticalis.*

Non enim alitèr ſe habet ſumma ex Logarithmis aggregati & differentiæ ſemi-baſis & ſemi-aggregati crurum hujus propoſitionis ad ſummam ex Logarithmis aggregati & differentiæ ſemi-baſis & ſemi-differentiæ crurum præcedentis propoſitionis, quàm duplum antilogarithmi dimidii anguli verticalis hîc, ad duplum Logarithmi ejuſdem dimidii anguli verticalis ſuperiùs: quod alterius loci eſt demonſtrare.

Admonitio.

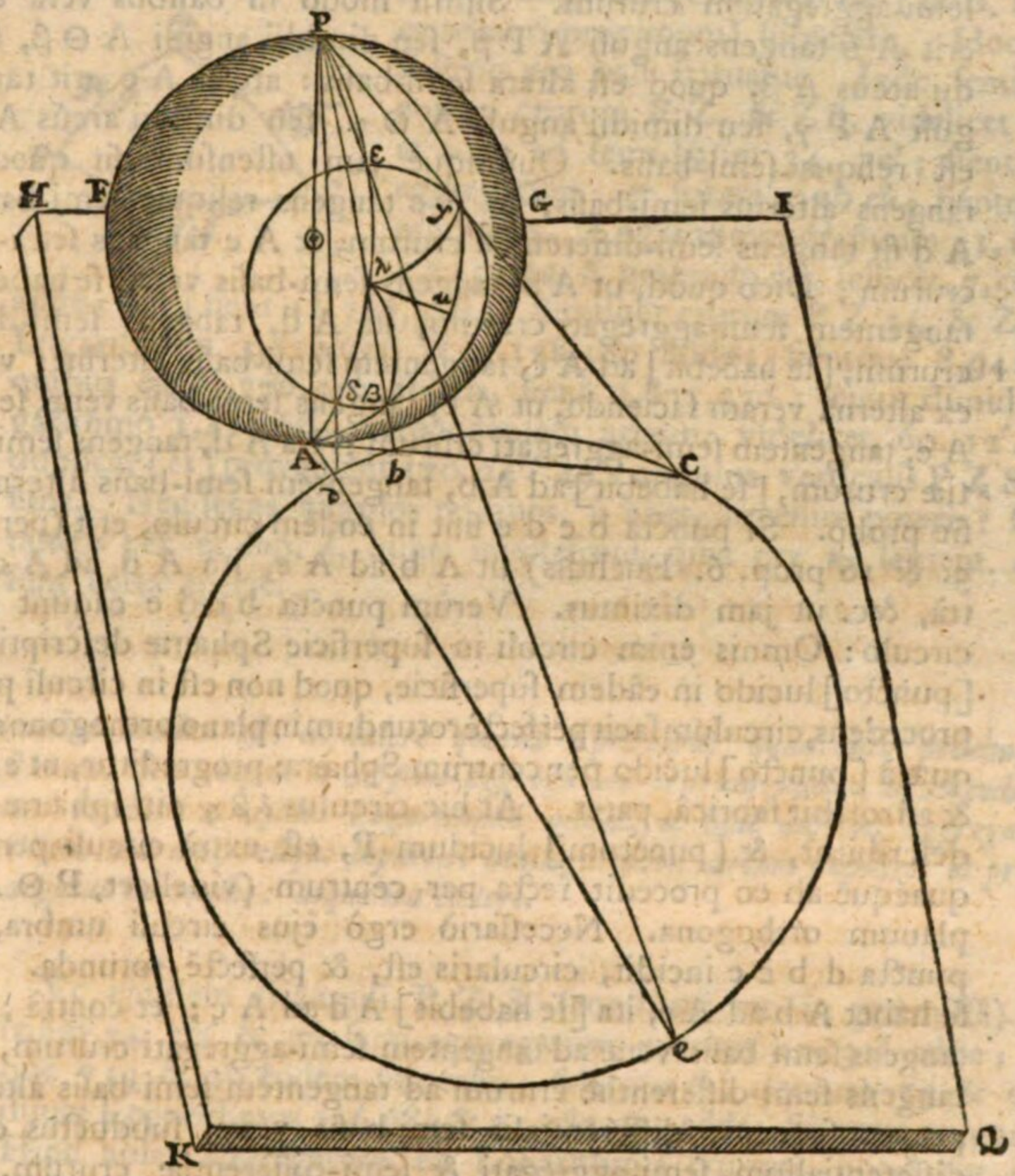
5. *IN Sphæricis etiàm, baſes veram & alternam, eodem ſenſu capimus, quo in rectilineis, nimirùm, alteram pro aggregato, alteram pro differentiâ, caſuum.*

6. *Tertiò, differentialis ſemi-baſis veræ datæ, ſubductus ex ſummâ differentialium ſemi-aggregati & ſemi-differentiæ crurum, relinquit differentialem ſemi-baſis alternæ.*

Hujus ratio fundamentalis eſt, quòd ut tangens ſemi-baſis veræ ſe habet ad tangentem ſemi-aggregati crurum, ita tangens ſemi-differentiæ crurum ſe habeat ad tangentem ſemi-baſis alternæ. Tangentium enim Logarithmi ſunt ſuorum arcuum differentiales per ſect. 22. & 25. cap. 3. lib. 1. Vnde hanc tangentium analogiam ſequetur illa ſuorum logarithmorum, ſeu differentialium æqualitas per prop. 4. cap. 2. lib. 1. Verùm, quia hujus analogiæ tangentium fundamentalis, hactenùs ignotæ, demonſtrationem à me fortè requirent Lectores, eam ideò, quantum hujus compendii brevitas patitur, hîc explicabimus.

Sphæra itaque A F P G incumbat plano H I K Q ut ſe invicèm tangant in communi puncto A; à quo, per Sphæræ centrum Θ, erigatur recta A Θ P ſecans ſupremum Sphæræ Hemiſphærium in puncto P. Erítque ita A Θ P perpendicularis plano H I K Q. Deinde angulo A deſcribatur in Sphæræ ſuperficie triangulum A λ γ in γ acutum, aut A λ β in β obtuſum; &, protractis ſemi-circulis A λ P, & A γ P, ſeu A β P, polo λ, intervallo λ γ, ſeu ei æquali λ β, ducatur circulus ε δ β γ, ſecans λ P in ε, & λ A in δ, & β γ in punctis β & γ. Ex puncto λ in arcum A β γ dimittatur perpendicularis arcus λ μ. Erunt itaque hîc A λ crus majus; λ γ, vel λ β, crus minus; A γ & A β baſes, altera vera, reliqua alterna; A δ differentia crurum, & A ε aggregatum crurum, quia λ ε, & λ δ, ex conſtructione, ſunt æqualia minori cruri λ γ ſeu λ β. His peractis, & ſuppoſito P vicem gerere oculi aut [puncti] lucidi cujuſpiam, ab eodem P in ſubjectum planum H I K Q dimittantur, radius P γ ſecans planum in c, & radius P β ſecans planum in b. Et, quia γ β A in eodem plano, ſeu circulo, ſunt cum [puncto] lucido P, erunt ſuæ umbræ c b A in eâdem rectâ. Similitèr ab eodem puncto P, in idem planum dimittantur radius P ε ſecans planum in e, & radius P δ ſecans planum in d. Et, quia ε δ A in eodem ſunt plano & circulo cum [puncto] lucido P:

ideò



ideò ſuæ umbræ c d A erunt in eâdem rectâ. Prætereà, quia P Θ A eſt plano orthogonalis ſeu rectangula, ideò triangula P A d, & P A e, atque P A b, & P A c, ſunt in A rectangula, atque ideò etiàm A d eſt tangens anguli A P δ, ſeu A P d, & A e eſt tangens anguli A P ε, vel A P e; ſic etiàm A b eſt tangens anguli A P β vel A P b, & A c eſt tangens anguli A P γ, vel A P c, poſito gnomone ſeu ſinu toto P A. Et, quia A d eſt tangens anguli A P δ, & A P δ eſt dimidium anguli A Θ δ per 20. prop. 3. Eucl. (quod hîc ſit in centro, ille in circumferentiâ) ideò A d eſt tangens dimidii anguli A Θ δ, ſeu (quod idem eſt) dimidii arcûs A δ, quod eſt ſemi-differentia crurum. Similitèr, quia A e eſt tangens anguli A P ε, angulus autèm A P ε in circumferentiâ ſit dimidium anguli A Θ ε in centro, ideò A e eſt tangens dimidii A Θ ε, ſeu dimidii arcûs A ε, quod eſt

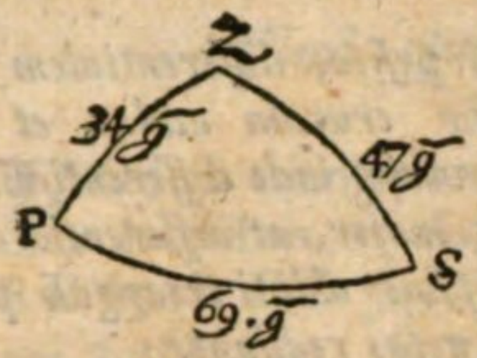
ſemi-

ſemi-aggregatum crurum. Simili modo in baſibus verâ & alternâ erit A b tangens anguli A P β, ſeu dimidii anguli A Θ β, ſeu dimidii arcûs A β, quod eſt altera ſemi-baſis : atque A c erit tangens anguli A P γ, ſeu dimidii anguli A Θ γ, ſeu dimidii arcûs A γ, quod eſt reliqua ſemi-baſis. Quúmque jam oſtenſum ſit quod A b ſit tangens alterius ſemi-baſis, & A c tangens reliquæ ſemi-baſis, atque A d ſit tangens ſemi-differentiæ crurum, & A e tangens ſemi-aggregati crurum ; Dico quòd, ut A b, tangens ſemi-baſis veræ, ſe habet ad A e, tangentem ſemi-aggregati crurum, ita A d, tangens ſemi-differentiæ crurum, [ſe habebit] ad A c, tangentem ſemi-baſis alternæ: vel contrà, ex alternâ veram faciendo, ut A c, tangens ſemi-baſis veræ, ſe habet ad A e, tangentem ſemi-aggregati crurum ; Ita A d, tangens ſemi-differentiæ crurum, [ſe habebit] ad A b, tangentem ſemi-baſis alternæ. Quod ſic probo. Si puncta b c d e ſint in eodem circulo, erit (per 36 prop. 3. & 16 prop. 6. Euclidis) ut A b ad A e, ita A d ad A c; & contrà, &c. ut jam diximus. Verùm puncta b c d e cadunt in eodem circulo: Omnis enim circuli in ſuperficie Sphæræ deſcripti umbra à [puncto] lucido in eâdem ſuperficie, quod non eſt in circuli peripheriâ, procedens, circulum facit perfectè rotundum in plano orthogono ad rectam, quæ à [puncto] lucido per centrum Sphæræ progreditur, ut ex Opticis, & aſtrolabii fabricâ, patet. At hic circulus δ β γ ε in ſphæræ ſuperficie deſcribitur, & [punctum] lucidum P, eſt extrà circuli peripheriam, quaéque ab eo procedit recta per centrum (videlicet, P Θ A) eſt ad planum orthogona. Neceſſariò ergò ejus circuli umbra, quæ in puncta d b c e incidit, circularis eſt, & perfectè rotunda. Ergò, ut ſe habet A b ad A e, ita [ſe habebit] A d ad A c ; et contrà ; id eſt, ut tangens ſemi-baſis veræ ad tangentem ſemi-aggregati crurum, ita [erit] tangens ſemi-differentiæ crurum ad tangentem ſemi-baſis alternæ : &, per conſequens, differentialis ſemi-baſis veræ, ſubductus ex ſummâ differentialium ſemi-aggregati & ſemi-differentiæ crurum, æquatur differentiali ſemi-baſis alternæ: quæ demonſtranda ſuſcepimus.

7. *Vnde, Trianguli Sphærici datis tribus lateribus, habetur triplici modo angulorum quivis.*

8. *Primus modus eſt, ut latus quodvis (præcipuè quadranti proximum) pro baſi ſtatuas. Inde ſemi-differentiam crurum, & ad ſemi-baſim addas, et à ſemi-baſi ſubtrahas : producti et reſidui Logarithmos addas,* [inter ſe ;] *hinc auferas aggregatum ex Logarithmis crurum ; reliqui bipartiti Logarithmi arcum duplices, et proveniet angulus verticalis, atque ita cæteri.*

Vt trianguli P Z S repetiti, dentur latera P Z 34 gr. & Z S 47 gr. & S P 69 gr. Quærantur anguli ; primóque quadranti proximus P Z S



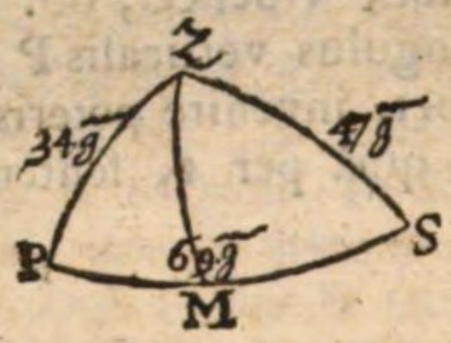
P Z S angulus, quem S P 69 (latus, ſcilicèt, quadranti proximum) ſubtendit. Hoc itaque S P 69 pro baſi ſtatuatur. Inde ſemi-differentiam crurum P Z, & Z S, videlicet 6. 30′, et adde ad ſemi-baſim 34. 30′; fiéntque 41 aggregatum : et ſubtrahe ab eâ; fiéntque 28. reſiduum. Logarithmos graduum 41, ſcilicèt, 4,215,044, & graduum 28, ſcilicèt, 7,561,472 adde; [et] fient 11,776,516. Similitèr crurum P Z 34, & Z S 47, Logarithmos 5,812,606 & 3,128,580 adde; fiéntque 8,941,186; quibus ex 11,776,516 ablatis, fient 2,835,330 : cujus dimidio Logarithmo 1,417,665, reſpondentem arcum, videlicet, 60. 12′. 24½″. duplica, [et] provenient 120. 24′. 49″. angulus verticalis P Z S quæſitus. Nec ſecùs angulos reliquos, ſi libet, invenire poteris : faciliùs tamen per 9. cap. 5. hujus innoteſcent, quia per 2. ſentent. cap. 3. ſunt certæ ſpeciei.

9. *Secundus modus eſt, ut latere quovis (præcipuè quadranti proximo) pro baſi ſtatuto, Semi-baſim et ad ſemi-aggregatum crurum addas, et ab eodem ſubtrahas : producti et reſidui Logarithmos addas, et hinc auferas aggregatum ex logarithmis crurum, reliqui bipartiti antilogarithmi arcum duplices, et proveniet inde angulus verticalis : atque ita cæteri.*

Vt ejuſdem trianguli P Z S, (conſtituti ut in præmiſſâ) ſemi-baſim 34. 30′ & ad ſemi-aggregatum crurum 40. 30′ adde; fiéntque 75; & ab eodem ſubtrahe; fiéntque 6 : quorum 75 & 6 graduum Logarithmos 346,683 & 22,582,951 adde; fiéntque 22,929,634. Hinc aufer aggregatum ex Logarithmis crurum, quod (ut ſuprà) eſt 8,941,186; fiéntque 13,988,448. Quæ bipartire, [et] fient inde 6,994,224 antilogarithmus conveniens arcui 60. 12′. 24½″. cujus duplum 120. 24′. 49″. eſt (ut ſuprà) quæſitus angulus P Z S verticalis. Cæteros, licet etiàm hoc modo, faciliùs tamen per 9. cap. 5. hujus, invenies angulos. Sunt enim per 2. ſentent. cap. 3. notæ ſpeciei.

10. *Tertius modus eſt, ut, latere quovis pro baſi poſito, differentialem ſemi-aggregati crurum ad differentialem ſemi-differentiæ crurum addas, et à producto auferas differentialem ſemi-baſis veræ; et proveniet inde differentialis ſemi-baſis alternæ: quarum ſemi-baſium ſumma eſt caſus major, et differentia caſus minor, duo diſtinguentes rectangula, quæ et ſuas, et ipſius oblati trianguli partes omnes (per nonam cap. 4. et octavam cap. 5. hujus) notas reddunt.*

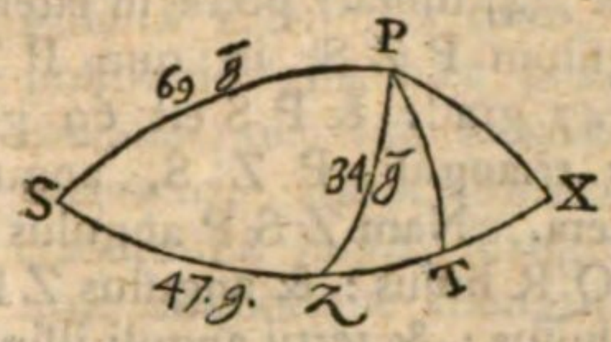
Vt propoſiti trianguli P Z S, datis lateribus ut ſuprà, quærantur anguli apud baſim, Z P S & Z S P. Semi-aggregatum crurum P Z & Z S, eſt 40, 30′. Semi-differentia crurum eſt 6, 30′. Illius differentialis eſt 1,577,296, hujus verò eſt 21,721,209. Quos adde, [et] fient 23,298,505. Hinc aufer ſemi-baſis veræ 34, 30′, differentialem 3,750,122; remanent 19,548,383, differentialis 8, 3′, 31″, pro ſemi-baſi alternâ. Adde ergò ſemi-baſes 34, 30′ & 8, 3′, 31″; fient inde 42, 33′, 31″, pro majore caſu M S; & ſubtrahe 8, 3′, 31″ à 34, 30′; relinquentur 26, 26′, 29″, pro minore caſu P M. Horum itaque caſuum officio habes duo jam [triangula] rectangula in M, ſcilicèt, P M Z & S M Z: quæ & perpendicularem Z M, & angulos verticales P Z M & S Z M, aut, ſi libet, ipſum P Z S, patefaciunt (per nonam cap. 4. & octavam cap. 5. hujus). Sed, his omiſſis, ad quæſitos baſis angulos Z P S & Z S P, redeamus. Caſûs P M, 26, 26′, 29″, jam acquiſiti, differentialem 6,985,518 (per 9. cap. 4.) adde ad differentialem complementi P Z, ſcilicèt, ad differentialem 56, qui eſt — 3,937,709; [et] provenient + 3,047,809, Logarithmus complementi anguli Z P S; quod complementum eſt 47, 30′, 1″. Similitèr caſûs S M 42, 33′, 31″, jam etiàm acquiſiti, differentialem 853,239 (per eandem nonam ſect.) adde ad differentialem complementi P Z, ſcilicèt, ad differentialem 43 gr. qui eſt 698,698; [et] provenient + 1,551,937, Logarithmus complementi anguli Z S P; quod complementum eſt 58, 53′, 55″. Memor autèm hîc ſis non ipſas partes P Z 34 & Z P S, aut P Z 47, & Z S P, ſed ſua complementa, viz. 56 gr. & 47, 30′, 1″, & 43 gr. & 58, 53′, 55″, *circulares partes* hîc dici per ſecundam cap. 4. hujus. Verus itaque angulus quæſitus Z P S eſt 42, 29′, 59″, & Z S P eſt 31, 6′, 5″; ut etiàm ex ſect. octavâ, cap. 5. hujus patet.



Aliud

Aliud ejusdem trianguli exemplum.

EOdem triangulo P Z S, alio ſitu conſtituto, ſit S Z baſis, &, datis lateribus ut ſuprà, quæratur angulus P Z S. Crurum itaque S P 69, & P Z 34, ſemi-aggregatum eſt 51, 30′, ejúſque differentialis — 2,288,650: ſemi-differentia verò eſt 17, 30′, ejúſque differentialis eſt + 11,542,341. Quos differentiales adde; [et] erit ſumma + 9,253,691; à quâ aufer differentialem dimidii baſis SZ, videlicet, differentialem 23, 30′, qui eſt 8,328,403, [et] remanebit 925,288 differentialis arcûs 42, 21′, 11″, pro ſemi-baſi alternâ. Adde ergò ſemi-baſes 42, 21′, 11″, & 23, 30′. [et] provenient 65, 51′, 11″, pro majore caſu S T; & tum ſubtrahe 23, 30′ à 42, 21′, 11″; remanent 18, 51′, 11″, pro minore caſu T X, vel T Z. Hujus ergò differentialem + 10,745,201, adde ad differentialem complem. Z P, ſcilicèt, ad differentialem grad. 56, qui eſt — 3,937,709, & provenient inde + 6,807,492, Logarithmus complementi anguli P Z T. Arcus autèm in tabulâ reſpondens huic Logarithmo 6,807,492 ex adverſo, eſt graduum 59, 35′, 11″, pro angulo P Z T, cujus anguli P Z T, quum angulus quæſitus P Z S ſit ad ſemi-circulum reliquus (quod ſemper occurrit quum baſis alterna eſt major verâ) erit neceſſe P Z S, eſſe graduum 120, 24 min. 49 ſec. alioquìn, ſi baſis vera alternam ſuperaverit, coincident anguli P Z T & P Z S, & æquales erunt.



Admonitio.

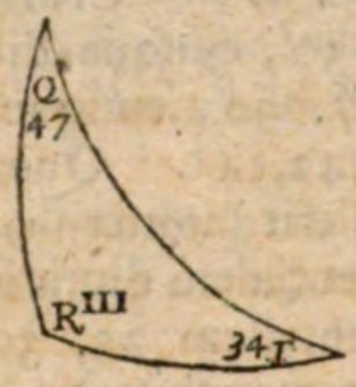
TRes jam habes veros modos inveniendi angulos ex datis lateribus, quorum unoquoque tres variæ ſolvuntur hujus, & cujuſque trianguli quæſtiones. Ex datis enim elevatione poli, altitudine ſolis, & declinatione ſolis, dubitantibus ſatisfit ad quæſtionem quâ vel [primò,] plaga ſolis, vel ſecundò, angulus ſitûs & poſitionis ſolis, vel, tertiò, hora diei quæritur.

Huc uſque ex lateribus invenimus angulos. Supereſt ex angulis invenire latera.

41. *In omni triangulo ſphærico mutari poſſunt latera in angulos, et anguli in latera: aſſumptis tamen priùs pro unico quovis angulo, et ſuo* ſubtendente latere, ſuis † ad ſemi-circulum reliquis.*

* Potiùs *ejus.*
† Potiùs *eorum.*

* Potiùs *ejus*.



Exempli gratiâ.

Esto triangulum Q R T, cujus sint anguli Q 47. R 111. & T 34. Sumamus, primò, pro angulo quovis, videlicet, pro R 111, suum * ad semi-circulum reliquum, quod est 69 grad. Dico hos angulos 47, 69, & 34, mutari posse in latera, & fiet superius triangulum P Z S, in quo P Z est 34 grad. Z S est 47 grad. & P S est 69 grad. Ut etiàm ex illius trianguli, P Z S, angulis, fient hujus mutuò latera. Nam Z S P angulus grad. 31, 6′, 5″, [trianguli] illius, est latus Q R hujus: & angulus Z P S grad. 42, 29′, 59″, illius, est latus R T hujus: & tertii anguli illius, qui est grad. 120, 24′, 49″, reliquus ad semi-circulum, qui est 59, 35′, 11″, est latus Q T hujus. Cujus rei demonstrationem exhibent Bartholomæus Pitiscus, Adrianus Metius, & alii. Eam igitùr hâc epitome minimè repetendam censeo.

12. *Vnde, Trianguli Sphærici datis tribus angulis, facili conversione acquiruntur latera.*

Vt, [si, exempli gratiâ] præcedentis trianguli Q R T dentur anguli Q 47, R 111, & T 34; quærantur autem latera. Pro angulo quovis unico, verbi gratiâ (ut suprà) pro R 111, sumatur suum ad semi-circulum reliquus, 69 grad. Inde, positis 47, 69, & 34, pro lateribus, ut in triangulo superiore P Z S factum est, per quemvis ex tribus modis suprascriptis, quære illius angulos, & invenies contrà latus 47, angulum 42, 29′, 59″; & contrà latus 34, angulum 31, 6′, 5″; & contrà latus 69 (quod pro 111 posuimus) reperies angulum 120, 24′, 49″. Ideò in triangulo oblato Q R T, pro latere R T, subtendente angulum Q 47. pone 42. 29′. 59″. Et pro latere Q R, subtendente angulum T 34, pone 31, 6′, 5″; verùm, pro latere Q T, subtendente angulum R 111, pone 59, 35′, 11″, quæ sunt [angulus] reliquus [anguli] graduum 120, 24′, 49″, ad semi-circulum; quia priùs 111 sumpseras suum ad semi-circulum reliquum, scilicèt, 69. Et ita ex angulis per conversionem acquires latera.

Admonitio.

Admonitio.

EX hâc laterum per angulos datos inventione tres variæ ſolvuntur hujus, & cujuſcunque trianguli, quæſtiones. Vt in triangulo P Z S. Ex datis, horâ diei, plagâ ſolis, & angulo ſitûs vel poſitionis ſolis, hæc præcedens ſatisfacit quæſtioni, quâ vel, primò, elevatio poli, vel, ſecundò, altitudo ſolis, vel, tertiò, declinatio ſolis quæratur. Ex octavâ itaque ſect. præcedentis, cap. 5. & ſeptimâ ac duodecimâ hujus, ſexagintà habes variarum quæſtionum ſolutiones, quæ in quodque triangulum cadunt. Nec his plures ex multiplici trium quarumlibet partium compoſitione oriri poſſunt variationes. Perfectam igitùr habes & abſolutam triangulorum, tam Sphæricorum quàm planorum, doctrinam.

CONCLVSIO.

SAtìs ergò jam oſtenſum eſt quod ſint, quid ſint, et cujus uſûs ſint, Logarithmi: Eorum enim beneficio abſque multiplicationis, diviſionis, aut radicum extractionis, moleſtiâ, omnis Geometricæ quæſtionis ſolutionem logiſticam promptiſſimè exhiberi, tum apodeictìcè demonſtravimus, tum exemplis utriuſque Trigonometriæ docuimus. Promiſſum itaque mirificum Logarithmorum canonem habetis, ejúſque ampliſſimum uſum: quæ ſi vobis eruditioribus grata fore ex reſcriptis veſtris intellexero, animus mihi addetur, ad tabulæ condendæ methodum in lucem etiàm proferendam. Interìm hoc brevi opuſculo fruamini, Deóque opifici ſummo, omniúmque operum bonorum opitulatori, laudem ſummam et gloriam tribuite.

Sequitur Tabula, ſeu canon, Logarithmorum.

Gr. o

o min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	0	Infinitum	Infinitum	0	10,000,000	60
1	2,909	81,425,681	81,425,680	1	10,000,000	59
2	5,818	74,494,213	74,494,211	2	9,999,998	58
3	8,727	70,439,564	70,439,560	4	9,999,996	57
4	11,636	67,562,746	67,562,739	7	9,999,993	56
5	14,544	65,331,315	65,331,304	11	9,999,989	55
6	17,453	63,508,099	63,508,083	16	9,999,986	54
7	20,362	61,966,595	61,966,573	22	9,999,980	53
8	23,271	60,631,284	60,631,256	28	9,999,974	52
9	26,180	59,453,453	59,453,418	35	9,999,967	51
10	29,088	58,399,857	58,399,814	43	9,999,959	50
11	31,997	57,446,759	57,446,707	52	9,999,950	49
12	34,906	56,576,646	56,576,584	62	9,999,940	48
13	37,815	55,776,222	55,776,149	73	9,999,928	47
14	40,724	55,035,148	55,035,064	84	9,999,917	46
15	43,632	54,345,225	54,345,129	96	9,999,905	45
16	46,541	53,699,843	53,699,734	109	9,999,892	44
17	49,450	53,093,600	53,093,577	123	9,999,878	43
18	52 359	52,522,019	52,521,881	138	9,999,863	42
19	55,268	51,981,356	51,981,202	154	9,999,847	41
20	58,177	51,468,431	51,468,361	170	9,999,831	40
21	61,086	50,980,537	50,980,450	187	9,999,813	39
22	63,995	50,515,342	50,515,137	205	9,999,795	38
23	66,904	50,070,827	50,070,603	224	9,999,776	37
24	69,813	49,645,239	49,644,995	244	9,999,756	36
25	72,721	49,237,030	49,236,765	265	9,999,736	35
26	75,630	48,844,826	48,844,539	287	9,999,714	34
27	78,539	48,467,431	48,467,122	309	9,999,692	33
28	81,448	48,103,763	48,103,431	332	9,999,668	32
29	84,357	47,752,859	47,752,503	356	9,999,644	31
30	87,265	47,413,852	47,413,471	381	9,999,619	30

Gr. o

o min.	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	87,265	47,413,852	47,413,471	381	9,999,619	30
31	90,174	47,085,961	47,085.554	407	9,999,593	29
32	93,083	46,768,483	46,768,049	434	9,999,566	28
33	95,992	46,460,773	46,460,312	461	9,999,539	27
34	98,901	46,162,254	46,161,765	489	9,999,511	26
35	101,809	45,872,392	45,871,874	518	9,999,482	25
36	104,718	45,590,688	45,590,140	548	9,999,452	24
37	107,627	45,316,714	45,316,135	579	9,999,421	23
38	110,536	45,050,041	45,049,430	611	9,999,389	22
39	113,445	44,790,296	44,789,652	644	9,999,357	21
40	116,353	44,537,132	44,536,455	677	9,999,323	20
41	119,262	44,290,216	44,289,505	711	9,999,289	19
42	122,171	44,049,255	44,048,509	746	9,999,254	18
43	125,079	43,813,959	43,813,177	782	9,999,218	17
44	127,988	43,584,078	43,583,259	819	9,999,181	16
45	130,896	43,359,360	43,358,503	857	9,999,143	15
46	133,805	43,139,582	43,138,686	896	9,999,105	14
47	136,714	42,924,534	42,923,599	935	9,999,065	13
48	139,622	42,714,014	42,713,039	975	9,999,025	12
49	142,531	42,507,833	42,506,817	1,016	9,998,984	11
50	145,439	42,305,826	42,304,768	1,058	9,998,942	10
51	148,348	42,107,812	42,106,711	1,101	9,998,900	9
52	151,257	41,913,644	41,912,499	1,145	9,998,856	8
53	154,165	41,723,175	41,721,986	1,189	9,998,811	7
54	157,074	41,536,271	41,535,037	1,234	9,998,766	6
55	159,982	41,352,795	41,351,515	1,280	9,998,720	5
56	162,891	41,172,626	41,171,299	1,327	9,998,673	4
57	165,799	41,006,643	41,005,268	1,375	9,998,025	3
58	168,708	40,821,746	40,820,322	1,424	9,998,577	2
59	171,616	40,650,816	40,649,343	1,473	9,998,527	1
60	174,524	40,482,764	40,481,241	1,523	9,998,477	0
						min.
						Gr.

Gr. 1

1 min.	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	174,524	40,482,764	40,481,241	1,523	9,998,477	60
1	177,433	40,317,483	40,315,909	1,574	9,998,426	59
2	180,341	40,154,899	40,153,273	1,626	9,998,374	58
3	183,250	39,994,918	39,993,239	1,679	9,998,321	57
4	186,158	39,837,448	39,835,715	1,733	9,998,267	56
5	189,066	39,682,421	39,680,633	1,788	9,998,212	55
6	191,975	39,529,765	39,527,922	1,843	9,998,157	54
7	194,883	39,379,407	39,377,508	1,899	9,998,101	53
8	197,792	39,231,274	39,229,318	1,956	9,998,044	52
9	200,700	39,085,307	39,083,293	2,014	9,997,986	51
10	203,608	38,941,441	38,939,368	2,073	9,997,927	50
11	206,517	38,799,612	38,797,479	2,133	9,997,867	49
12	209,425	38,659,767	38,657,573	2,194	9,997,806	48
13	212,333	38,521,858	38,519,603	2,255	9,997,745	47
14	215,241	38,385,824	38,383,507	2,317	9,997,683	46
15	218,149	38,251,613	38,249,233	2,380	9,997,620	45
16	221,057	38,119,183	38,116,739	2,444	9,997,556	44
17	223,965	37,988,481	37,985,972	2,509	9,997,491	43
18	226,873	37,859,471	37,856,896	2,575	9,997,425	42
19	229,781	37,732,105	37,729,464	2,641	9,997,359	41
20	232,689	37,606,339	37,603,631	2,708	9,997,292	40
21	235,597	37,482,135	37,479,359	2,776	9,997,224	39
22	238,505	37,359,458	37,356,613	2,845	9,997,155	38
23	241,413	37,238,269	37,235,354	2,915	9,997,085	37
24	244,321	37,118,532	37,115,546	2,986	9,997,014	36
25	247,229	37,000,208	36,997,150	3,058	9,996,943	35
26	250,137	36,883,272	36,880,142	3,130	9,996,871	34
27	253,045	36,767,690	36,764,487	3,203	9,996,798	33
28	255,953	36,653,428	36,650,151	3,277	9,996,724	32
29	258,861	36,540,448	36,537,096	3,352	9,996,649	31
30	261,769	36,428,748	36,425,320	3,428	9,996,573	30

Gr. 1

1 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	261,769	36,428,748	36,425,320	3,428	9,996,573	30
31	264,677	36,318,272	36,314,768	3,504	9,996,496	29
32	267,585	36,209,009	36,205,427	3,582	9,996,419	28
33	270,493	36,100,924	36,097,264	3,660	9,996,341	27
34	273,401	35,994,000	35,990,261	3,739	9,996,262	26
35	276,308	35,888,207	35,884,388	3,819	9,996,182	25
36	279,216	35,783,520	35,779,620	3,900	9,996,101	24
37	282,124	35,679,917	35,675,935	3,982	9,996,019	23
38	285,032	35,577,380	35,573,316	4,064	9,995,937	22
39	287,940	35,475,892	35,471,745	4,147	9,995,854	21
40	290,847	35,375,415	35,371,184	4,231	9,995,770	20
41	293,755	35,275,935	35,271,619	4,316	9,995,685	19
42	296,663	35,177,444	35,173,042	4,402	9,995,599	18
43	299,570	35,079,909	35,075,420	4,489	9,995,512	17
44	302,478	34,983,320	34,978,743	4,577	9,995,424	16
45	305,385	34,887,652	34,882,987	4,665	9,995,336	15
46	308,293	34,792,895	34,788,141	4,754	9,995,247	14
47	311,200	34,699,029	34,694,185	4,844	9,995,157	13
48	314,108	34,606,036	34,601,101	4,935	9,995,066	12
49	317,015	34,513,899	34,508,872	5,027	9,994,974	11
50	319,922	34,422,606	34,417,486	5,120	9,994,881	10
51	322,830	34,332,140	34,326,926	5,214	9,994,787	9
52	325,737	34,242,484	34,237,176	5,308	9,994,693	8
53	328,645	34,153,629	34,148,226	5,403	9,994,598	7
54	331,552	34,076,549	34,071,050	5,499	9,994,502	6
55	334,459	33,978,246	33,972,650	5,596	9,994,405	5
56	337,367	33,891,701	33,886,007	5,694	9,994,307	4
57	340,274	33,805,893	33,800,100	5,793	9,994,208	3
58	343,181	33,720,820	33,714,927	5,893	9,994,109	2
59	346,088	33,636,464	33,630,471	5,993	9,994,009	1
60	348,995	33,552,817	33,546,723	6,094	9,993,908	0
						min
						Gr.
						88

Gr. 2

2 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	348,995	33,552,817	33,546,723	6,094	9,993,908	60
1	351,902	33,469,860	33,463,664	6,196	9,993,806	59
2	354,809	33,387,588	33,381,289	6,299	9,993,703	58
3	357,716	33,305,993	33,299,590	6,403	9,993,599	57
4	360,623	33,225,056	33,218,549	6,507	9,993,495	56
5	363,530	33,144,770	33,138,158	6,612	9,993,390	55
6	366,437	33,065,128	33,058,410	6,718	9,993,284	54
7	369,344	32,986,107	32,979,282	6,825	9,993,177	53
8	372,251	32,907,712	32,900,779	6,933	9,993,069	52
9	375,158	32,829,923	32,822,881	7,042	9,992,960	51
10	378,064	32,752,740	32,745,588	7,152	9,992,850	50
11	380,971	32,676,149	32,668,887	7,262	9,992,740	49
12	383,878	32,600,139	32,592,866	7,373	9,992,629	48
13	386,785	32,524,706	32,517,221	7,485	9,992,517	47
14	389,692	32,449,837	32,442,239	7,598	9,992,404	46
15	392,598	32,375,526	32,367,814	7,712	9,992,290	45
16	395,505	32,301,761	32,293,934	7,827	9,992,175	44
17	398,412	32,228,539	32,220,596	7,943	9,992,060	43
18	401,318	32,155,852	32,147,793	8,059	9,991,944	42
19	404,225	32,083,692	32,075,516	8,176	9,991,827	41
20	407,131	32,012,045	32,003,751	8,294	9,991,709	40
21	410,038	31,940,909	31,932,496	8,413	9,991,590	39
22	412,944	31,870,276	31,861,743	8,533	9,991,470	38
23	415,851	31,800,141	31,791,487	8,654	9,991,349	37
24	418,757	31,730,492	31,721,716	8,776	9,991,228	36
25	421,663	31,661,332	31,652,434	8,898	9,991,106	35
26	424,570	31,592,644	31,583,623	9,021	9,990,983	34
27	427,476	31,524,424	31,515,279	9,145	9,990,859	33
28	430,382	31,456,672	31,447,402	9,270	9,990,734	32
29	433,288	31,389,371	31,379,975	9,396	9,990,608	31
30	436,194	31,322,524	31,313,001	9,523	9,990,482	30

Gr. 2

2 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	436,194	31,322,524	31,313,001	9,523	9,990,482	30
31	439,100	31,256,121	31,246,471	9,650	9,990,355	29
32	442,006	31,190,158	31,180,380	9,778	9,990,227	28
33	444,912	31,124,626	31,114,719	9,907	9,990,098	27
34	447,818	31,059,521	31,049,484	10,037	9,989,968	26
35	450,724	30,994,841	30,984,673	10,168	9,989,837	25
36	453,630	30,930,577	30,920,277	10,300	9,989,706	24
37	456,536	30,866,722	30,856,290	10,432	9,989,574	23
38	459,442	30,803,277	30,792,712	10,565	9,989,441	22
39	462,348	30,740,230	30,729,531	10,699	9,989,307	21
40	465,253	30,677,578	30,666,744	10,834	9,989,172	20
41	468,159	30,615,317	30,604,347	10,970	9,989,036	19
42	471,065	30,553,442	30,542,335	11,107	9,988,899	18
43	473,970	30,491,949	30,480,704	11,245	9,988,761	17
44	476,876	30,430,834	30,419,451	11,383	9,988,623	16
45	479,781	30,370,090	30,358,568	11,522	9,988,484	15
46	482,687	30,309,715	30,298,053	11,662	9,988,344	14
47	485,592	30,249,702	30,237,899	11,803	9,988,203	13
48	488,498	30,190,049	30,178,104	11,945	9,988,061	12
49	491,403	30,130,749	30,118,661	12,088	9,987,918	11
50	494,308	30,071,797	30,059,565	12,232	9,987,775	10
51	497,214	30,013,193	30,000,817	12,376	9,987,631	9
52	500,119	29,954,933	29,942,412	12,521	9,987,486	8
53	503,024	29,897,014	29,884,347	12,667	9,987,340	7
54	505,929	29,839,424	29,826,610	12,814	9,987,193	6
55	508,834	29,782,165	29,769,203	12,962	9,987,045	5
56	511,740	29,725,236	29,712,125	13,111	9,986,897	4
57	514,645	29,668,628	29,655,367	13,261	9,986,748	3
58	517,550	29,612,331	29,598,920	13,411	9,986,598	2
59	520,455	29,556,358	29,542,796	13,562	9,986,447	1
60	523,360	29,500,706	29,486,992	13,714	9,986,295	0

min.
Gr.
87
87.

Gr. 3

3 min	Sinus	Logarithmi	Differentiæ	+ | − Logarithmi	Sinus	
0	523,360	29,500,706	29,486,992	13,714	9,986,295	60
1	526,265	29,445,354	29,431,487	13,867	9,986,143	59
2	529,170	29,390,307	29,376,286	14,021	9,985,989	58
3	532,075	29,335,565	29,321,389	14,176	9,985,835	57
4	534,980	29,281,122	29,266,791	14,331	9,985,680	56
5	537,884	29,226,973	29,212,486	14,487	9,985,524	55
6	540,789	29,173,115	29,158,471	14,644	9,985,367	54
7	543,694	29,119,548	29,104,746	14,802	9,985,209	53
8	546,598	29,066,270	29,051,309	14,961	9,985,050	52
9	549,503	29,013,273	28,998,152	15,121	9,984,891	51
10	552,407	28,960,557	28,945,276	15,281	9,984,731	50
11	555,312	28,908,117	28,892,675	15,442	9,984,570	49
12	558,216	28,855,951	28,840,347	15,604	9,984,408	48
13	561,120	28,804,057	28,788,290	15,767	9,984,245	47
14	564,024	28,752,430	28,736,499	15,931	9,984,081	46
15	566,928	28,701,071	28,684,975	16,096	9,983,917	45
16	569,832	28,649,975	28,633,714	16,261	9,983,752	44
17	572,736	28,599,142	28,582,715	16,427	9,983,586	43
18	575,640	28,548,570	28,531,976	16,594	9,983,419	42
19	578,544	28,498,247	28,481,485	16,762	9,983,251	41
20	581,448	28,448,177	28,431,246	16,931	9,983,082	40
21	584,352	28,398,354	28,381,253	17,101	9,982,912	39
22	587,256	28,348,782	28,331,510	17,272	9,982,742	38
23	590,160	28,299 459	28,282,015	17,444	9,982,571	37
24	593,064	28,250,377	28,232,761	17,616	9,982,399	36
25	595,967	28,201,535	28,183,746	17,789	9,982,226	35
26	598,871	28,152,930	28,134,967	17,963	9,982,052	34
27	601,775	28,104,561	28,086,423	18,138	9,981,877	33
28	604,678	28,056,428	28,038,114	18,314	9,981,701	32
29	607,582	28,008,524	27,990,033	18,491	9,981,525	31
30	610,485	27,960,848	27,942,178	18,670	9,981,348	30

Gr. 3

3 min	Sinus	Logarithmi	+ Differentiæ −	Logarithmi	Sinus	
30	610,485	27,960,848	27,942,178	18,670	9,981,348	30
31	613,389	27,913,400	27,894,552	18,848	9,981,170	29
32	616,292	27,866,180	27,847,153	19,027	9,980,991	28
33	619,196	27,819,184	27,799,977	19,207	9,980,811	27
34	622,099	27,772,408	27,753,020	19,388	9,980,631	26
35	625,002	27,725,848	27,706,278	19,570	9,980,450	25
36	627,905	27,679,504	27,659,752	19,752	9,980,268	24
37	630,808	27,633,374	27,613,439	19,935	9,980,085	23
38	633,711	27,587,457	27,567,338	20,119	9,979,901	22
39	636,614	27,541,753	27,521,449	20,304	9,979,716	21
40	639,517	27,496,257	27,475,767	20,490	9,979,530	20
41	642,420	27,450,968	27,430,291	20,677	9,979,343	19
42	645,323	27,405,885	27,385,020	20,865	9,979,156	18
43	648,226	27,361,003	27,339,950	21,053	9,978,968	17
44	651,129	27,316,323	27,295,081	21,242	9,978,779	16
45	654,031	27,271,843	27,250,411	21,432	9,978,589	15
46	656,934	27,227,563	27,205,940	21,623	9,978,398	14
47	659,837	27,183,476	27,161,661	21,815	9,978,207	13
48	662,739	27,139,581	27,117,573	22,008	9,978,015	12
49	665,642	27,095,878	27,073,676	22,202	9,977,822	11
50	668,544	27,052,373	27,029,976	22,397	9,977,628	10
51	671,447	27,009,057	26,986,465	22,592	9,977,433	9
52	674,349	26,965,926	26,943,138	22,788	9,977,237	8
53	677,251	26,922,980	26,899,995	22,985	9,977,040	7
54	680,153	26,880,218	26,857,035	23,183	9,976,843	6
55	683,055	26,837,639	26,814,257	23,382	9,976,645	5
56	685,957	26,795,243	26,771,661	23,582	9,976,446	4
57	688,859	26,753,027	26,729,244	23,783	9,976,246	3
58	691,761	26,710,988	26,687,003	23,985	9,976,045	2
59	694,663	26,669,126	26,644,939	24,187	9,975,843	1
60	697,565	26,627,442	26,603,052	24,390	9,975,640	0
						min
						Gr.
						86

Gr. 4

4 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	697,565	26,627,442	26,603,052	24,390	9,975,640	60
1	700,467	26,585 929	26,561,335	24,594	9,975,437	59
2	703,369	26,544,587	26,519,788	24,799	9,975,233	58
3	706,270	26,503,416	26,478,411	25,005	9,975,028	57
4	709,172	26,462,418	26,437,207	25,211	9,974,822	56
5	712,073	26,421,589	26,396,171	25,418	9,974,615	55
6	714,975	26,380,927	26,355,301	25,626	9,974,408	54
7	717,876	26,340,428	26,314,593	25,835	9,974,200	53
8	720,777	26,300,094	26,274,050	26,044	9,973,991	52
9	723,678	26,259,923	26,233,669	26,254	9,973,781	51
10	726,579	26,219,913	26,193,448	26,465	9,973,570	50
11	729,480	26,180,067	26,153,390	26,677	9,973,358	49
12	732,381	26,140,377	26,113,487	26,890	9,973,145	48
13	735,282	26,100,842	26,073,738	27,104	9,972,931	47
14	738,183	26,061,465	26,034,146	27,319	9,972,717	46
15	741,084	26,022,244	25,994,709	27,535	9,972,502	45
16	743,985	25,983,176	25,955,424	27,752	9,972,286	44
17	746,886	25,944,260	25,916,290	27,970	9,972,069	43
18	749,787	25,905,496	25,877,308	28,188	9,971,851	42
19	752,688	25,866,884	25,838,477	28,407	9,971,633	41
20	755,588	25,828,423	25,799,796	28,627	9,971,414	40
21	758,489	25,790,110	25,751,262	28,848	9,971,194	39
22	761,389	25,751,942	25,722,872	29,070	9,970,973	38
23	764,290	25,713,920	25,684,727	29,293	9,970,751	37
24	767,180	25,676,043	25,646,527	29,516	9,970,528	36
25	770,090	25,638,310	25,608,570	29,740	9,970,304	35
26	772,991	25,600,722	25,570,757	29,965	9,970,079	34
27	775,891	25,563,273	25,533,082	30,191	9,969,854	33
28	778,791	25,525,966	25,495,548	30,418	9,969,628	32
29	781,691	25,488,798	25,458,152	30,646	9,969,401	31
30	784,591	25,451,769	25,420,894	30,875	9,969,173	30

Gr. 4

4 min	Sinus	Logarithmi	Differentiæ	+ | − Logarithmi	Sinus	
30	784,591	25,451,769	25,420,894	30,875	9,969,173	30
31	787,491	25,414,876	25,383,772	31,104	9,968,944	29
32	790,391	25,378,119	25,346,785	31,334	9,968,715	28
33	793,291	25,341,498	25,309,933	31,565	9,968,485	27
34	796,191	25,305,013	25,273,216	31,797	9,968,254	26
35	799,090	25,268,662	25,236,632	32,030	9,968,022	25
36	801,990	25,232,442	25,200,178	32,264	9,967,789	24
37	804,889	25,196,355	25,163,857	32,498	9,967,555	23
38	807,789	25,160,399	25,127,666	32,733	9,967,320	22
39	810,688	25,124,571	25,091,602	32,969	9,967,085	21
40	813,587	25,088,870	25,055,664	33,206	9,966,849	20
41	816,486	25,053,298	25,019,854	33,444	9,966,612	19
42	819,385	25,017,853	24,984,170	33,683	9,966,374	18
43	822,284	24,982,533	24,948,610	33,923	9,966,135	17
44	825,183	24,947,340	24,913,177	34,163	9,965,895	16
45	828,082	24,912,272	24,877,868	34,404	9,965,655	15
46	830,981	24,877,326	24,842,680	34,646	9,965,414	14
47	833,880	24,842,502	24,807,613	34,889	9,965,172	13
48	836,778	24,807,799	24,772,666	35,133	9,964,929	12
49	839,677	24,773,219	24,737,841	35,378	9,964,685	11
50	842,575	24,738,761	24,703,138	35,623	9,964,440	10
51	845,474	24,704,420	24,668,551	35,869	9,964,194	9
52	848,372	24,670,196	24,634,080	36,116	9,963,948	8
53	851,271	24,636,090	24,599,726	36,364	9,963,701	7
54	854,169	24,602,100	24,565,487	36,613	9,963,453	6
55	857,067	24,568,228	24,531,365	36,863	9,963,204	5
56	859,965	24,534,473	24,497,359	37,114	9,962,954	4
57	862,863	24,500,829	24,463,463	37,366	9,962,703	3
58	865,761	24,467,298	24,429,679	37,619	9,962,452	2
59	868,659	24,433,880	24,396,008	37,872	9,962,200	1
60	871,557	24,400,578	24,362,452	38,126	9,961,947	0

min. *Gr.*

85 85

Gr. 5

5 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	871,557	24,400,578	24,362,452	38,126	9,961,947	60
1	874,455	24,367,384	24,329,005	38,381	9,961,693	59
2	877,353	24,334,302	24,295,665	38,637	9,961,438	58
3	880,250	24,301,329	24,262,435	38,894	9,961,183	57
4	883,148	24,268,467	24,229,316	39,151	9,960,927	56
5	886,045	24,235,712	24,196,303	39,409	9,960,670	55
6	888,943	24,203,064	24,163,396	39,668	9,960,412	54
7	891,840	24,170,523	24,130,595	39,928	9,960,153	53
8	894,737	24,138,089	24,097,900	40,189	9,959,893	52
9	897,634	24,105,760	24,065,309	40,451	9,959,632	51
10	900,531	24,073,540	24,032,827	40,713	9,959,370	50
11	903,428	24,041,422	24,000,446	40,976	9,959,107	49
12	906,325	24,009,408	23,968,168	41,240	9,958,844	48
13	909,222	23,977,495	23,935,990	41,505	9,958,580	47
14	912,119	23,945,685	23,903,914	41,771	9,958,315	46
15	915,016	23,913,978	23,871,940	42,038	9,958,049	45
16	917,913	23,882,373	23,840,067	42,306	9,957,782	44
17	920,809	23,850,867	23,808,292	42,575	9,957,515	43
18	923,706	23,819,460	23,776,615	42,845	9,957,247	42
19	926,602	23,788,153	23,745,038	43,115	9,956,978	41
20	929,498	23,756,943	23,713,557	43,386	9,956,708	40
21	932,395	23,725,832	23,682,174	43,656	9,956,437	39
22	935,291	23,694,818	23,650,887	43,931	9,956,165	38
23	938,187	23,663,900	23,619,695	44,205	9,955,893	37
24	941,083	23,633,080	23,588,601	44,479	9,955,620	36
25	943,979	23,602,355	23,557,601	44,754	9,955,346	35
26	946,875	23,571,725	23,526,695	45,030	9,955,071	34
27	949,771	23,541,190	23,495,883	45,307	9,954,795	33
28	952,667	23,510,748	23,465,163	45,585	9,954,518	32
29	955,563	23,480,399	23,434,535	45,864	9,954,240	31
30	958,458	23,450,143	23,403,999	46,144	9,953,962	30

Gr. 5

5 min	Sinus	Logarithmi	Differentiæ +	Logarithmi −	Sinus	
30	958,458	23,450,143	23,403,999	46,144	9,953,962	30
31	961,354	23,419,980	23,373,556	46,424	9,953,683	29
32	964,249	23,389,908	23,343,203	46,705	9,953,403	28
33	967,144	23,359,927	23,312,940	46,987	9,953,122	27
34	970,039	23,330,036	23,282,766	47,270	9,952,840	26
35	972,934	23,300,235	23,252,681	47,554	9,952,557	25
36	975,829	23,270,525	23,222,686	47,839	9,952,274	24
37	978,724	23,240,903	23,192,778	48,125	9,951,990	23
38	981,619	23,211,368	23,162,956	48,412	9,951,705	22
39	984,514	23,181,920	23,133,220	48,700	9,951,419	21
40	987,408	23,152,560	23,103,572	48,988	9,951,132	20
41	990,303	23,123,287	23,074,010	49,277	9,950,844	19
42	993,198	23,094,100	23,044,533	49,567	9,950,555	18
43	996,092	23,064,999	23,015,141	49,858	9,950,266	17
44	998,987	23,035,985	22,985,836	50,149	9,949,976	16
45	1,001,881	23,007,056	22,956,615	50,441	9,949,685	15
46	1,004,775	22,978,212	22,927,478	50,734	9,949,393	14
47	1,007,669	22,949,449	22,898,421	51,028	9,949,100	13
48	1,010,563	22,920,769	22,869,446	51,323	9,948,807	12
49	1,013,457	22,892,172	22,840,553	51,619	9,948,513	11
50	1,016,351	22,863,658	22,811,742	51,916	9,948,218	10
51	1,019,245	22,835,227	22,783,013	52,214	9,947,922	9
52	1,022,139	22,806,878	22,754,366	52,512	9,947,625	8
53	1,025,032	22,778,609	22,725,798	52,811	9,947,327	7
54	1,027,926	22,750,420	22,697,309	53,111	9,947,028	6
55	1,030,819	22,722,311	22,668,899	53,412	9,946,729	5
56	1,033,713	22,694,283	22,640,569	53,714	9,946,429	4
57	1,036,606	22,666,333	22,612,316	54,017	9,946,128	3
58	1,039,499	22,638,461	22,584,140	54,321	9,945,826	2
59	1,042,392	22,610,667	22,556,041	54,626	9,945,523	1
60	1,045,285	22,582,951	22,528,019	54,932	9,945,219	0
						min.
						Gr. 84

Gr. 6

6 min	Sinus	Logarithmi	+ | − Differentiæ	Logarithmi	Sinus	
0	1,045,285	22,582,951	22,528,019	54,932	9,945,219	60
1	1,048,178	22,555,313	22,500,075	55,238	9,944.914	59
2	1,051,071	22,527,752	22,472,207	55,545	9,944,609	58
3	1,053,964	22,500,267	22,444,414	55,853	9,944,303	57
4	1,056,857	22,472,859	22,416,697	56,162	9,943,996	56
5	1,059,749	22,445,527	22,389,055	56,472	9,943,688	55
6	1,062,642	22,418,272	22,361,490	56,782	9,943,379	54
7	1,065,534	22,391,091	22,333,998	57,093	9,943,069	53
8	1,068,426	22,363,984	22,306,579	57,405	9,942,759	52
9	1,071,318	22,336,951	22,279,233	57,718	9,942,448	51
10	1,074,210	22,309,991	22,251,959	58,032	9,942,136	50
11	1,077,102	22,283,104	22,224,757	58,347	9,941,823	49
12	1,079,994	22,256,290	22,197,627	58,663	9,941,509	48
13	1,082,886	22,229,549	22,170,570	58,979	9,941,194	47
14	1,085,778	22,202,881	22,143,585	59,296	9,940,879	46
15	1,088,669	22,176,285	22,116,671	59,614	9,940,563	45
16	1,091,561	22,149,762	22,089,829	59,933	9,940,246	44
17	1,094,452	22,123,308	22,063,055	60,253	9,939,928	43
18	1,097,344	22,096,925	22,036,351	60,574	9,939,609	42
19	1,100,235	22,070,612	22,009,717	60,895	9,939,290	41
20	1,103,126	22,044,368	21,983,151	61,217	9,938,970	40
21	1,106,017	22,018,195	21,956,655	61,540	9,938,649	39
22	1,108,908	21,992,090	21,930,226	61,864	9,938,327	38
23	1,111,799	21,966,054	21,903,865	62,189	9,938,004	37
24	1,114,690	21,940,086	21,877,571	62,515	9,937,680	36
25	1,117,580	21,914,186	21,851,344	62,842	9,937,355	35
26	1,120,471	21,888,355	21,825,185	63,170	9,937,029	34
27	1,123,361	21,862,590	21,799,091	63,499	9,936,703	33
28	1,126,252	21,836,892	21,773,064	63,828	9,936,376	32
29	1,129,142	21,811,261	21,747,103	64,158	9,936,048	31
30	1,132,032	21,785,698	21,721,209	64,489	9,935,719	30

Gr. 6

6 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	1,132,032	21,785,698	21,721,209	64,489	9,935,719	30
31	1,134,922	21,760,195	21,695,378	64,821	9,935,389	29
32	1,137,812	21,734,767	21,669,613	65,154	9,935,058	28
33	1,140,702	21,709,400	21,643,912	65,488	9,934,727	27
34	1,143,592	21,684,100	21,618,278	65,822	9,934,395	26
35	1,146,482	21,658,865	21,592,708	66,157	9,934,062	25
36	1,149,372	21,633,695	21,567,202	66,493	9,933,728	24
37	1,152,261	21,608,586	21,541,756	66,830	9,933,393	23
38	1,155,151	21,583,540	21,516,372	67,168	9,933,057	22
39	1,158,040	21,558,557	21,491,050	67,507	9,932,721	21
40	1,160,929	21,533,639	21,465,793	67,846	9,932,384	20
41	1,163,818	21,508,781	21,440,595	68,186	9,932,046	19
42	1,166,707	21,483,986	21,415,459	68,527	9,931,707	18
43	1,169,596	21,459,254	21,390,385	68,869	9,931,367	17
44	1,172,485	21,434,585	21,365,373	69,212	9,931,026	16
45	1,175,374	21,409,979	21,340,423	69,556	9,930,685	15
46	1,178,263	21,385,434	21,315,533	69,901	9,930,343	14
47	1,181,151	21,360,949	21,290,702	70,247	9,930,000	13
48	1,184,040	21,336,524	21,265,931	70,593	9,929,656	12
49	1,186,928	21,312,160	21,241,220	70,940	9,929,311	11
50	1,189,816	21,287,855	21,216,567	71,288	9,928,965	10
51	1,192,704	21,263,609	21,191,972	71,637	9,928,618	9
52	1,195,592	21,239,423	21,167,436	71,987	9,928,271	8
53	1,198,480	21,215,297	21,142,959	72,338	9,927,923	7
54	1,201,368	21,191,230	21,118,540	72,690	9,927,574	6
55	1,204,255	21,167,222	21,094,179	73,043	9,927,224	5
56	1,207,143	21,143,273	21,069,877	73,396	9,926,873	4
57	1,210,031	21,119,381	21,045,631	73,750	9,926,521	3
58	1,212,918	21,095,546	21,021,441	74,105	9,926,169	2
59	1,215,806	21,071,769	20,997,308	74,461	9,925,816	1
60	1,218,693	21,048,049	20,973,231	74,818	9,925,461	0

min
Gr.

Gr. 7

7 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	1,218,693	21,048,049	20,973,231	74,818	9,925,461	60
1	1,221,580	21,024,385	20,949,209	75,176	9,925,106	59
2	1,224,467	21,000,779	20,925,245	75,534	9,924,750	58
3	1,227,354	20,977,230	20,901,337	75,893	9,924,393	57
4	1,230,241	20,953,738	20,877,485	76,253	9,924,036	56
5	1,233,128	20,930,302	20,853,688	76,614	9,923,678	55
6	1,236,015	20,906,922	20,829,946	76,976	9,923,319	54
7	1,238,901	20,883,595	20,806,256	77,339	9,922,959	53
8	1,241,788	20,860,323	20,782,620	77,703	9,922,598	52
9	1,244,674	20,837,106	20,759,038	78,068	9,922,236	51
10	1,247,560	20,813,945	20,735,512	78,433	9,921,874	50
11	1,250,446	20,790,838	20,712,039	78,799	9,921,511	49
12	1,253,332	20,767,785	20,688,619	79,166	9,921,147	48
13	1,256,218	20,744,785	20,665,251	79,534	9,920,782	47
14	1,259,104	20,721,828	20,641,935	79,903	9,920,416	46
15	1,261,990	20,698,946	20,618,674	80,272	9,920,049	45
16	1,264,876	20,676,107	20,595,465	80,642	9,929,682	44
17	1,267,761	20,653,321	20,572,308	81,01	9,929,314	43
18	1,270,647	20,630,588	20,549,203	81,385	9,918,945	42
19	1,273,532	20,607,906	20,526,148	81,758	9,918,575	41
20	1,276,417	20,585,278	20,503,149	82,132	9,918,200	40
21	1,279,302	20,562,701	20,480,194	82,507	9,917,832	39
22	1,282,187	20,540,176	20,457,293	82,883	9,917,459	38
23	1,285,072	20,517,703	20,434,444	83,259	9,917,086	37
24	1,287,957	20,495,281	20,411,645	83,636	9,916,712	36
25	1,290,841	20,472,909	20,388,895	84,014	9,916,337	35
26	1,293,726	20,450,587	20,366,194	84,393	9,91 961	34
27	1,296,610	20,428,316	20,343,543	84,773	9,915,584	33
28	1,299,494	20,406,096	20,320,942	85,154	9,915,206	32
29	1,302,378	20,383,925	20,298,389	85,536	9,914,828	31
30	1,305,262	20,361,806	20,275,887	85,919	9,914,449	30

Gr. 7

7 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
30	1,305,262	20,361,806	20,275,887	85,919	9,914,449	30
31	1,308,146	20,339,737	20,253,435	86,302	9,914,069	29
32	1,311,030	20,317,717	20,231,031	86,686	9,913,688	28
33	1,313,914	20,295,746	20,208,675	87,071	9,913,306	27
34	1,316,798	20,273,822	20,186,365	87,457	9,912,923	26
35	1,319,681	20,251,947	20,164,103	87,844	9,912,540	25
36	1,322,564	20,230,120	20,141,888	88,232	9,912,156	24
37	1,325,447	20,208,341	20,119,720	88,621	9,911,771	23
38	1,328,330	20,186,611	20,097,600	89,011	9,911,385	22
39	1,331,213	20,164,931	20,075,530	89,401	9,910,998	21
40	1,334,096	20,143,301	20,053,509	89,792	9,910,610	20
41	1,336,979	20,121,717	20,031,533	90,184	9,910,221	19
42	1,339,862	20,100,180	20,009,603	90,577	9,909,832	18
43	1,342,744	20,078,689	19,987,718	90,971	9,909,442	17
44	1,345,627	20,057,245	19,965,880	91,365	9,909,051	16
45	1,348,509	20,035,846	19,944,086	91,760	9,908,659	15
46	1,351,392	20,014,494	19,922,338	92,156	9,908,266	14
47	1,354,274	19,993,189	19,900,636	92,553	9,907,873	13
48	1,357,156	19,971,931	19,878,980	92,951	9,907,479	12
49	1,360,038	19,950,718	19,857,368	93,350	9,907,084	11
50	1,362,920	19,929,552	19,835,802	93,750	9,906,688	10
51	1,365,802	19,908,432	19,814,281	94,151	9,906,291	9
52	1,368,683	19,887,357	19,792,805	94,552	9,905,893	8
53	1,371,564	19,866,327	19,771,373	94,954	9,905,494	7
54	1,374,446	19,845,341	19,749,984	95,357	9,905,095	6
55	1,377,327	19,824,400	19,728,639	95,761	9,904,695	5
56	1,380,208	19,803,504	19,707,338	96,166	9,904,294	4
57	1,383,089	19,782,652	19,686,080	96,572	9,903,892	3
58	1,385,970	19,761,844	19,664,865	96,979	9,903,489	2
59	1,388,851	19,741,081	19,643,694	97,387	9,903,085	1
60	1,391,731	19,720,362	19,622,566	97,796	9,902,681	0

min.
Gr.
82 82

Gr. 8

8

min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	1,391,731	19,720,362	19,622,566	97,796	9,902,681	60
1	1,394,612	19,699,687	19,601,482	98,205	9,902,276	59
2	1,397,492	19,679,054	19,580,439	98,615	9,901,870	58
3	1,400,373	19,658,464	19,559,438	99,026	9,901,463	57
4	1,403,253	19,637,917	19,538,479	99,438	9,901,055	56
5	1,406,133	19,617,413	19,517,562	99,851	9,900,646	55
6	1,409,013	19,596,952	19,496,687	100,265	9,900,237	54
7	1,411,893	19,576,535	19,475,856	100,679	9,899,827	53
8	1,414,772	19,556,160	19,455,066	101,094	9,899,416	52
9	1,417,652	19,535,827	19,434,317	101,510	9,899,004	51
10	1,420,531	19,515,538	19,413,611	101,927	9,898,591	50
11	1,423,410	19,495,290	19,392,945	102,345	9,898,177	49
12	1,426,289	19,475,084	19,372,320	102,764	9,897,762	48
13	1,429,168	19,454,918	19,351,734	103,184	9,897,347	47
14	1,432,047	19,434,794	19,331,190	103,604	9,896,931	46
15	1,434,926	19,414,711	19,310,686	104,025	9,896,514	45
16	1,437,805	19,394,669	19,290,222	104,447	9,896,096	44
17	1,440,684	19,374,668	19,269,798	104,870	9,895,677	43
18	1,443,562	19,354,708	19,249,414	105,294	9,895,257	42
19	1,446,441	19,334,787	19,229,068	105,719	9,894,837	41
20	1,449,319	19,314,908	19,208,763	106,145	9,894,416	40
21	1,452,197	19,295,072	19,188,501	106,571	9,893,994	39
22	1,455,075	19,275,275	19,168,277	106,998	9,893,571	38
23	1,457,953	19,255,517	19,148,091	107,426	9,893,147	37
24	1,460,831	19,235,798	19,127,943	107,855	9,892,723	36
25	1,463,708	19,216,118	19,107,833	108,285	9,892,298	35
26	1,466,586	19,196,477	19,087,761	108,716	9,891,872	34
27	1,469,463	19,176,875	19,067,727	109,148	9,891,445	33
28	1,472,340	19,157,313	19,047,732	109,581	9,891,017	32
29	1,475,217	19,137,792	19,027,777	110,015	9,890,588	31
30	1,478,094	19,118,310	19,007,861	110,449	9,890,159	30

Gr. 8

min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	1,478,094	19,118,310	19,007,861	110,449	9,890,159	30
31	1,480,971	19,098,865	18,987,981	110,884	9,889,729	29
32	1,483,848	19,079,459	18,968,139	111,320	9,889,298	28
33	1,486,724	19,060,091	18,908,334	111,757	9,888,866	27
34	1,489,601	19,040,761	18,928,566	112,195	9,888,433	26
35	1,492,477	19,021,469	18,908,835	112,634	9,887,999	25
36	1,495,353	19,002,215	18,889,141	113,074	9,887,564	24
37	1,498,229	18,982,999	18,869,485	113,514	9,887,128	23
38	1,501,105	18,963,822	18,849,867	113,955	9,886,692	22
39	1,503,981	18,944,682	18,830,285	114,397	9,886,255	21
40	1,506,857	18,925,581	18,810,741	114,840	9,885,817	20
41	1,509,733	18,906,517	18,791,233	115,284	9,885,378	19
42	1,512,608	18,887,489	18,771,760	115,729	9,884,938	18
43	1,515,484	18,868,498	18,752,323	116,175	9,884,498	17
44	1,518,359	18,849,543	18,732,921	116,622	9,884,057	16
45	1,521,234	18,830,625	18,713,556	117,069	9,883,615	15
46	1,524,109	18,811,744	18,694,227	117,517	9,883,172	14
47	1,526,984	18,792,899	18,674,933	117,966	9,882,728	13
48	1,529,859	18,774,090	18,655,674	118,416	9,882,283	12
49	1,532,734	18,755,318	18,636,451	118,867	9,881,838	11
50	1,535,608	18,736,581	18,617,262	119,319	9,881,392	10
51	1,538,482	18,717,882	18,598,111	119,771	9,880,945	9
52	1,541,356	18,699,218	18,578,994	120,224	9,880,497	8
53	1,544,230	18,680,589	18,559,911	120,678	9,880,048	7
54	1,547,104	18,661,995	18,540,862	121,133	9,879,598	6
55	1,549,978	18,643,437	18,521,848	121,589	9,879,148	5
56	1,552,852	18,624,915	18,502,869	122,046	9,878,697	4
57	1,555,725	18,606,428	18,483,924	122,504	9,878,245	3
58	1,558,599	18,587,975	18,465,013	122,962	9,877,792	2
59	1,561,472	18,569,557	18,446,136	123,421	9,877,338	1
60	1,564,345	18,551,174	18,427,293	123,881	9,876,883	0
						min.
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				81		81

Gr. 9

9 min	Sinus	Logarithmi	Differentiæ +	Differentiæ −	Logarithmi	Sinus	
0	1,564,345	18,551,174	18,427,293		123,881	9,876,883	60
1	1,567,218	18,532,826	18,408,484		124,342	9,876,427	59
2	1,570,091	18,514,511	18,389,707		124,804	9,875,971	58
3	1,572,964	18,496,231	18,370,964		125,267	9,875,514	57
4	1,575,837	18,477,984	18,352,253		125,731	9,875,056	56
5	1,578,709	18,459,772	18,333,576		126,196	9,874,597	55
6	1,581,581	18,441,594	18,314,933		126,661	9,874,137	54
7	1,584,453	18,423,451	18,296,324		127,127	9,873,677	53
8	1,587,325	18,405,341	18,277,747		127,594	9,873,216	52
9	1,590,197	18,387,265	18,259,203		128,062	9,872,754	51
10	1,593,069	18,369,223	18,240,692		128,531	9,872,291	50
11	1,595,941	18,351,214	18,222,213		129,001	9,871,827	49
12	1,598,812	18,333,237	18,203,765		129,472	9,871,362	48
13	1,601,684	18,315,294	18,185,351		129,943	9,870,897	47
14	1,604,555	18,297,384	18,166,969		130,415	9,870,431	46
15	1,607,426	18,279,507	18,148,619		130,888	9,869,964	45
16	1,610,297	18,261,663	18,130,301		131,362	9,869,496	44
17	1,613,168	18,243,851	18,112,014		131,837	9,869,027	43
18	1,616,038	18,226,071	18,093,758		132,313	9,868,557	42
19	1,618,909	18,208,323	18,075,533		132,790	9,868,087	41
20	1,621,779	18,190,606	18,057,338		133,268	9,867,616	40
21	1,624,649	18,172,924	18,039,177		133,747	9,867,144	39
22	1,627,519	18,155,273	18,021,047		134,226	9,866,671	38
23	1,630,389	18,137,654	18,002,948		134,706	9,866,197	37
24	1,633,259	18,120,067	17,984,880		135,187	9,865,722	36
25	1,636,129	18,102,511	17,966,842		135,669	9,865,246	35
26	1,638,999	18,084,987	17,948,835		136,152	9,864,770	34
27	1,641,868	18,067,495	17,930,859		136,636	9,864,293	33
28	1,644,738	18,050,034	17,912,913		137,121	9,863,815	32
29	1,647,607	18,032,604	17,894,997		137,607	9,863,336	31
30	1,650,476	18,015,207	17,877,114		138,093	9,862,856	30

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Gr. 9

9 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
30	1,650,476	18,015,207	17,877,114	138,093	9,862,856	30
31	1,653,345	17,997,839	17,859,259	138,580	9,862,376	29
32	1,656,214	17,980,503	17,841,435	139,068	9,861,895	28
33	1,659,082	17,963,198	17,823,641	139,557	9,861,413	27
34	1,661,951	17,945,922	17,805,875	140,047	9,860,930	26
35	1,664,819	17,928,677	17,788,139	140,538	9,860,446	25
36	1,667,687	17,911,463	17,770,433	141,030	9,859,961	24
37	1,670,555	17,894,281	17,752,759	141,522	9,859,475	23
38	1,673,423	17,877,128	17,735,113	142,015	9,858,989	22
39	1,676,291	17,860,006	17,717,497	142,509	9,858,502	21
40	1,679,159	17,842,915	17,699,911	143,004	9,858,014	20
41	1,682,027	17,825,852	17,682,352	143,500	9,857,525	19
42	1,684,894	17,808,820	17,664,823	143,997	9,857,035	18
43	1,687,761	17,791,817	17,647,322	144,495	9,856,544	17
44	1,690,628	17,774,843	17,629,849	144,994	9,856,053	16
45	1,693,495	17,757,899	17,612,406	145,493	9,855,561	15
46	1,696,362	17,740,985	17,594,992	145,993	9,855,068	14
47	1,699,229	17,724,100	17,577,606	146,494	9,854,574	13
48	1,702,095	17,707,244	17,560,248	146,996	9,854,079	12
49	1,704,962	17,690,418	17,542,919	147,499	9,853,583	11
50	1,707,828	17,673,622	17,525,619	148,003	9,853,087	10
51	1,710,694	17,656,856	17,508,349	148,507	9,852,590	9
52	1,713,560	17,640,118	17,491,106	149,012	9,852,092	8
53	1,716,426	17,623,408	17,473,890	149,518	9,851,593	7
54	1,719,292	17,606,726	17,456,701	150,025	9,851,093	6
55	1,722,157	17,590,073	17,439,540	150,533	9,850,593	5
56	1,725,022	17,573,448	17,422,406	151,042	9,850,092	4
57	1,727,887	17,556,851	17,405,299	151,552	9,849,590	3
58	1,730,752	17,540,28	17,388,220	152,063	9,849,087	2
59	1,733,617	17,523,744	17,371,169	152,575	9,848,583	1
60	1,736,482	17,507,234	17,354,146	153,088	9,848,078	0

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Gr. 10

10 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
0	1,736,482	17,507,234	17,354,146	153,088	9,848,078	60
1	1,739,347	17,490,751	17,337,150	153,601	9,847,572	59
2	1,742,211	17,474,296	17,320,181	154,115	9,847,066	58
3	1,745,075	17,457,869	17,303,239	154,630	9,846,559	57
4	1,747,939	17,441,470	17,286,324	155,146	9,846,051	56
5	1,750,803	17,425,098	17,269,435	155,663	9,845,542	55
6	1,753,667	17,408,754	17,252,573	156,181	9,845,032	54
7	1,756,531	17,392,438	17,235,738	156,700	9,844,521	53
8	1,759,394	17,376,149	17,218,929	157,220	9,844,010	52
9	1,762,258	17,359,887	17,202,147	157,740	9,843,498	51
10	1,765,121	17,343,652	17,185,391	158,261	9,842,985	50
11	1,767,984	17,327,444	17,168,661	158,783	9,842,471	49
12	1,770,847	17,311,263	17,151,957	159,306	9,841,956	48
13	1,773,710	17,295,109	17,135,279	159,830	9,841,440	47
14	1,776,573	17,278,982	17,118,627	160,355	9,840,924	46
15	1,779,435	17,262,882	17,102,001	160,881	9,840,407	45
16	1,782,298	17,246,809	17,085,401	161,408	9,839,889	44
17	1,785,160	17,230,762	17,068,827	161,935	9,839,370	43
18	1,788,022	17,214,742	17,052,279	162,463	9,838,850	42
19	1,790,884	17,198,749	17,035,757	162,992	9,838,329	41
20	1,793,746	17,182,783	17,019,261	163,522	9,837,808	40
21	1,796,608	17,166,843	17,002,790	164,053	9,837,286	39
22	1,799,469	17,150,929	16,986,344	164,585	9,836,763	38
23	1,802,331	17,135,041	16,969,924	165,117	9,836,239	37
24	1,805,192	17,119,179	16,953,529	165,650	9,835,714	36
25	1,808,053	17,103,342	16,937,158	166,184	9,835,189	35
26	1,810,914	17,087,531	16,920,812	166,719	9,834,663	34
27	1,813,774	17,071,746	16,904,491	167,255	9,834,136	33
28	1,816,634	17,055,987	16,888,195	167,792	9,833,608	32
29	1,819,495	17,040,254	16,871,924	168,330	9,833,079	31
30	1,822,355	17,024,542	16,855,678	168,869	9,832,549	30

Gr. 10

10 min	Sinus	Logarithmi	+ Differentiæ −	Logarithmi	Sinus	
30	1,822,355	17,024,547	16,855,678	168,869	9,832,549	30
31	1,825,215	17,008,866	16,839,458	169,408	9,832,019	29
32	1,828,075	16,993,210	16,823,262	169,948	9,831,488	28
33	1,830,935	16,977,579	16,807,090	170,489	9,830,956	27
34	1,833,795	16,961,973	16,790,942	171,031	9,830,423	26
35	1,836,654	16,946,392	16,774,818	171,574	9,829,889	25
36	1,839,513	16,930,836	16,758,718	172,118	9,829,354	24
37	1,842,372	16,915,305	16,742,642	172,663	9,828,818	23
38	1,845,231	16,899,799	16,726,590	173,209	9,828,282	22
39	1,848,090	16,884,317	16,710,561	173,756	9,827,745	21
40	1,850,949	16,868,860	16,694,557	174,303	9,827,207	20
41	1,853,808	16,853,428	16,678,577	174,851	9,826,668	19
42	1,856,666	16,838,021	16,662,621	175,400	9,826,128	18
43	1,859,524	16,822,638	16,646,688	175,950	9,825,587	17
44	1,862,382	16,807,280	16,630,779	176,501	9,825,046	16
45	1,865,240	16,791,946	16,614,893	177,053	9,824,504	15
46	1,868,098	16,776,636	16,599,030	177,606	9,823,961	14
47	1,870,956	16,761,351	16,583,191	178,160	9,823,417	13
48	1,873,813	16,746,090	16,567,375	178,715	9,822,872	12
49	1,876,670	16,730,853	16,551,583	179,270	9,822,227	11
50	1,879,527	16,715,640	16,535,814	179,826	9,821,781	10
51	1,882,384	16,700,451	16,520,068	180,383	9,821,234	9
52	1,885,241	16,685,286	16,504,345	180,941	9,820,686	8
53	1,888,098	16,670,145	16,488,645	181,500	9,820,137	7
54	1,890,954	16,655,028	16,472,968	182,060	9,819,587	6
55	1,893,810	16,639,934	16,457,313	182,621	9,819,037	5
56	1,896,666	16,624,864	16,441,681	183,183	9,818,486	4
57	1,899,522	16,609,817	16,426,072	183,745	9,817,934	3
58	1,902,378	16,594,794	16,410,486	184,308	9,817,381	2
59	1,905,234	16,579,794	16,394,922	184,872	9,816,827	1
60	1,908,090	16,564,818	16,379,381	185,437	9,816,272	0
						min.
						Gr.

Gr. 11

11 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
0	1,908,090	16,564,818	16,379,381	185,437	9,816,272	60
1	1,910,945	16,549,865	16,363,862	186,003	9,815,716	59
2	1,913,800	16,534,935	16,348,365	186,570	9,815,160	58
3	1,916,655	16,520,028	16,332,890	187,138	9,814,603	57
4	1,919,510	16,505,144	16,317,438	187,706	9,814,045	56
5	1,922,365	16,490,283	16,302,008	188,275	9,813,486	55
6	1,925,220	16,475,445	16,286,600	188,845	9,812,926	54
7	1,928,074	16,460,630	16,271,214	189,416	9,812,366	53
8	1,930,928	16,445,837	16,255,849	189,988	9,811,850	52
9	1,933,782	16,431,067	16,240,506	190,561	9,811,243	51
10	1,936,636	16,416,320	16,225,185	191,135	9,810,680	50
11	1,939,490	16,401,596	16,209,886	191,710	9,810,116	49
12	1,942,344	16,386,895	16,194,610	192,285	9,809,551	48
13	1,945,197	16,372,216	16,179,355	192,861	9,808,986	47
14	1,948,050	16,357,559	16,164,121	193,438	9,808,420	46
15	1,950,903	16,342,924	16,148,908	194,016	9,807,853	45
16	1,953,756	16,328,311	16,133,716	194,595	9,807,285	44
17	1,956,609	16,313,720	16,118,545	195,175	9,806,716	43
18	1,959,462	16,299,151	16,103,395	195,756	9,806,147	42
19	1,962,314	16,284,604	16,088,266	196,338	9,805,577	41
20	1,965,166	16,270,079	16,073,159	196,920	9,805,006	40
21	1,968,018	16,255,576	16,058,073	197,503	9,804,434	39
22	1,970,870	16,241,095	16,043,008	198,087	9,803,861	38
23	1,973,722	16,226,636	16,027,964	198,672	9,803,287	37
24	1,976,574	16,212,198	16,012,940	199,258	9,802,712	36
25	1,979,425	16,197,782	15,997,937	199,845	9,802,137	35
26	1,982,276	16,183,388	15,982,955	200,433	9,801,561	34
27	1,985,127	16,169,016	15,967,994	201,022	9,800,984	33
28	1,987,978	16,154,665	15,953,053	201,612	9,800,406	32
29	1,990,829	16,140,336	15,938,133	202,203	9,799,827	31
30	1,993,679	16,126,028	15,923,233	202,795	9,799,247	30

Gr. 11

min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	1,993,679	16,126,028	15,923,233	202,795	9,799,247	30
31	1,996,530	16,111,742	15,908,355	203,387	9,798,667	29
32	1,999,380	16,097,477	15,893,497	203,980	9,798,086	28
33	2,002,230	16,083,232	15,878,658	204,574	9,797,504	27
34	2,005,080	16,069,008	15,863,839	205,169	9,796,921	26
35	2,007,930	16,054,805	15,849,040	205,765	9,796,337	25
36	2,010,780	16,040,623	15,834,261	206,362	9,795,753	24
37	2,013,629	16,026,462	15,819,502	206,960	9,795,168	23
38	2,016,478	16,012,322	15,804,764	207,558	9,794,582	22
39	2,019,327	15,998,203	15,790,046	208,157	9,793,995	21
40	2,022,176	15,984,105	15,775,348	208,757	9,793,407	20
41	2,025,025	15,970,028	15,760,670	209,358	9,792,818	19
42	2,027,874	15,955,972	15,746,012	209,960	9,792,228	18
43	2,030,722	15,941,936	15,731,373	210,563	9,791,638	17
44	2,033,570	15,927,921	15,716,754	211,167	9,791,047	16
45	2,036,418	15,913,926	15,702,154	211,772	9,790,455	15
46	2,039,266	15,899,951	15,687,573	212,378	9,789,862	14
47	2,042,114	15,885,996	15,673,012	212,984	9,789,268	13
48	2,044,962	15,872,062	15,658,461	213,591	9,788,674	12
49	2,047,809	15,858,148	15,643,949	214,199	9,788,079	11
50	2,050,656	15,844,254	15,629,446	214,808	9,787,483	10
51	2,053,503	15,830,371	15,614,953	215,418	9,786,886	9
52	2,056,350	15,816,518	15,600,489	216,029	9,786,288	8
53	2,059,197	15,802,685	15,586,044	216,641	9,785,689	7
54	2,062,043	15,788,871	15,571,617	217,254	9,785,090	6
55	2,064,889	15,775,077	15,557,210	217,867	9,784,490	5
56	2,067,735	15,761,303	15,542,822	218,481	9,783,889	4
57	2,070,581	15,747,559	15,528,463	219,096	9,783,287	3
58	2,073,427	15,733,824	15,514,112	219,712	9,782,684	2
59	2,076,272	15,720,109	15,499,780	220,329	9,782,080	1
60	2,079,117	15,706,414	15,485,467	220,947	9,781,476	0
						min.
						Gr.
						78

Gr. 12

12 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	2,079,117	15,706,414	15,485,467	220,947	9,781,476	60
1	2,081,962	15,692,738	15,471,172	221,566	9,780,871	59
2	2,084,807	15,679,082	15,456,896	222,186	9,780,265	58
3	2,087,652	15,665,445	15,442,639	222,806	9,779,658	57
4	2,090,497	15,651,828	15,428,401	223,427	9,779,050	56
5	2,093,342	15,638,230	15,414,181	224,049	9,778,442	55
6	2,096,180	15,624,651	15,399,979	224,672	9,777,833	54
7	2,099,030	15,611,092	15,385,796	225,296	9,777,223	53
8	2,101,874	15,597,552	15,371,631	225,921	9,776,612	52
9	2,104,718	15,584,031	15,357,484	226,547	9,776,000	51
10	2,107,562	15,570,530	15,343,356	227,174	9,775,387	50
11	2,110,405	15,557,048	15,329,246	227,802	9,774,773	49
12	2,113,248	15,543,585	15,315,155	228,430	9,774,159	48
13	2,116,091	15,530,141	15,301,082	229,059	9,773,544	47
14	2,118,934	15,516,715	15,287,026	229,689	9,772,928	46
15	2,121,777	15,503,308	15,272,988	230,320	9,772,311	45
16	2,124,620	15,489,920	15,258,968	230,952	9,771,693	44
17	2,127,462	15,476,551	15,244,966	231,585	9,771,075	43
18	2,130,304	15,463,200	15,230,981	232,219	9,770,456	42
19	2,133,146	15,449,868	15,217,014	232,854	9,769,836	41
20	2,135,988	15,436,554	15,203,064	233,490	9,769,215	40
21	2,138,830	15,423,259	15,189,133	234,126	9,768,593	39
22	2,141,671	15,409,982	15,175,219	234,763	9,767,970	38
23	2,144,512	15,396,724	15,161,323	235,401	9,767,347	37
24	2,147,353	15,383,484	15,147,444	236,040	9,766,723	36
25	2,150,194	15,370,262	15,133,582	236,680	9,766,098	35
26	2,153,035	15,357,059	15,119,738	237,321	9,765,472	34
27	2,155,876	15,343,874	15,105,911	237,963	9,764,845	33
28	2,158,716	15,330,708	15,092,102	238,606	9,764,217	32
29	2,161,556	15,317,560	15,078,310	239,250	9,763,589	31
30	2,164,396	15,304,430	15,064,535	239,895	9,762,960	30

Gr. 12

12 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	2,164,396	15,304,430	15,064,535	239,895	9,762,960	30
31	2,167,236	15,291,319	15,050,779	240,540	9,762,330	29
32	2,170,076	15,278,226	15,037,040	241,186	9,761,699	28
33	2,172,916	15,265,150	15,023,317	241,833	9,761,067	27
34	2,175.755	15,252,092	15,009,611	242,481	9,760,435	26
35	2,178,594	15,239,052	14,995,922	243,130	9,759,80	25
36	2,181,433	15,226,030	14,982,250	243,780	9,759,168	24
37	2,184,272	15,213,025	14,968,594	244,431	9,758,533	23
38	2,187,111	15,200,038	14,954,955	245,083	9,757,897	22
39	2,189,949	15,187,068	14,941,333	245,735	9,757,260	21
40	2,192,787	15,174,116	14,927,728	246,388	9,756,623	20
41	2,195,625	15,161,182	14,914,140	247,042	9,755,985	19
42	2,198,463	15,148,266	14,900,569	247,697	9,755,346	18
43	2,201,300	15,135,367	14,887,014	248,353	9,754,706	17
44	2,204,137	15,122,485	14,873,475	249,010	9,754,065	16
45	2,206,974	15,109,621	14,859,953	249,668	9,753,423	15
46	2,209,811	15,096,774	14,846,447	250,327	9,752,781	14
47	2,212,648	15,083,944	14,832,957	250,987	9,752,138	13
48	2,215,485	15,071,132	14,819,485	251,647	9,751,494	12
49	2,218,322	15,058,337	14,806,029	252,308	9,750,849	11
50	2,221,158	15,045,559	14,792,589	252,970	9,750,203	10
51	2,223,994	15,032,799	14,779,166	253,633	9,749,557	9
52	2,226,830	15,020,056	14,765,759	254,297	9,748,910	8
53	2,229,666	15,007,330	14,752,368	254,962	9,748,262	7
54	2,232,502	14,994,620	14,738,992	255,628	9,747,613	6
55	2,235,337	14,981,927	14,725,632	256,295	9,746,963	5
56	2,238,172	14,969,251	14,712,288	256,963	9,746,312	4
57	2,241,007	14,956,592	14,698,960	257,632	9,745,660	3
58	2,243,842	14,943,950	14,685,649	258,301	9,745,008	2
59	2,246,677	14,931,325	14,672,354	258,971	9,744,355	1
60	2,249,511	14,918,717	14,659,075	259,642	9,743,700	0
				77		min. Gr. 77

Gr. 13

13

min	*Sinus*	*Logarithmi*	*Differentiæ* + | −	*Logarithmi*	*Sinus*	
0	2,249,511	14,918,717	14,659,075	259,642	9,743,700	60
1	2,252,345	14,906,126	14,645,812	260,314	9,743,045	59
2	2,255,179	14,893,551	14,632,564	260,987	9,742,389	58
3	2,258,013	14,880,993	14,619,332	261,661	9,741,733	57
4	2,260,847	14,868,452	14,606,116	262,336	9,741,076	56
5	2,263,680	14,855,927	14,592,916	263,011	9,740,418	55
6	2,266,513	14,843,419	14,579,732	263,687	9,739,759	54
7	2,269,346	14,830,928	14,566,564	264,364	9,739,099	53
8	2,272,179	14,818,453	14,553,411	265,042	9,738,439	52
9	2,275,012	14,805,995	14,540,274	265,721	9,737,778	51
10	2,277,844	14,793,553	14,527,152	266,401	9,737,116	50
11	2,280,676	14,781,128	14,514,046	267,082	9,736,453	49
12	2,283,508	14,768,719	14,500,955	267,764	9,735,789	48
13	2,286,340	14,756,325	14,487,878	268,447	9,735,124	47
14	2,289,172	14,743,947	14,474,817	269,130	9,734,459	46
15	2,292,004	14,731,585	14,461,771	269,814	9,733,793	45
16	2,294,835	14,719,239	14,448,740	270,499	9,733,126	44
17	2,297,666	14,706,909	14,435,724	271,185	9,732,458	43
18	2,300,497	14,694,595	14,422,723	271,872	9,731,789	42
19	2,303,328	14,682,297	14,409,737	272,560	9,731,120	41
20	2,306,159	14,670,015	14,396,766	273,249	9,730,450	40
21	2,308,949	14,657,749	14,383,810	273,939	9,729,779	39
22	2,311,819	14,645,498	14,370,868	274,630	9,729,107	38
23	2,314,649	14,633,263	14,357,941	275,322	9,728,434	37
24	2,317,479	14,621,044	14,345,029	276,015	9,727,760	36
25	2,320,309	14,608,841	14,332,132	276,709	9,727,085	35
26	2,323,138	14,596,654	14,319,250	277,404	9,726,409	34
27	2,325,967	14,584,483	14,306,384	278,099	9,725,733	33
28	2,328,799	14,572,328	14,293,533	278,795	9,725,056	32
29	2,331,625	14,560,189	14,280,697	279,492	9,724,378	31
30	2,334,454	14,548,066	14,267,876	280,190	9,723,699	30

Gr. 13

13 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
30	2,334,454	14,548,066	14,267,876	280,190	9,723,699	30
31	2,337,282	14,535,958	14,255,069	280,889	9,723,020	29
32	2,340,110	14,523,866	14,242,277	281,589	9,722,340	28
33	2,342,938	14,511,789	14,229,500	282,289	9,721,659	27
34	2,345,766	14,499,727	14,216,737	282,990	9,720,977	26
35	2,348,594	14,487,680	14,203,988	283,692	9,720,294	25
36	2,351,421	14,475,648	14,191,253	284,395	9,719,610	24
37	2,354,248	14,463,632	14,178,533	285,099	9,718,926	23
38	2,357,075	14,451,631	14,165,827	285,804	9,718,241	22
39	2,359,902	14,439,645	14,153,135	286,510	9,717,555	21
40	2,362,729	14,427,674	14,140,457	287,217	9,716,868	20
41	2,365,555	14,415,718	14,127,793	287,925	9,716,180	19
42	2,368,381	14,403,777	14,115,143	288,634	9,715,491	18
43	2,371,207	14,391,851	14,102,507	289,344	9,714,802	17
44	2,374,033	14,379,941	14,089,887	290,054	9,714,112	16
45	2,376,859	14,368,046	14,077,281	290,765	9,713,421	15
46	2,379,684	14,356,166	14,064,689	291,477	9,712,729	14
47	2,382,509	14,344,301	14,052,111	292,190	9,712,036	13
48	2,385,334	14,332,451	14,039,547	292,904	9,711,343	12
49	2,388,159	14,320,616	14,026,997	293,619	9,710,649	11
50	2,390,983	14,308,796	14,014,461	294,335	9,709,954	10
51	2,393,808	14,296,991	14,001,939	295,052	9,709,258	9
52	2,396,632	14,285,200	13,989,430	295,770	9,708,561	8
53	2,399,456	14,273,424	13,976,935	296,489	9,707,863	7
54	2,402,280	14,261,662	13,964,453	297,209	9,707,165	6
55	2,405,104	14,249,915	13,951,986	297,929	9,706,466	5
56	2,407,927	14,238,182	13,939,532	298,650	9,705,766	4
57	2,410,750	14,226,464	13,927,092	299,372	9,705,065	3
58	2,413,573	14,214,761	13,914,666	300,095	9,704,363	2
59	2,416,396	14,203,072	13,902,253	300,819	9,703,660	1
60	2,419,219	14,191,398	13,889,854	301,544	9,702,957	0
						min.
						Gr.
						76

Gr. 14

14 min	Sinus	Logarithmi	+ Differentiæ −	Logarithmi	Sinus	
0	2,419,219	14,191,398	13,889,854	301,544	9,702,957	60
1	2,422,041	14,179,738	13,877,468	302,270	9,702,253	59
2	2,424,863	14,168,092	13,865,095	302,997	9,701,548	58
3	2,427,685	14,156,461	13,852,737	303,724	9,700,842	57
4	2,430,507	14,144,844	13,840,392	304,452	9,700,135	56
5	2,433,329	14,133,242	13,828,061	305,181	9,699,428	55
6	2,436,150	14,121,654	13,815,743	305,911	9,698,720	54
7	2,438,971	14,110,081	13,803,439	306,642	9,698,011	53
8	2,441,792	14,098,522	13,791,148	307,374	9,697,301	52
9	2,444,613	14,086,977	13,778,870	308,107	9,696,590	51
10	2,447,434	14,075,447	13,766,606	308,841	9,695,879	50
11	2,450,254	14,063,931	13,754,355	309,576	9,695,167	49
12	2,453,074	14,052,429	13,742,117	310,312	9,694,454	48
13	2,455,894	14,040,940	13,729,891	311,049	9,693,740	47
14	2,458,714	14,029,465	13,717,679	311,786	9,693,025	46
15	2,461,533	14,018,004	13,705,480	312,524	9,692,309	45
16	2,464,352	14,006,557	13,693,294	313,263	9,691,593	44
17	2,467,171	13,995,124	13,681,121	314,003	9,690,876	43
18	2,469,990	13,983,705	13,668,961	314,744	9,690,158	42
19	2,472,809	13,972,300	13,656,814	315,486	9,689,439	41
20	2,475,628	13,960,909	13,644,680	316,229	9,688,719	40
21	2,478,446	13,949,532	13,632,559	316,973	9,687,998	39
22	2,481,264	13,938,168	13,620,450	317,718	9,687,277	38
23	2,484,082	13,926,818	13,608,354	318,464	9,686,555	37
24	2,486,900	13,915,482	13,596,272	319,210	9,685,832	36
25	2,489,717	13,904,159	13,584,202	319,957	9,685,108	35
26	2,492,534	13,892,850	13,572,145	320,705	9,684,383	34
27	2,495 351	13,881,554	13,560,100	321,454	9,683,657	33
28	2,498,168	13,870,272	13,548,068	322,204	9,682,931	32
29	2,500,984	13,859,004	13,536,049	322,955	9,682,204	31
30	2,503,800	13,847,749	13,524,042	323,707	9,681,476	30

Gr. 14

14 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	2,503,800	13,847,749	13,524,042	323,707	9,681,476	30
31	2,506,626	13,836,508	13,512,048	324,460	9,680,747	29
32	2,509,431	13,825,280	13,500,066	325,214	9,680,017	28
33	2,512,248	13,814,066	13,488,097	325,969	9,679,287	27
34	2,515,064	13,802,865	13,476,141	326,724	9,678,556	26
35	2,517,879	13,791,678	13,464,198	327,480	9,677,824	25
36	2,520,694	13,780,504	13,452,267	328,237	9,677,091	24
37	2,523,509	13,769,343	13,440,348	328,995	9,676,357	23
38	2,526,324	13,758,195	13,428,441	329,754	9,675,623	22
39	2,529,138	13,747,061	13,416,547	330,514	9,674,888	21
40	2,531,952	13,735,940	13,404,665	331,275	9,674,152	20
41	2,524,766	13,724,833	13,392,796	332,037	9,673,415	19
42	2,537,580	13,713,739	13,380,939	332,800	9,672,677	18
43	2,540,393	13,702,658	13,369,094	333,564	9,671,938	17
44	2,543,206	13,691,590	13,357,262	334,328	9,671,199	16
45	2,546,019	13,680,535	13,345,442	335,093	9,670,459	15
46	2,548,832	13,669,493	13,333,634	335,859	9,669,718	14
47	2,551,645	13,658,464	13,321,838	336,626	9,668,976	13
48	2,554,458	13,647,448	13,310,054	337,394	9,668,233	12
49	2,557,270	13,636,445	13,298,282	338,163	9,667,490	11
50	2,560,082	13,625,454	13,286,521	338,933	9,666,746	10
51	2,562,894	13,614,476	13,274,772	339,704	9,666,001	9
52	2,565,706	13,603,511	13,263,035	340,476	9,665,255	8
53	2,568,517	13,592,559	13,251,310	341,249	9,664,508	7
54	2,571,328	13,581,620	13,239,597	342,023	9,663,761	6
55	2,574,139	13,570,694	13,227,896	342,798	9,663,013	5
56	2,576,950	13,559,781	13,216,208	343,573	9,662,264	4
57	2,579,760	13,548,880	13,204,531	344,349	9,661,514	3
58	2,582,570	13,537,992	13,192,866	345,126	9,660,763	2
59	2,585,380	13,527,117	13,181,213	345,904	9,660,011	1
60	2,588,190	13,516,255	13,165,572	346,683	9,659,258	0
				75		min. Gr. 75

Gr. 15

15 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	2,588,190	13,516,255	13,165,572	346,683	9,650,268	60
1	2,591,000	13,505,406	13,157,943	347,463	9,658,505	59
2	2,5 3,809	13,494,570	13,146,326	348,244	9,657,751	58
3	2,596,618	13,483,746	13,134,720	349,026	9,646,996	57
4	2,599,427	13,472,934	13,123,126	349,808	9,656,240	56
5	2,602,236	13,462,135	13,111,544	350,591	9,655,484	55
6	2,605,045	13,451,348	13,099,973	351,375	9,654,727	54
7	2,607,853	13,440,573	13,088,413	352,160	9,653,969	53
8	2,610,661	13,429,810	13,076,864	352,946	9,653,210	52
9	2,613,469	13,419,060	13,065,327	353,733	9,652,450	51
10	2,616,277	13,408,322	13,053,801	354,521	9,651,689	50
11	2,619,084	13,397,596	13,042,286	355,310	9,650,927	49
12	2,621,891	13,386,883	13,030,783	356,100	9,650,165	48
13	2,624,698	13,376,182	13,019,291	356,891	9,649,402	47
14	2,627,505	13,365,493	13,007,810	357,683	9,648,638	46
15	2,630,312	13,354,817	12,996,341	358,476	9,647,873	45
16	2,633,118	13,344,153	12,984,883	359,270	9,647,108	44
17	2,635,924	13,333,502	12,973,438	360,064	9,646,342	43
18	2,638,730	13,322,863	12,962,004	360,859	9,645,575	42
19	2,641,536	13,312,237	12,950,582	361,655	9,644,807	41
20	2,644,342	13,301,623	12,939,171	362,452	9,644,038	40
21	2,647,147	13,291,022	12,927,772	363,250	9,643,268	39
22	2,649,952	13,280,432	12,916,383	364,049	9,642,498	38
23	2,652,757	13,269,854	12,905,005	364,849	9,641,727	37
24	2,655,562	13,259,288	12,893,638	365,650	9,640,955	36
25	2,658,366	13,248,734	12,882,282	366,452	9,640,182	35
26	2,661,170	13,238,191	12,870,936	367,255	9,639,408	34
27	2,663,974	13,227,660	12,859,601	368,059	9,638,633	33
28	2,666,777	13,217,141	12,848,278	368,863	9,637,858	32
29	2,669,580	13,206,633	12,836,965	369,668	9,637,082	31
30	2,672,383	13,196,137	12,825,663	370,474	9,636,305	30

Gr. 15

15 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	2,672,383	13,196,137	12,825,663	370,474	9,636,305	30
31	2,675,186	13,185,653	12,814,372	371,281	9,635,52	29
32	2,677,989	13,175,181	12,803,092	372,089	9,634,748	28
33	2,680,792	13,164,721	12,791,823	372,898	9,633,969	27
34	2,683,595	13,154,273	12,780,565	373,708	9,633,189	26
35	2,686,397	13,143,837	12,769,318	374,519	9,632,408	25
36	2,689,199	13,133,413	12,758,082	375,331	9,931,628	24
37	2,692,001	13,123,000	12,746,856	376,144	9,630,843	23
38	2,694,802	13,112,599	12,735,641	376,958	9,630,059	22
39	2,697,603	13,102,210	12,724,438	377,772	9,629,275	21
40	2,700,404	13,091,833	12,713,246	378,587	9,628,490	20
41	2,703,205	13,081,468	12,702,065	379,403	9,627,704	19
42	2,706,005	13,071,114	12,690,894	380,220	9,626,917	18
43	2,708,805	13,060,771	12,679,733	381,038	9,626,129	17
44	2,711,605	13,050,440	12,668,583	381,857	9,625,341	16
45	2,714,405	13,040,120	12,657,443	382,677	9,624,552	15
46	2,717,204	13,029,812	12,646,314	383,498	9,623,762	14
47	2,720,003	13,019,515	12,635,195	384,320	9,622,971	13
48	2,722,802	13,009,229	12,624,086	385,143	9,622,179	12
49	2,725,601	12,998,955	12,612,988	385,967	9,621,387	11
50	2,728,400	12,988,692	12,601,901	386,791	9,620,594	10
51	2,731,198	12,978,441	12,590,825	387,616	9,619,800	9
52	2,733,996	12,968,201	12,579,759	388,442	9,619,085	8
53	2,736,794	12,957,972	12,568,703	389,269	9,618,209	7
54	2,739,592	12,947,755	12,557,658	390,097	9,617,413	6
55	2,742,389	12,937,549	12,546,623	390,926	9,616,616	5
56	2,745,186	12,927,354	12,535,598	391,756	9,615,818	4
57	2,747,983	12,917,171	12,524,584	392,587	9,615,019	3
58	2,750,780	12,906,999	12,513,580	393,419	9,614,219	2
59	2,753,577	12,896,838	12,502,586	394,252	9,613,418	1
60	2,756,373	12,886,689	12,491,603	395,086	9,612,617	0
						min.
						Gr.
				74		74

Gr. 16

16

min	*Sinus*	*Logarithmi*	*Differentiæ*	*Logarithmi*	*Sinus*	
			+ | −			
0	2,756,373	12,886,689	12,491,603	395,086	9,612,617	60
1	2,759,169	12,876,551	12,480,630	395,921	9,611,815	59
2	2,761,965	12,866,423	12,469,667	396,756	9,611,012	58
3	2,764,761	12,856,306	12,458,714	397,592	9,610,208	57
4	2,767,556	12,846,200	12,447,771	398,429	9,609,403	56
5	2,770,351	12,836,105	12,436,838	399,267	9,608,598	55
6	2,773,146	12,826,021	12,425,915	400,106	9,607,792	54
7	2,775,941	12,815,948	12,415,002	400,946	9,606,985	53
8	2,778,735	12,805,886	12,404,099	401,787	9,606,177	52
9	2,781,529	12,795,835	12,393,206	402,629	9,605,368	51
10	2,784,323	12,785,795	12,382,323	403,472	9,604,559	50
11	2,787,117	12,775,766	12,371,450	404,316	9,603,749	49
12	2,789,911	12,765,748	12,360,587	405,161	9,602,938	48
13	2,792,704	12,755,741	12,349,734	406,007	9,602,126	47
14	2,795,497	12,745,745	12,338,891	406,854	9,601,313	46
15	2,798,290	12,735,760	12,328,059	407,701	9,600,499	45
16	2,801,082	12,725,785	12,317,236	408,549	9,599,685	44
17	2,803,874	12,715,821	12,306,423	409,398	9,598,870	43
18	2,806,666	12,705,868	12,295,620	410,248	9,598,054	42
19	2,809,458	12,695,926	12,284,827	411,099	9,597,237	41
20	2,812,250	12,685,995	12,274,044	411,951	9,596,419	40
21	2,815,041	12,676,075	12,263,271	412,804	9,595,600	39
22	2,817,832	12,666,166	12,252,508	413,658	9,594,781	38
23	2,820,623	12,656,267	12,241,754	414,513	9,593,961	37
24	2,823,414	12,646,379	12,231,010	415,369	9,593,140	36
25	2,826,244	12,636,501	12,220,275	416,226	9,592,318	35
26	2,828,994	12,626,633	12,209,550	417,083	9,591,495	34
27	2,831,784	12,616,776	12,197,835	417,941	9,590,672	33
28	2,834,574	12,606,929	12,188,129	418,800	9,589,848	32
29	2,837,364	12,597,093	12,177,433	419,660	9,589,023	31
30	2,840,153	12,587,267	12,166,746	420,521	9,588,197	30

Gr. 16

16 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
30	2,840,153	12,587,267	12,166,746	420,521	9,588,197	30
31	2,842,942	12,577,452	12,156,069	421,383	9,587,371	29
32	2,845,731	12,567,647	12,145,401	422,246	9,586,544	28
33	2,848,520	12,557,853	12,134,743	423,110	9,585,716	27
34	2,851,308	12,548,069	12,124,094	423,975	9,584,887	26
35	2,854,096	12,538,296	12,113,455	424,841	9,584,057	25
36	2,856,884	12,528,533	12,102,825	425,708	9,583,226	24
37	2,859,672	12,518,780	12,092,204	426,576	9,582,395	23
38	2,862,459	12,509,038	12,081,593	427,445	9,581,563	22
39	2,865,246	12,499,306	12,070,992	428,314	9,580,730	21
40	2,868,033	12,489,585	12,060,401	429,184	9,579,896	20
41	2,870,819	12,479,874	12,049,819	430,055	9,579,061	19
42	2,873,605	12,470,174	12,039,247	430,927	9,578,225	18
43	2,876,391	12,460,484	12,028,684	431,800	9,577,389	17
44	2,879,177	12,450,804	12,018,130	432,674	9,576,552	16
45	2,881,963	12,441,134	12,007,585	433,549	9,575,714	15
46	2,884,748	12,431,474	11,997,049	434,425	9,574,875	14
47	2,887,533	12,421,824	11,986,522	435,302	9,574,036	13
48	2,890,318	12,412,184	11,976,004	436,180	9,573,196	12
49	2,893,103	12,402,554	11,965,495	437,059	9,572,355	11
50	2,895,888	12,392,934	11,954,996	437,938	9,571,513	10
51	2,898,672	12,383,324	11,944,506	438,818	9,570,670	9
52	2,901,456	12,373,724	11,934,025	439,699	9,569,826	8
53	2,904,240	12,364,134	11,923,553	440,581	9,568,982	7
54	2,907,023	12,354,554	11,913,090	441,464	9,568,137	6
55	2,909,806	12,344,984	11,902,636	442,348	9,567,291	5
56	2,912,589	12,335,425	11,892,192	443,233	9,566,444	4
57	2,915,371	12,325,876	11,881,757	444,119	9,565,596	3
58	2,918,153	12,316,337	11,871,330	445,007	9,564,747	2
59	2,920,935	12,306,808	11,860,912	445,896	9,563,898	1
60	2,923,717	12,297,289	11,850,503	446,786	9,563,048	0

min. / *Gr.*

73 | 37

Gr. 17

17 min	Sinus	Logarithmi	Differentiæ	+ | − Logarithmi	Sinus	
0	2,923,717	12 297,289	11,850,503	446,786	9,563,048	60
1	2,926,499	12,287,780	11,840,104	447,676	9,562,197	59
2	2,929,280	12,278,280	11,829,713	448,567	9,561,345	58
3	2,932,061	12,268,790	11,819,331	449,459	9,560,492	57
4	2,934,842	12,259,310	11,808,958	450,352	9,559,639	56
5	2,937,623	12,249,840	11,798,594	451,246	9,558,785	55
6	2,940,403	12,240,379	11,788,239	452,140	9,557,930	54
7	2,943,183	12,230,928	11,777,893	453,035	9,557,074	53
8	2,945,963	12,221,487	11,767,556	453,931	9,556,217	52
9	2,948,743	12,212,056	11,757,228	454,828	9,555,360	51
10	2,951,523	12,202,634	11,746,908	455,726	9,554,502	50
11	2,954,302	12,193,222	11,736,597	456,625	9,553,643	49
12	2,957,081	12,183,820	11,726,295	457,525	9,552,783	48
13	2,959,860	12,174,427	11,716,001	458,426	9,551,922	47
14	2,962,630	12,165,044	11,705,716	459,328	9,551,061	46
15	2,965,416	12,155,671	11,695,440	460,231	9,550,199	45
16	2,968,194	12,146,308	11,685,173	461,135	9,549,336	44
17	2,970,972	12,136,954	11,674,914	462,040	9,548,472	43
18	2,973,750	12,127,610	11,664,665	462,945	9,547,607	42
19	2,976,527	12,118,276	11,654,425	463,851	9,546,742	41
20	2,979,304	12,108,952	11,644,194	464,758	9,545,876	40
21	2,982,081	12,099,637	11,633,971	465,666	9,545,009	39
22	2,984,857	12,090,332	11,623,757	466,575	9,544,141	38
23	2,987,633	12,081,036	11,613,551	467,485	9,543,272	37
24	2,990,409	12,071,749	11,603,353	468,396	9,542,403	36
25	2,993,185	12,062,472	11,593,164	469,308	9,541,533	35
26	2,995,960	12,053,204	11,582,983	470,221	9,540,662	34
27	2,998,735	12,043,945	11,572,810	471,135	9,539,790	33
28	3,001,510	12,034,696	11,562,646	472,050	9,538,917	32
29	3,004,284	12,025,456	11,552,490	472,966	9,538,043	31
30	3,007,058	12,016,225	11,542,341	473,884	9,537,169	30

Gr. 17

17 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	3,007,058	12,016,225	11,542,341	473,884	9,537,169	30
31	3,009,832	12,007,004	11,532,202	474,802	9,536,294	29
32	3,012,606	11,997,792	11,522,071	475,721	9,535,418	28
33	3,015,380	11,988,589	11,511,948	476,641	9,534,541	27
34	3,018,153	11,979,396	11,501,835	477,561	9,533,664	26
35	3,020,926	11,970,212	11,491,730	478,482	9,532,786	25
36	3,023,699	11,961,037	11,481,633	479,404	9,531,907	24
37	3,026,472	11,951,872	11,471,545	480,327	9,531,027	23
38	3,029,244	11,942,716	11,461,465	481,251	9,530,146	22
39	3,032,016	11,933,569	11,451,393	482,176	9,529,264	21
40	3,034,788	11,924,431	11,441,329	483,102	9,528,382	20
41	3,037,559	11,915,303	11,431,274	484,029	9,527,499	19
42	3,040,330	11,906,184	11,421,227	484,957	9,526,615	18
43	3,043,101	11,897,074	11,411,188	485,886	9,525,730	17
44	3,045,872	11,887,973	11,401,157	486,816	9,524,844	16
45	3,048,643	11,878,881	11,391,134	487,747	9,523,958	15
46	3,051,413	11,869,798	11,381,119	488,679	9,523,071	14
47	3,054,183	11,860,724	11,371,113	489,611	9,522,183	13
48	3,056,953	11,851,659	11,361,115	490,544	9,521,294	12
49	3,059,723	11,842,603	11,351,125	491,478	9,520,404	11
50	3,062,492	11,833,557	11,341,144	492,413	9,519,514	10
51	3,065,261	11,824,520	11,331,171	493,349	9,518,623	9
52	3,068,030	11,815,492	11,321,206	494,286	9,517,731	8
53	3,070,798	11,806,473	11,311,249	495,224	9,516,838	7
54	3,073,566	11,797,463	11,301,300	496,163	9,515,944	6
55	3,076,334	11,788,461	11,291,358	497,103	9,515,050	5
56	3,079,102	11,779,468	11,281,424	498,044	9,514,155	4
57	3,081,869	11,770,484	11,271,498	498,986	9,513,259	3
58	3,084,636	11,761,509	11,261,580	499,929	9,512,362	2
59	3,087,403	11,752,543	11,251,670	500,873	9,511,464	1
60	3,090,170	11,743,586	11,241,768	501,818	9,510,565	0
				72		min. Gr. 72

Gr. 18

18 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	3,090,170	11,743,586	11,241,768	501,818	9,510,565	60
1	3,092,936	11,734,638	11,231,874	502,764	9,509,666	59
2	3,095,702	11,725,699	11,221,988	503,711	9,508,766	58
3	3,098,468	11,716,768	11,212,109	504,659	9,507,865	57
4	3,101,234	11,707,846	11,202,239	505,607	9,506,963	56
5	3,103,999	11,698,933	11,192,377	506,556	9,506,061	55
6	3,106,764	11,690,029	11,182,523	507,506	9,505,158	54
7	3,109,529	11,681,133	11,172,676	508,457	9,504,254	53
8	3,112,294	11,672,246	11,162,837	509,409	9,503,349	52
9	3,115,058	11,663,368	11,153,006	510,362	9,502,443	51
10	3,117,822	11,654,499	11,143,183	511,316	9,501,536	50
11	3,120,586	11,645,638	11,133,367	512,271	9,500,629	49
12	3,123,349	11,636,786	11,123,559	513,227	9,499,721	48
13	3,126,112	11,627,943	11,113,759	514,184	9,498,812	47
14	3,128,875	11,619,109	11,103,967	515,142	9,497,902	46
15	3,131,638	11,610,283	11,094,182	516,101	9,496,991	45
16	3,134,400	11,601,466	11,084,405	517,061	9,496,080	44
17	3,137,162	11,592,658	11,074,637	518,021	9,495,168	43
18	3,139,924	11,583,858	11,064,876	518,982	9,494,255	42
19	3,142,686	11,575,067	11,055,123	519,944	9,493,341	41
20	3,145,448	11,566,285	11,045,378	520,907	9,492,427	40
21	3,148,209	11,557,511	11,035,640	521,871	9,491,512	39
22	3,150,970	11,548,746	11,025,910	522,836	9,490,596	38
23	3,153,731	11,539,989	11,016,178	523,802	9,489,679	37
24	3,156,491	11,531,240	11,006,471	524,769	9,488,761	36
25	3,159,251	11,522,500	10,996,763	525,737	9,487,842	35
26	3,162,011	11,513,768	10,987,062	526,706	9,486,923	34
27	3,164,770	11,505,045	10,977,369	527,676	9,486,003	33
28	3,167,529	11,496,330	10,967,683	528,647	9,485,082	32
29	3,170,288	11,487,624	10,958,004	529,620	9,484,160	31
30	3,173,047	11,478,926	10,948,332	530,594	9,483,237	30

Gr. 18

18 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	3,173,047	11,478,926	10,948,332	530,594	9,483,237	30
31	3,175,805	11,470,237	10,938,669	531,568	9,482,314	29
32	3,178,563	11,461,556	10,929,013	532,543	9,481,390	28
33	3,181,321	11,452,883	10,919,364	533,519	9,480,465	27
34	3,184,079	11,444,219	10,909,723	534,496	9,479,539	26
35	3,186,837	11,435,563	10,900,090	535,473	9,478,612	25
36	3,189,594	11,426,915	10,890,464	536,451	9,477,685	24
37	3,192,351	11,418,275	10,880,845	537,430	9,476,757	23
38	3,195,108	11,409,644	10,871,234	538,410	9,475,828	22
39	3,197,864	11,401,021	10,861,630	539,391	9,474,898	21
40	3,200,620	11,392,406	10,852,033	540,373	9,473,967	20
41	3,203,375	11,383,800	10,842,444	541,356	9,473,035	19
42	3,206,130	11,375,202	10,832,862	542,340	9,472,103	18
43	3,208,885	11,366,612	10,823,287	543,325	9,471,170	17
44	3,211,640	11,358,030	10,813,719	544,311	9,470,236	16
45	3,214,395	11,349,456	10,804,158	545,298	9,469,301	15
46	3,217,150	11,340,891	10,794,605	546,286	9,468,366	14
47	3,219,904	11,332,334	10,785,059	547,275	9,467,430	13
48	3,222,658	11,323,785	10,775,520	548,265	9,466,493	12
49	3,225,412	11,315,244	10,765,988	549,256	9,465,555	11
50	3,228,165	11,306,711	10,756,462	550,249	9,464,616	10
51	3,230,918	11,298,186	10,746,944	551,242	9,463,677	9
52	3,233,671	11,289,670	10,737,434	552,236	9,462,737	8
53	3,236,423	11,281,162	10,727,931	553,231	9,461,796	7
54	3,239,175	11,272,662	10,718,436	554,226	9,460,854	6
55	3,241,927	11,264,170	10,708,948	555,222	9,459,911	5
56	3,244,679	11,255,686	10,699,467	556,219	9,458,968	4
57	3,247,430	11,247,210	10,689,993	557,217	9,458,024	3
58	3,250,181	11,238,742	10,680,526	558,216	9,457,079	2
59	3,252,932	11,230,282	10,671,066	559,216	9,456,133	1
60	3,255,682	11,221,830	10,661,613	560,217	9,455,186	0
				71		mins. Gr. 71

Gr. 19

19 min	Sinus	Logarithmi	Differentiæ	Logarithmi	Sinus	
			+ | −			
0	3,255,682	11,221,830	10,661,613	560,217	9,455,186	60
1	3,258,432	11,213,386	10,652,167	561,219	9,454,239	59
2	3,261,182	11,204,950	10,642,728	562,222	9,453,291	58
3	3,263,931	11,196,522	10,633,296	563,226	9,452,342	57
4	3,266,681	11,188,102	10,623,871	564,231	9,451,392	56
5	3,269,430	11,179,690	10,614,453	565,237	9,450,441	55
6	3,272,179	11,171,286	10,605,042	566,244	9,449,490	54
7	3,274,927	11,162,889	10,595,637	567,252	9,448,538	53
8	3,277,675	11,154,500	10,586,239	568,261	9,447,585	52
9	3,280,423	11,146,119	10,576,849	569,270	9,446,631	51
10	3,283,171	11,137,746	10,567,466	570,280	9,445,676	50
11	3,285,918	11,129,381	10,558,090	571,291	9,444,720	49
12	3,288,665	11,121,024	10,548,721	572,303	9,443,764	48
13	3,291,412	11,112,675	10,539,359	573,316	9,442,807	47
14	3,294,159	11,104,334	10,530,004	574,330	9,441,849	46
15	3,296,906	11,096,000	10,520,655	575,345	9,440,890	45
16	3,299,652	11,087,674	10,511,313	576,361	9,439,931	44
17	3,302,398	11,079,356	10,501,977	577,379	9,438,971	43
18	3,305,144	11,071,046	10,492,648	578,398	9,438,010	42
19	3,307,889	11,062,744	10,483,326	579,418	9,437,048	41
20	3,310,634	11,054,449	10,474,010	580,439	9,436,085	40
21	3,313,379	11,046,162	10,464,702	581,460	9,435,122	39
22	3,316,123	11,037,883	10,455,401	582,482	9,434,158	38
23	3,318,867	11,029,612	10,446,107	583,505	9,433,193	37
24	3,321,611	11,021,348	10,436,819	584,529	9,432,227	36
25	3,324,355	11,013,092	10,427,538	585,554	9,431,260	35
26	3,327,098	11,004,843	10,418,263	586,580	9,430,292	34
27	3,329,841	10,996,602	10,408,995	587,607	9,429,325	33
28	3,332,585	10,988,368	10,399,733	588,635	9,428,356	32
29	3,335,327	10,980,142	10,390,479	589,663	9,427,386	31
30	3,338,069	10,971,923	10,381,231	590,692	9,426,415	30

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Gr. 19

19 ′ min	Sinus	Logarithmi	Differentiæ	Logarithmi	Sinus	
			− +	−		
30	3,338,069	10,971,923	10,381,231	590,692	9,426,415	30
31	3,340,811	10,963,712	10,371,990	591,722	9,425,444	29
32	3,343,553	10,955,509	10,362,756	592,753	9,424,472	28
33	3,346,294	10,947,313	10,353,528	593,785	9,423,499	27
34	3,349,035	10,939,125	10,344,307	594,818	9,422,525	26
35	3,351,776	10,930,944	10,335,092	595,852	9,421,550	25
36	3,354,516	10,922,771	10,325,884	596,887	9,420,575	24
37	3,357,256	10,914,606	10,316,682	597,924	9,419,599	23
38	3,359,996	10,906,448	10,307,486	598,962	9,418,622	22
39	3,362,736	10,898,298	10,298,297	600,001	9,417,644	21
40	3,365,475	10,890,156	10,289,115	601,041	9,416,665	20
41	3,368,214	10,882,021	10,279,940	602,081	9,415,685	19
42	3,370,953	10,873,894	10,270,772	603,122	9,414,705	18
43	3,373,691	10,865,774	10,261,610	604,164	9,413,724	17
44	3,376,429	10,857,661	10,252,454	605,207	9,412,742	16
45	3,379,167	10,849,555	10,243,304	606,251	9,411,760	15
46	3,381,905	10,841,457	10,234,161	607,296	9,410,777	14
47	3,384,642	10,833,366	10,225,024	608,342	9,409,793	13
48	3,387,379	10,825,282	10,215,893	609,389	9,408,808	12
49	3,390,116	10,817,206	10,206,770	610,436	9,407,822	11
50	3,392,852	10,809,137	10,197,653	611,484	9,406,836	10
51	3,395,588	10,801,075	10 188,542	612,533	9,405,849	9
52	3,398,324	10,793,021	10,179,438	613,583	9,404,861	8
53	3,401,060	10,784,974	10,170,340	614,634	9,403,872	7
54	3,403,795	10,776,934	10,161,248	615,686	9,402,882	6
55	3,406,530	10,768,902	10,152,162	616,740	9,401,891	5
56	3,409,265	10,760,877	10,143,082	617,795	9,400,900	4
57	3,411,999	10,752,860	10,134,009	618,851	9,399,908	3
58	3,414,733	10,744,850	10,124,942	619,908	9,398,915	2
59	3,417,467	10,736,847	10,115,881	620,966	9,397,921	1
60	3,420,201	10,728,852	10,106,827	622,025	9,396,926	0
				70		min.
						Gr.
						70

Gr. 20

20 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	3,420,201	10,728,852	10,106,827	622,025	9,396,926	60
1	3,422,934	10,720,865	10,097,781	623,084	9,395,931	59
2	3,425,667	10,712,885	10,088,741	624,144	9,394,935	58
3	3,428,400	10,704,912	10,079,707	625,205	9,393,938	57
4	3 431,133	10,696,945	10,070,678	626,267	9,392,940	56
5	3,433,865	10,688,984	10,061,654	627,330	9,391,941	55
6	3,436,597	10,681,030	10,052,636	628,394	9,390,942	54
7	3,439,329	10,673,085	10,043,626	629,459	9,389,942	53
8	3,442,060	10,665,147	10,034,622	630,525	9,388,941	52
9	3,444,791	10,657,216	10,025,624	631,592	9,387,939	51
10	3,447,522	10,649,292	10,016,632	632,660	9,386,937	50
11	3,450,253	10,641,375	10 007,646	633,729	9,385,934	49
12	3,452,983	10,633,465	9,998,666	634,799	9,384,930	48
13	3,455,713	10,625,562	9,989,692	635,870	9,383,925	47
14	3,458,442	10,617,667	9,980,725	636,942	9,382,919	46
15	3,461,171	10,609,779	9,971,764	638,015	9,381,913	45
16	3,463,900	10,601,898	9,962,810	639,088	9,380,906	44
17	3,466,629	10,594,024	9,953,862	640,162	9 379,898	43
18	3,469,357	10,586,157	9,944,920	641,237	9,378,889	42
19	3,472,085	10,578,297	9,935,984	642,313	9,377,880	41
20	3,474,813	10,570,444	9,927,054	643,390	9,376,870	40
21	3,477,540	10,562,598	9,918,130	644,468	9,375,859	39
22	3,480,267	10,554,760	9,909,213	645,547	9,374,847	38
23	3,482,994	10,546,929	9,900,302	646,627	9,373,834	37
24	3,485,721	10,539,104	9,891,396	647,708	9,372,820	36
25	3,488,447	10,531,286	9,882,496	648,790	9,371,806	35
26	3,491,173	10,523,474	6,873,601	649,873	9,370,791	34
27	3,493,899	10,515,669	9,864,711	650,958	9,369,775	33
28	3,496,624	10,507,871	9,855,827	652,044	9,368,758	32
29	3,499,343	10,500,080	9,846,949	653,131	9,367,740	31
30	3,502,075	10,492,295	9,838,076	654,219	9,366,722	30

Gr. 20

20 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	3,502,075	10,492,295	9,838,076	654,219	9,366,722	30
31	3,504,799	10,484,516	9,829,209	655,307	9,365,703	29
32	3,507,523	10,476,745	9,820,349	656,396	9,364,683	28
33	3,510,247	10,468,981	9,811,495	657,486	9,363,662	27
34	3,512,971	10,461,225	9,802,648	658,577	9,362,640	26
35	3,515,694	10,453,476	9,793,807	659,669	9,361,618	25
36	3,518,417	10,445,734	9,784,972	660,762	9,360,595	24
37	3,521,140	10 437,999	9,776,143	661,856	9,359,571	23
38	3,523,862	10,430,271	9,767,320	662,951	9,358,546	22
39	3,526,584	10,422,550	9,758,503	664,047	9,357,521	21
40	3,529,306	10,414,836	9,749,693	665,143	9,356,495	20
41	3,532,027	10,407,129	9,740,889	666,240	9,355,468	19
42	3,534,748	10,399,429	9,732,091	667,338	9,354,440	18
43	3,537,469	10,391,735	9,723,298	668,437	9,353,411	17
44	3,540,190	10,384,047	9,714,510	669,537	9,352,382	16
45	3,542,910	10,376,366	9,705,728	670,638	9,351,352	15
46	3,545,630	10,368,692	9,696,951	671,741	9,350,321	14
47	3,548,350	10,361,024	9,688,179	672,845	9,349,289	13
48	3,551,070	10,353,362	9,679,412	673,950	9,348,257	12
49	3,553,789	10,345,706	9,670,650	675,056	9,347,224	11
50	3,556,508	10,338,057	9,661,894	676,163	9,346,190	10
51	3,559,227	10,330,415	9,653,144	677,271	9,345,155	9
52	3,561,945	10,322,780	9,644,400	678,380	9,344,119	8
53	3,564,663	10,315,152	9,635,662	679,490	9,343,082	7
54	3,567,380	10,307,531	9,626,930	680,601	9,342,045	6
55	3,570,097	10,299,916	9,618,204	681,712	9,341,007	5
56	3,572,814	10,292,308	9,609,484	682,824	9,339,968	4
57	3,575,531	10,284,707	9,600,770	683,937	9,338,928	3
58	3,578,247	10,277,113	9,592,062	685,051	9,337,887	2
59	3,580,963	10,269,526	9,583,360	686,166	9,336,846	1
60	3,583,679	10,261,946	9,574,664	687,282	9,335,804	0

min. Gr.

Gr. 21

21 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	3,583,679	10,261,946	9,574,664	687,282	9,335,804	60
1	3,586,395	10,254,372	9,565,973	688,399	9,334,761	59
2	3,589,110	10,246,804	9,557,287	689,517	9,333,717	58
3	3,591,825	10,239,243	9,548,607	690,636	9,332,673	57
4	3,594,540	10,231,688	9,539,932	691,756	9,331,628	56
5	3,597,254	10,224,140	9,531,263	692,877	9,330,582	55
6	3,599,968	10,216,598	9,522,599	693,999	9,329,535	54
7	3,602,682	10,209,063	9,513,941	695,122	9,328,488	53
8	3,605,395	10,201,534	9,505,288	696,246	9,327,440	52
9	3,608,108	10,194,012	9,496,642	697,370	9,326,391	51
10	3,610,821	10,186,496	9,488,001	698,495	9,325,341	50
11	3,613,533	10,178,987	9,479,366	699,621	9,324,290	49
12	3,616,245	10,171,484	9,470,736	700,748	9,323,238	48
13	3,618,957	10,163,988	9,462,111	701,877	9,322,186	47
14	3,621,669	10,156,498	9,453,491	703,007	9,321,133	46
15	3,624,380	10,149,015	9,444,877	704,138	9,320,079	45
16	3,627,091	10,141,538	9,436,268	705,270	9,319,024	44
17	3,629,802	10,134,067	9,427,664	706,403	9,317,969	43
18	3,632,512	10,126,603	9,419,066	707,537	9,316,913	42
19	3,635,222	10,119,145	9,410,473	708,672	9,315,856	41
20	3,637,932	10,111,694	9,401,886	709,808	9,314,798	40
21	3,640,642	10,104,249	9,393,305	710,944	9,313,739	39
22	3,643,351	10,096,811	9,384,730	712,081	9,312,680	38
23	3,646,060	10,089,379	9,376,160	713,219	9,311,620	37
24	3,648,768	10,081,953	9,367,595	714,358	9,310,559	36
25	3,651,476	10,074,533	9,359,035	715,498	9,309,497	35
26	3,654,184	10,067,120	9,350,481	716,639	9,308,434	34
27	3,656,892	10,059,713	9,341,931	717,781	9,307,371	33
28	3,659,599	10,052,312	9,333,386	718,926	9,306,307	32
29	3,662,306	10,044,918	9,324,847	720,071	9,305,242	31
30	3,665,012	10,037,530	9,316,313	721,217	9,304,176	30

Gr. 21

21 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	3,665,012	10,037,530	9,316,313	721,217	9,304,176	30
31	3,667,718	10,030,148	9,307,784	722,364	9,303,109	29
32	3,670,424	10,022,773	9,299,261	723,512	9,302,042	28
33	3,673,130	10,015,404	9,290,744	724,660	9,300,974	27
34	3,675,835	10,008,041	9,282,232	725,809	9,299,905	26
35	3,678,541	10,000,685	9,273,726	726,959	9,298,836	25
36	3,681,246	9,993,335	9,265,215	728,110	9,297,766	24
37	3,683,951	9,985,991	9,256,729	729,262	9,296,695	23
38	3,686,655	9,978,653	9,248,238	730,415	9,295,623	22
39	3,689,359	9,971,322	9,239,753	731,569	9,294,550	21
40	3,692,062	9,963,997	9,231,273	732,724	9,293,476	20
41	3,694,765	9,956,678	9,222,798	733,880	9,292,401	19
42	3,697,468	9,949,366	9,214,326	735,037	9,291,326	18
43	3,700,170	9,942,060	9,205,865	736,195	9,290,250	17
44	3,702,872	9,934,760	9,197,406	737,354	9,289,173	16
45	3,705,574	9,927,466	9,188,952	738,514	9,288,096	15
46	3,708,276	9,920,178	9,180,503	739,675	9,287,018	14
47	3,710,977	9,912,896	9,172,059	740,837	9,285,939	13
48	3,713,678	9,905,620	9,163,620	742,000	9,284,859	12
49	3,716,379	9,898,350	9,155,186	743,164	9,283,778	11
50	3,719,080	9,891,086	9,146,757	744,329	9,282,697	10
51	3,721,780	9,883,828	9,138,333	745,495	9,281,615	9
52	3,724,480	9,876,577	9,129,915	746,662	9,280,532	8
53	3,727,179	9,869,332	9,121,502	747,830	9,279,448	7
54	3,729,878	9,862,093	9,113,094	748,999	9,278,363	6
55	3,732,577	9,854,860	9,104,691	750,169	9,277,278	5
56	3,735,275	9,847,633	9,096,293	751,340	9,276,192	4
57	3,737,973	9,840,412	9,087,900	752,512	9,275,105	3
58	3,740,671	9,833,192	9,079,512	753,685	9,274,017	2
59	3,743,369	9,825,988	9,071,129	754,859	9,272,928	1
60	3,746,066	9,818,785	9,062,752	756,033	9,271,839	0

min. Gr.

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Gr. 22

22 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
0	3,746,066	9,818,785	9,062,752	756,033	9,271,839	60
1	3,748,763	9,811,589	9,054,381	757,208	9,270,749	59
2	3,751,460	9,804,399	9,046,015	758,384	9,269,658	58
3	3,754,156	9,797,215	9,037,654	759,561	9,268,566	57
4	3,756,852	9,790,036	9,029,296	760,740	9,267,474	56
5	3,759,548	9,782,863	9,020,943	761,920	9,266,381	55
6	3,762,243	9,775,696	9,012,595	763,101	9,265,287	54
7	3,764,938	9,768,53[illegible]	9,004,252	764,283	9,264,192	53
8	3,767,633	9,761,380	8,995,914	765,466	9,263,096	52
9	3,770,327	9,754,231	8,987,581	766,650	9,262,000	51
10	3,773,021	9,747,088	8,979,253	767,835	9,260,903	50
11	3,775,715	9,739,950	8,970,929	769,021	9,259,805	49
12	3,778,408	9,732,818	8,962,610	770,208	9,258,706	48
13	3,781,101	9,725,69[illegible]	8,954,297	771,396	9,257,606	47
14	3,783,794	9,718,574	8,945,989	772,585	9,256,506	46
15	3,786,486	9,711,461	8,937,686	773,775	9,255,405	45
16	3,789,178	9,704,354	8,929,388	774,966	9,254,303	44
17	3,791,870	9,697,353	8,921,196	776,157	9,253,200	43
18	3,794,562	9,690,158	8 912,809	777,349	9,252,097	42
19	3,797,253	9,683,069	8,904,527	778,542	9,250,993	41
20	3,799,944	9,675,986	8,896,250	779,736	9,249,888	40
21	3,802,635	9,668,908	8,887,977	780,931	9,248,782	39
22	3,805,325	9,661,836	8,879,709	782,127	9,247,676	38
23	3,808,015	9,654,770	8,871,446	783,324	9,246,569	37
24	3,810,704	9,647,709	8,863,18[illegible]	784,522	9,245,461	36
25	3,813,393	9,640,654	8,854,933	785,721	9,244,352	35
26	3,816,082	9,633,605	8,846,683	786,922	9,243,242	34
27	3,818,771	9,626,562	8,838,438	788,124	9,242,131	33
28	3,821,459	9,619,525	8,830,198	789,327	9,241,020	32
29	3,824,147	9,612,494	8,821,963	790,531	9,239,908	31
30	3,826,834	9,605,468	8,813,732	791,736	9,238,795	30

Gr. 22

22 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	3,826,834	9,605,468	8,813,732	791,736	9,238,795	30
31	3,829,521	9,598,448	8,805,506	792,942	9,237,682	29
32	3,832,208	9,591,434	8,797,285	794,149	9,236,568	28
33	3,834,895	9,584,426	8,789,069	795,357	9,235,453	27
34	3,837,581	9,577,424	8,780,859	796,565	9,234,337	26
35	3,840,267	9,570,427	8,772,653	797,774	9,233,220	25
36	3,842,953	9,563,436	8,764,452	798,984	9,232,103	24
37	3,845,638	9,556,451	8,756,256	800,195	9,230,985	23
38	3,848,323	9,549,472	8,748,065	801,407	9,229,866	22
39	3,851,008	9,542,498	8,739,878	802,620	9,228,746	21
40	3,853,692	9,535,530	8,731,696	803,834	9,227,625	20
41	3,856,376	9,528,567	8,723,518	805,049	9,226,504	19
42	3,859,060	9,521,610	8,715,345	806,265	9,225,382	18
43	3,861,743	9,514,659	8,707,177	807,482	9,224,259	17
44	3,864,426	9,507,713	8,699,013	808,700	9,223,135	16
45	3,867,109	9,500,773	8,690,854	809,919	9,222,010	15
46	3,869,791	9 493,839	8,682,700	811,139	9,220,884	14
47	3,872,473	9,486,911	8,674,551	812,360	9,219,758	13
48	3,875,155	9,479,988	8,666,405	813,583	9,218,631	12
49	3,877,837	9,473,071	8,658,264	814,807	9,217,504	11
50	3,880,518	9,466,160	8,650,128	816,032	9,216,376	10
51	3,883,199	9,459,254	8,641,996	817,258	9,215,247	9
52	3,885,880	9,452,354	8,633,870	818,484	9,214,117	8
53	3,888,560	9,445,460	8,625,749	819,711	9,212,986	7
54	3,891,240	9,438,571	8,617,632	820,939	9,211,855	6
55	3,893,919	9,431,688	8,609,520	822,168	9,210,723	5
56	3,896,598	9,424,810	8,601,412	823,398	9,209,590	4
57	3,899,277	9,417,938	8,593,309	824,629	9,208,456	3
58	3,901,955	9,411,071	8,585,210	825,861	9,207,321	2
59	3,904,633	9,404,210	8,577,116	827,094	9,206,185	1
60	3,907,311	9,397,354	8,569,026	828,328	9,205,048	0

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Gr. 23

23 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	3,907,311	9,397,354	8,569,026	828,328	9,205,049	60
1	3,909,989	9,390,504	8,560,941	829,563	9,203,912	59
2	3,912,666	9,383,660	8,552,861	830,799	9,202,774	58
3	3,915,343	9,376,821	8,544,785	832,036	9,201,635	57
4	3,918,020	9,369,988	8,536,714	833,274	9,200,496	56
5	3,920,696	9,363,160	8,528,647	834,513	9,199,356	55
6	3,923,372	9,356,337	8,520,584	835,753	9,198,215	54
7	3,926,048	9,349,520	8,512,525	836,995	9,197,073	53
8	3,928,723	9,342,708	8,504,470	838,238	9,195,931	52
9	3,931,398	9,335,902	8,496,420	839,482	9,194,788	51
10	3,934,072	9,329,101	8,488,374	840,727	9,193,644	50
11	3,936,746	9,322,306	8,480,333	841,973	9,192,499	49
12	3,939,420	9,315,516	8,472,296	843,220	9,191,353	48
13	3,942,093	9,308,731	8,464,263	844,468	9,190,207	47
14	3,944,766	9,301,952	8,456,236	845,716	9,189,060	46
15	3,947,439	9,295,178	8,448,213	846,965	9,187,912	45
16	3,950,112	9,288,410	8,440,195	848,215	9,186,763	44
17	3,952,784	9,281,647	8,432,181	849,466	9,185,614	43
18	3,955,456	9,274,890	8,424,172	850,718	9,184,464	42
19	3,958,128	9,268,138	8,416,167	851,971	9,183,313	41
20	3,960,799	9,261,392	8,408,167	853,225	9,182,161	40
21	3,963,470	9,254,651	8,400,171	854,480	9,181,009	39
22	3,966,140	9,247,915	8,392,179	855,736	9,179,856	38
23	3,968,810	9,241,185	8,384,192	856,993	9,178,702	37
24	3,971,480	9,234,460	8,376,209	858,251	9,177,547	36
25	3,974,149	9,227,741	8,368,231	859,510	9,176,391	35
26	3,976,818	9,221,027	8,360,257	860,770	9,175,235	34
27	3,979,487	9,214,319	8,352,288	862,031	9,174,078	33
28	3,982,155	9,207,616	8,344,322	863,294	9,172,920	32
29	3,984,823	9,200,918	8,336 360	864,558	9,171,761	31
30	3,987,491	9,194,226	8,328,403	865,823	9,170,601	30

Gr. 23

23 min	Sinus	Logarithmi	+ Differentiæ −	Logarithmi	Sinus	
30	3,987,491	9,194,226	8,328,403	865,823	9,170,601	30
31	3,990,159	9,187,539	8,320,450	867,089	9,169,440	29
32	3,992,826	9,180,857	8,312,501	868,356	9,168,279	28
33	3,995,493	9,174,181	8,304,558	869,623	9,167,117	27
34	3,998,159	9,167,510	8,296,619	870,891	9,165,955	26
35	4,000,825	9,160,844	8,288,684	872,160	9,164,792	25
36	4,003,491	9,154,183	8,280,753	873,430	9,163,628	24
37	4,006,156	9,147,528	8,272,827	874,701	9,162,463	23
38	4,008,821	9,140,878	8,264,905	875,973	9,161,297	22
39	4,011,486	9,134,233	8,256,987	877,246	9,160,131	21
40	4,014,150	9,127,593	8,249,073	878,520	9,158,964	20
41	4,016,814	9,120,959	8,241,164	879,795	9,157,796	19
42	4,019,478	9,114,330	8,233,259	881,071	9,156,627	18
43	4,022,141	9,107,706	8,225,358	882,348	9,155,457	17
44	4,024,804	9,101,087	8,217,461	883,626	9,154,286	16
45	4,026,467	9,094,473	8,209,568	884,905	9,153,115	15
46	4,030,130	9,087,865	8,201,679	886,186	9,151,943	14
47	4,032,792	9,081,262	8,193,794	887,468	9,150,770	13
48	4,035,454	9,074,664	8,185,913	888,751	9,149,597	12
49	4,038,115	9,068,071	8,178,036	890,035	9,148,423	11
50	4,040,776	9,061,483	8,170,163	891,320	9,147,248	10
51	4,043,437	9,054,901	8,162,295	892,606	9,146,072	9
52	4,046,097	9,048,324	8,154,431	893,893	9,144,895	8
53	4,048,757	9,041,752	8,146,571	895,181	9,143,717	7
54	4,051,416	9,035,185	8,138,715	896,470	9,142,540	6
55	4,054,075	9,028,623	8,130,863	897,760	9,141,361	5
56	4,056,734	9,022,066	8,123,015	899,051	9,140,181	4
57	4,059,392	9,015,514	8,115,172	900,342	9,139,001	3
58	4,062,050	9,008,968	8,107,334	901,634	9,137,820	2
59	4,064,708	9,002,427	8,099,500	902,927	9,136,638	1
60	4,067,366	8,995,891	8,091,670	904,221	9,135,455	0

min. Gr.

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Gr. 24

24 min	Sinus	Logarithmi	Differentiæ	Logarithmi	Sinus	
			+ | −			
0	4,067,366	8,995,891	8,091,670	904,221	9,135,455	60
1	4,070,023	8,989,360	8,083,844	905,516	9,134,271	59
2	4,072,680	8,982,834	8,076,022	906,812	9,133,087	58
3	4,075,337	8,976,313	8,068,204	908,109	9,131,902	57
4	4,077,993	8,969,797	8,060,389	909,408	9,130,716	56
5	4,080,649	8,963,286	8,052,578	910,708	9,129,529	55
6	4,083,305	8,956,780	8,044,771	912,009	9,128,342	54
7	4,085,960	8,950,280	8,036,969	913,311	9,127,154	53
8	4,088,615	8,943,785	8,029,171	914,614	9,125,965	52
9	4,091,269	8,937,295	8,021,377	915,918	9,124,775	51
10	4,093,923	8,930,810	8,013,587	917,223	9,123,584	50
11	4,096,577	8,924,330	8,005,801	918,529	9,122,392	49
12	4,099,231	8,917,855	7,998,019	919,836	9,121,200	48
13	4,101,884	8,911,385	7,990,241	921,144	9,120,007	47
14	4,104,537	8,904,920	7,982,467	922,453	9,118,814	46
15	4,107,189	8,898,460	7,974,697	923,763	9,117,620	45
16	4,109,841	8,892,005	7,966,931	925,074	9,116,425	44
17	4,112,493	8,885,555	7,959,169	926,386	9,115,229	43
18	4,115,144	8,879,110	7,951,411	927,699	9,114,032	42
19	4,117,795	8,872,670	7,943,657	929,013	9,112,835	41
20	4,120,446	8,866,235	7,935,908	930,327	9,111,637	40
21	4,123,096	8,859,804	7,928,161	931,643	9,110,438	39
[illegible]	4,125,746	8,853,379	7,920,419	932,960	9,109,238	38
23	4,128,395	8,846,959	7,912,681	934,278	9,108,038	37
24	4,131,044	8,840,544	7,904,947	935,597	9,106,837	36
25	4,133,693	8,834,134	7,897,217	936,917	9,105,635	35
26	4,136,341	8,827,729	7,889,491	938,238	9,104,432	34
27	4,138,989	8,821,329	7,881,769	939,560	9,103,228	33
28	4,141,637	8,814,934	7,874,051	940,883	9,102,024	32
29	4,144,285	8,808,544	7,866,337	942,207	9,100,819	31
30	4,146,932	8,802,159	7,858,627	943,532	9,099,613	30

Gr. 24

24 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
30	4,146,932	8,802,159	7,858,627	943,532	9,099,613	30
31	4,149,579	8,795,779	7,850,92	944,858	9,098,406	29
32	4,152,226	8,789,404	7,843,219	946,185	9,097,198	28
33	4,154,872	8,783,033	7,835,520	947,513	9,095,990	27
34	4,157,518	8,776,667	7,827,825	948,842	9,094,781	26
35	4,160,163	8,770,306	7,820,134	950,172	9,093,572	25
36	4,162,808	8,763,959	7,812,456	951,503	9,092,362	24
37	4,165,453	8,757,599	7,804,764	952,835	9,091,151	23
38	4,168,097	8,751,253	7,797,085	954,168	9,089,939	22
39	4,170,741	8,744,912	7,789,409	955,503	9,088,726	21
40	4,173,385	8,738,575	7,781,736	956,839	9,087,512	20
41	4,176,028	8,732,243	7,774,067	958,176	9,086,297	19
42	4,178,671	8,725,916	7,766,402	959,514	9,085,082	18
43	4,181,313	8,719,594	7,758,741	960,853	9,083,866	17
44	4,183,955	8,713,277	7,751,084	962,193	9,082,649	16
45	4,186,597	8,706,965	7,743,431	963,534	9,081,432	15
46	4,189,239	8,700,657	7,735,782	964,875	9,080,214	14
47	4,191,880	8,694,354	7,728,137	966,217	9,078,995	13
48	4,194,521	8,688,056	7,720,496	967,560	9,077,775	12
49	4,197,162	8,681,763	7,712,859	968,904	9,076,555	11
50	4,199,802	8,675,475	7,705,226	970,249	9,075,334	10
51	4,202,442	8,669,192	7,697,597	971,595	9,074,112	9
52	4,205,081	8,662,913	7,689,970	972,943	9,072,889	8
53	4,207,720	8,656,639	7,682,347	974,292	9,071,665	7
54	4,210,359	8,650,370	7,674,728	975,642	9,070,441	6
55	4,212,997	8,644,106	7,667,113	976,993	9,069,216	5
56	4,215,635	8,637,846	7,659,501	978,345	9,067,990	4
57	4,218,273	8,631,591	7,651,893	979,698	9,066,763	3
58	4,220,910	8,625,341	7,644,289	981,052	9,065,535	2
59	4,223,547	8,619,096	7,636,689	982,407	9,064,307	1
60	4,226,183	8,612,856	7,629,093	983,763	9,063,078	0

min.
Gr.

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Gr. 25

25 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	4,226,183	8,612,856	7,629,093	983,763	9,063,078	60
1	4,228,819	8,606,620	7,621,500	985,120	9,061,848	59
2	4,231,455	8,600,389	7,613,911	986,478	9,060,618	58
3	4,234,090	8,594,163	7,606,326	987,837	9,059,387	57
4	4,236,725	8,587,942	7,598,745	989,197	9,058,155	56
5	4,239,360	8,581,725	7,591,167	990,558	9,056,922	55
6	4,241,994	8,575,513	7,583,593	991,920	9,055,688	54
7	4,244,628	8,569,306	7,576,023	993,283	9,054,454	53
8	4,247,262	8,563,103	7,568,456	994,647	9,053,219	52
9	4,249,895	8,556,905	7,560,893	996,012	9,051,983	51
10	4,252,528	8,550,712	7,553,333	997,379	9,050,746	50
11	4,255,161	8,544,523	7,545,776	998,747	9,049,508	49
12	4,257,793	8,538,339	7,538,223	1,000,116	9,048,270	48
13	4,260,425	8,532,160	7,530,674	1,001,486	9,047,031	47
14	4,263,056	8,525,985	7,523,129	1,002,856	9,045,791	46
15	4,265,687	8,519,815	7,515,588	1,004,227	9,044,551	45
16	4,268,318	8,513,650	7,508,051	1,005,599	9,043,310	44
17	4,270,949	8,507,489	7,500,517	1,006,972	9,042,068	43
18	4,273,579	8,501,333	7,492,987	1,008,346	9,040,825	42
19	4,276,209	8,495,181	7,485,460	1,009,721	9,039,582	41
20	4,278,838	8,489,034	7,477,937	1,011,097	9,038,338	40
21	4,281,467	8,482,892	7,470,418	1,012,474	9,037,093	39
22	4,284,096	8,476,754	7,462,902	1,013,852	9,035,847	38
23	4,286,724	8,470,621	7,455,389	1,015,232	9,034,600	37
24	4,289,352	8,464,493	7,447,880	1,016,613	9,033,353	36
25	4,291,979	8,458,369	7,440,374	1,017,995	9,032,105	35
26	4,294,606	8,452,250	7,432,872	1,019,378	9,030,856	34
27	4,297,233	8,446,135	7,425,373	1,020,762	9,029,606	33
28	4,299,859	8,440,025	7,417,878	1,022,147	9,028,356	32
29	4,302,485	8,433,919	7,410,386	1,023,533	9,027,105	31
30	4,305,111	8,427,818	7,402,898	1,024,920	9,025,853	30

Gr. 27

27 min	Sinus	Logarithmi	+ Differentiæ −	Logarithmi	Sinus	
30	4,617,486	7,727,344	6,528,363	1,198,981	8,870,108	30
31	4,620,066	7,721,757	6,521,261	1,200,496	8,868,765	29
32	4,622,646	7,716,174	6,514,162	1,202,012	8,867,421	28
33	4,625,225	7,710,596	6,507,067	1,203,529	8,866,076	27
34	4,627,804	7,705,022	6,499,975	1,205,047	8,864,730	26
35	4,630,382	7,699,452	6,492,886	1,206,566	8,863,383	25
36	4,632,960	7,693,886	6,485,800	1,208,086	8,862,035	24
37	4,635,538	7,688,324	6,478,717	1,209,607	8,860,687	23
38	4,638,115	7,682,766	6,471,637	1,211,129	8,859,338	22
39	4,640,692	7,677,212	6,464,560	1,212,652	8,857,989	21
40	4,643,268	7,671,662	6,457,485	1,214,177	8,856,639	20
41	4,645,844	7,666,116	6,450,413	1,215,703	8,855,288	19
42	4,648,420	7,660,574	6,443,344	1,217,230	8,852,936	18
43	4,650,995	7,655,035	6,436,277	1,218,758	8,852,583	17
44	4,653,570	7,649,500	6,429,213	1,220,287	8,851,230	16
45	4,656,145	7,643,969	6,422,152	1,221,817	8,849,876	15
46	4,658,719	7,638,442	6,415,094	1,223,348	8,848,521	14
47	4,661,293	7,632,919	6,408,039	1,224,880	8,847,165	13
48	4,663,866	7,627,400	6,400,987	1,226,413	8,845,809	12
49	4,666,439	7,621,885	6,393,938	1,227,947	8,844,452	11
50	4,669,012	7,616,374	6,386,893	1,229,481	8,843,095	10
51	4,671,584	7,610,867	6,379,850	1,231,017	8,841,737	9
52	4,674,156	7,605,363	6,372,809	1,232,554	8,840,378	8
53	4,676,727	7,599,863	6,365,771	1,234,092	8,839,018	7
54	4,679,298	7,594,367	6,358,735	1,235,632	8,837,657	6
55	4,681,869	7,588,875	6,351,702	1,237,173	8,836,295	5
56	4,684,439	7,583,387	6,344,672	1,238,715	8,834,932	4
57	4,687,009	7,577,903	6,337,645	1,240,258	8,833,569	3
58	4,689,578	7,572,422	6 330,620	1,241,802	8,832,205	2
59	4,692,147	7,566,945	6,323,598	1,243,347	8,830,841	1
60	4,694,716	7,561,472	6,316,578	1,244,894	8,829,476	0
						min.
						Gr.
					62	62

Gr. 28

28 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	4,694,716	7,561,472	6,316,578	1,244,894	8,829,476	60
1	4,697,284	7,556,003	6,309,561	1,246,442	8,828,110	59
2	4,699,852	7,550,538	6,302,547	1,247,991	8,826,743	58
3	4,702,419	7,545,076	6,295,535	1,249,541	8,825,375	57
4	4,704,986	7,539,618	6,288,526	1,251,092	8,824,007	56
5	4,707,553	7,534,164	6,281,520	1,252,644	8,822,638	55
6	4,710,119	7,528,714	6,274,517	1,254,197	8,821,268	54
7	4,712,685	7,523,268	6,267,517	1,255,751	8,819,898	53
8	4,715,250	7,517,826	6,260,521	1,257,305	8,818,527	52
9	4,717,815	7,512,388	6,253,528	1,258,860	8,817,155	51
10	4,720,380	7,506,954	6,246,538	1,260,416	8,815,783	50
11	4,722,944	7,501,524	6,239,550	1,261,974	8,814,408	49
12	4,725,508	7,496,097	6,232,564	1,263,533	8,813,034	48
13	4,728,071	7,490,674	6,225,581	1,265,093	8,811,659	47
14	4,730,634	7,485,255	6,218,601	1,266,654	8,810,285	46
15	4,733,197	7,479,840	6,211,624	1,268,216	8,808,907	45
16	4,735,759	7,474,428	6,204,649	1,269,779	8,807,530	44
17	4,738,321	7,469,020	6,197,676	1,271,344	8,806,152	43
18	4,740,882	7,463,616	6,190,706	1,272,910	8,804,773	42
19	4,743,443	7,458,216	6,183,739	1,274,477	8,803,394	41
20	4,746,004	7,452,819	6,176,774	1,276,045	8,802,014	40
21	4,748,564	7,447,426	6,169,812	1,277,614	8,800,633	39
22	4,751,124	7,442,037	6,162,853	1,279,184	8,799,251	38
23	4,753,683	7,436,651	6,155,896	1,280,755	8,797,869	37
24	4,756,242	7,431,269	6,148,942	1,282,327	8,796,486	36
25	4,758,801	7,425,891	6,141,991	1,283,900	8,795,102	35
26	4,761,359	7,420,517	6,135,063	1,285,474	8,793,717	34
27	4,763,917	7,415,146	6,128,096	1,287,050	8,792,332	33
28	4,766,474	7,409,779	6,121,152	1,288,627	8,790,946	32
29	4,769,031	7,404,416	6,114,211	1,290,205	8,789,559	31
30	4,771,588	7,399,057	6,107,273	1,291,784	8,788,171	30

Gr. 28

28 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	4,771,588	7,399,057	6,107,273	1,291,784	8,788,171	30
31	4,774,144	7,393,701	6,100,337	1,293,364	8,786,782	29
32	4,776,700	7,388,349	6,093,404	1,294,945	8,785,393	28
33	4,779,255	7,383,001	6,086,474	1,296,527	8,784,003	27
34	4,781,810	7,377,657	6,079,547	1,298,110	8,782,613	26
35	4,784,365	7,372,316	6,072,622	1,299,694	8,781,222	25
36	4,786,919	7,366,979	6,065,700	1,301,279	8,779,830	24
37	4,789,473	7,361,646	6,058,781	1,302,865	8,778,437	23
38	4,792,026	7,356,316	6,051,863	1,304,453	8,777,044	22
39	4,794,579	7,350,990	6,044,948	1,306,042	8,775,650	21
40	4,797,132	7,345,668	6,038,036	1,307,632	8,774,255	20
41	4,799,684	7,340,349	6,031,126	1,309,223	8,772,859	19
42	4,802,236	7,335,034	6,024,219	1,310,815	8,771,462	18
43	4,804,787	7,329,723	6,017,315	1,312,408	8,770,065	17
44	4,807,338	7,324,415	6,010,413	1,314,002	8,768,667	16
45	4,809,888	7,319,111	6,003,514	1,315,597	8,767,267	15
46	4,812,438	7,313,811	5,996,618	1,317,193	8,765,868	14
47	4,814,988	7,308,514	5,989,723	1,318,791	8,764,468	13
48	4,817,537	7,303,221	5,982,831	1,320,390	8,763,068	12
49	4,820,086	7,297,931	5,975,941	1,321,990	8,761,665	11
50	4,822,635	7,292,645	5,969,054	1,323,591	8,760,263	10
51	4,825,183	7,287,363	5,962,170	1,325,193	8,758,860	9
52	4,827,731	7,282,084	5,955,288	1,326,796	8,757,456	8
53	4,830,278	7,276,809	5,948,409	1,328,400	8,756,051	7
54	4,832,825	7,271,538	5,941,533	1,330,005	8,754,646	6
55	4,835,371	7,266,270	5,934,659	1,331,611	8,753,240	5
56	4,837,917	7,261,006	5,927,787	1,333,219	8,751,833	4
57	4,840,462	7,255,746	5,920,918	1,334,828	8,750,425	3
58	4,843,007	7,250,489	5,914,051	1,336,438	8,749,016	2
59	4,845,552	7,245,236	5,907,187	1,338,049	8,741,607	1
60	4,848,096	7,239,987	5,900,326	1,339,661	8,746,197	0

min. Gr.

Gr. 29

29 min	Sinus	Logarithmi	Differentiæ	Logarithmi	Sinus	
			+ | −			
0	4,848,096	7,239,987	5,900,326	1,339,661	8,746,197	60
1	4,850,640	7,234,742	5,893,468	1,341,274	8,744,787	59
2	4,853,184	7,229,500	5,886,612	1,342,888	8,743,376	58
3	4,855,727	7,224,262	5,879,759	1,344,503	8,741,964	57
4	4,858,270	7,219,027	5,872,906	1,346,119	8,740,551	56
5	4,860,812	7,213,795	5,866,059	1,347,736	8,739,137	55
6	4,863,354	7,208,567	5,859,213	1,349,354	8,737,722	54
7	4,865,895	7,203,342	5,852,368	1,350,974	8,736,307	53
8	4,868,436	7,198,121	5,845,526	1,352,595	8,734,891	52
9	4,870,977	7,192,903	5,838,686	1,354,217	8,733,475	51
10	4,873,517	7,187,689	5,831,849	1,355,840	8,732,058	50
11	4,876,057	7,182,478	5,825,014	1,357,464	8,730,640	49
12	4,878,596	7,177,271	5,818,182	1,359,089	8,729,221	48
13	4,881,135	7,172,068	5,811,353	1,360,715	8,727,801	47
14	4,883,674	7,166,868	5,804,526	1,362,342	8,726,381	46
15	4,886,212	7,161,672	5,797,701	1,363,971	8,724,960	45
16	4,888,750	7,156,480	5,790,879	1,365,601	8,723,538	44
17	4,891,287	7,151,291	5,784,059	1,367,232	8,722,116	43
18	4,893,824	7,146,106	5,777,242	1,368,864	8,720,693	42
19	4,896,361	7,140,924	5,770,427	1,370,497	8,719,269	41
20	4,898,897	7,135,746	5,763,615	1,372,131	8,717,844	40
21	4,901,433	7,130,572	5,756,806	1,373,766	8,716,418	39
22	4,903,968	7,125,401	5,749,999	1,375,402	8,714,992	38
23	4,906,503	7,120,234	5,743,195	1,377,039	8,713,565	37
24	4,909,037	7,115,070	5,736,392	1,378,678	8,712,138	36
25	4,911,571	7,109,909	5,729,591	1,380,318	8,710,710	35
26	4,914,105	7,104,752	5,722,793	1,381,959	8,709,281	34
27	4,916,638	7,099,598	5,715,997	1,383,601	8,707,851	33
28	4,919,171	7,094,448	5,709,204	1,385,244	8,706,420	32
29	4,921,703	7,089,301	5,702,413	1,386,888	8,704,989	31
30	4,924,235	7,084,158	5,695,625	1,388,533	8,703,557	30

60

Gr. 29

29 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	4,924,235	7,084,158	5,695,625	1,388,533	8,703,557	30
31	4,926,767	7,079,018	5,688,839	1,390,179	8,702,124	29
32	4,929,298	7,073,882	5,682,056	1,391,826	8,700,691	28
33	4,931,829	7,068,749	5,675,275	1,393,474	8,699,257	27
34	4,934,359	7,063,620	5,668,496	1,395,124	8,697,822	26
35	4,936,889	7,058,494	5,661,719	1,396,775	8,696,386	25
36	4,939,418	7,053,372	5,654,945	1,398,427	8,694,949	24
37	4,941,947	7,048,253	5,648,173	1,400,080	8,693,512	23
38	4,944,476	7,043,138	5,641,404	1,401,734	8,692,074	22
39	4,947,004	7,038,026	5,634,637	1,403,389	8,690,636	21
40	4,949,532	7,032,918	5,627,873	1,405,045	8,689,197	20
41	4,952,059	7,027,814	5,621,111	1,406,703	8,687,757	19
42	4,954,586	7,022,713	5,614,351	1,408,362	8,686,316	18
43	4,957,113	7,017,615	5,607,593	1,410,022	8,684,873	17
44	4,959,639	7,012,521	5,600,838	1,411,683	8,683,431	16
45	4,962,165	7,007,430	5,594,085	1,413,345	8,681,988	15
46	4,964,690	7,002,342	5,587,334	1,415,008	8,680,544	14
47	4,967,215	6,997,258	5,580,586	1,416,672	8,679,100	13
48	4,969,740	6,992,177	5,573,840	1,418,337	8,677,655	12
49	4,972,264	6,987,099	5,567,095	1,420,004	8,676,209	11
50	4,974,788	6,982,025	5,560,353	1,421,672	8,674,762	10
51	4,977,311	6,976,954	5,553,613	1,423,341	8,673,314	9
52	4,979,834	6,971,886	5,546,875	1,425,011	8,671,866	8
53	4,982,356	6,966,822	5,540,140	1,426,682	8,670,417	7
54	4,984,878	6,961,761	5,533,407	1,428,354	8,668,968	6
55	4,987,399	6,956,704	5,526,677	1,430,027	8,667,518	5
56	4,989,920	6,951,650	5,519,949	1,431,701	8,666,067	4
57	4,992,441	6,946,600	5,513,224	1,433,376	8,664,615	3
58	4,994,961	6,941,553	5,506,500	1,435,053	8,663,162	2
59	4,997,481	6,936,509	5,499,778	1,436,731	8,661,708	1
60	5,000,000	6,931,469	5,493,059	1,438,410	8,660,254	0
						min.
						Gr.
			60			60

Gr. 30

30 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	5,000,000	6,931,469	5,493,059	1,438,410	8,660,254	60
1	5,002,519	6,926,432	5,486,342	1,440,090	8,658,799	59
2	5,005,038	6,921,399	5,479,628	1,441,771	8,657,344	58
3	5,007,556	6,916,369	5,472,916	1,443,453	8,655,888	57
4	5,010,074	6,911,342	5,466,206	1,445,136	8,654,431	56
5	5,012,591	6,906,319	5,459,498	1,446,821	8,652,973	55
6	5,015,108	6,901,299	5,452,792	1,448,507	8,651,514	54
7	5,017,624	6,896,282	5,446,088	1,450,194	8,650,055	53
8	5,020,140	6,891,269	5,439,387	1,451,882	8,648,595	52
9	5,022,656	6,886,259	5,432,688	1,453,571	8,647,134	51
10	5,025,171	6,881,253	5,425,992	1,455,261	8,645,673	50
11	5,027,686	6,876,250	5,419,298	1,456,952	8,644,211	49
12	5,030,200	6,871,250	5,412,605	1,458,645	8,642,748	48
13	5,032,714	6,866,254	5,405,915	1,460,339	8,641,284	47
14	5,035,227	6,861,261	5,399,227	1,462,034	8,639,820	46
15	5,037,740	6,856,271	5,392,541	1,463,730	8,638.355	45
16	5,040,253	6,851,285	5,385,858	1,465,427	8,636,889	44
17	5,042,765	6,846,302	5,379,177	1,467,125	8,635,423	43
18	5,045,277	6,841,323	5,372,499	1,468,824	8,633,956	42
19	5,047,788	6,836,347	5,365,822	1,470,525	8,632,488	41
20	5,050,299	6,831,374	5,359,147	1,472,227	8,631,019	40
21	5,052,809	6,826,405	5,352,475	1,473,930	8,629,549	39
22	5,055,319	6,821,439	5,345,805	1,475,634	8,628,079	38
23	5,057,829	6,816,476	5,339,137	1,477,339	8,626,608	37
24	5,060,338	6,811,516	5,332,471	1,479,045	8,625,137	36
25	5,062,847	6,806,560	5,325,808	1,480,752	8,623,665	35
26	5,065,355	6,801,607	5,319,147	1,482,460	8,622,192	34
27	5,067,863	6,796,657	5,312,488	1,484,169	8,620,718	33
28	5,070,370	6,791,710	5,305,831	1,485,879	8,619,243	32
29	5,072,877	6,786,767	5,299,177	1,487,590	8,617,768	31
30	5,075,384	6,781,827	5,292,525	1,489,302	8,616,292	30

Gr. 30

30 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	5,075,384	6,781,827	5,292,525	1,489,302	8,616,292	30
31	5,077,890	6,776,890	5,285,874	1,491,016	8,614,815	29
32	5,080,396	6,771,956	5,279,225	1,492,731	8,613,338	28
33	5,082,901	6,767,026	5,272,579	1,494,447	8,611,860	27
34	5,085,406	6,762,099	5,265,934	1,496,165	8,610,381	26
35	5,087,911	6,757,175	5,259,291	1,497,884	8,608,901	25
36	5,090,415	6,752,255	5,252,651	1,499,604	8,607,420	24
37	5,092,919	6,747,338	5,246,013	1,501,325	8,605,939	23
38	5,095,422	6,742,424	5,239,377	1,503,047	8,604,457	22
39	5,097,925	6,737,513	5,232,743	1,504,770	8,602,975	21
40	5,100,427	6,732,606	5,226,112	1,506,494	8,601,492	20
41	5,102,929	6,727,702	5,219,482	1,508,220	8,600,008	19
42	5,105,430	6,722,802	5,212,855	1,509,947	8,598,523	18
43	5,107,932	6,717,905	5,206,230	1,511,675	8,597,037	17
44	5,110,431	6,713,011	5,199,607	1,513,404	8,595,551	16
45	5,112,931	6,708,120	5,192,986	1,515,134	8,594,064	15
46	5,115,431	6,703,232	5,186,367	1,516,865	8,592,577	14
47	5,117,930	6,698,348	5,179,751	1,518,597	8,591,089	13
48	5,120,429	6,693,467	5,173,137	1,520,330	8,589,600	12
49	5,122,927	6,688,589	5,166,525	1,522,064	8,588,110	11
50	5,125,425	6,683,714	5,159,914	1,523,800	8,586,619	10
51	5,127,922	6,678,842	5,153,305	1,525,537	8,585,127	9
52	5,130,419	6,673,974	5,146,699	1,527,275	8,583,635	8
53	5,131,916	6,669,109	5,140,095	1,529,014	8,582,142	7
54	5,135,412	6,664,247	5,133,493	1,530,754	8,580,649	6
55	5,137,908	6,659,388	5,126,892	1,532,496	8,579,155	5
56	5,140,403	6,654,532	5,120,293	1,534,239	8,577,660	4
57	5,142,898	6,649,680	5,113,697	1,535,983	8,576,164	3
58	5,145,393	6,644,831	5,107,103	1,537,728	8,574,668	2
59	5,147,887	6,639,985	5,100,511	1,539,474	8,573,171	1
60	5,150,381	6,635,142	5,093,921	1,541,221	8,571,673	0
						min.
						Gr.

Gr. 31

31 min	Sinus	Logarithmi	+ | − Differentiæ	Logarithmi	Sinus	
0	5,150,381	6,635,142	5,093,921	1,541,221	8,571,673	60
1	5,152,874	6,630,302	5,087,332	1,542,970	8,570,175	59
2	5,155,367	6,625,465	5,080,745	1,544,720	8,568,676	58
3	5,157,859	6,620,631	5,074,160	1,546,471	8,567,176	57
4	5,160,351	6,615,801	5,067,578	1,548,223	8,565,675	56
5	5,162,843	6,610,974	5,060,998	1,549,976	8,564,173	55
6	5,165,334	6,606,150	5,054,420	1,551,730	8,562,671	54
7	5,167,825	6,601,329	5,047,844	1,553,485	8,561,168	53
8	5,170,315	6,596,512	5,041,271	1,555,241	8,559,664	52
9	5,172,805	6,591,698	5,034,700	1,556,998	8,558,160	51
10	5,175,294	6,586,887	5,028,130	1,558,757	8,556,655	50
11	5,177,783	6,582,079	5,021,562	1,560,517	8,555,149	49
12	5,180,271	6,577,275	5,014,997	1,562,278	8,553,643	48
13	5,182,759	6,572,474	5,008,434	1,564,040	8,552,136	47
14	5,185,246	6,567,676	5,001,873	1,565,803	8,550,628	46
15	5,187,733	6,562,881	4,995,313	1,567,568	8,549,119	45
16	5,190,220	6,558,089	4,988,755	1,569,334	8,547,609	44
17	5,192,706	6,553,300	4,982,199	1,571,101	8,546,096	43
18	5,195,192	6,548,514	4,975,645	1,572,869	8,544,588	42
19	5,197,667	6,543,731	4,969,093	1,574,638	8,543,077	41
20	5,200,162	6,538,951	4,962,543	1,576,408	8,541,565	40
21	5,202,646	6,534,174	4,955,994	1,578,180	8,540,052	39
22	5,205,130	6,529,400	4,949,447	1,579,953	8,538,538	38
23	5,207,614	6,524,629	4,942,902	1,581,727	8,537,024	37
24	5,210,097	6,519,862	4,936,360	1,583,502	8,535,509	36
25	5,212,580	6,515,098	4,929,820	1,585,278	8,533,993	35
26	5,215,062	6,510,337	4,923,282	1,587,055	8,532,476	34
27	5,217,544	6,505,580	4,916,747	1,588,833	8,530,958	33
28	5,220,025	6,500,826	4,910,213	1,590,613	8,529,440	32
29	5,222,506	6,496,075	4,903,681	1,592,394	8,527,921	31
30	5,224,986	6,491,327	4,897,151	1,594,176	8,526,402	30

Gr. 31

31 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
30	5,224,986	6,491,327	4,897,151	1,594,176	8,526,402	30
31	5,227,466	6,486,583	4,890,624	1.595,959	8,524,882	29
32	5,229,946	6,481,842	4,884.098	1,597,744	8,523,361	28
33	5,232,425	6,477,103	4,877,573	1,599,530	8,521,839	27
34	5,234,904	6,472,367	4,871,050	1,601,317	8,520,317	26
35	5,237,382	6,467,634	4,867,529	1,603,105	8,518,794	25
36	5,239,86	6,462,904	4,858,010	1,604,894	8,517,270	24
37	5,242,337	6,458,177	4,851,493	1,606,684	8,515,745	23
38	5,244,814	6,453,453	4,844,978	1,608,475	8,514,220	22
39	5,247,290	6,448,732	4,838,465	1,610,267	8,512,694	21
40	5,249,766	6,444 014	4,831,954	1,612,060	8,511,167	20
41	5,252,241	6,439,299	4,825,444	1,613,855	8,509,639	19
42	5,254,716	6,434,588	4,818,937	1,615,651	8,508,111	18
43	5,257,191	6,429,880	4,812,432	1,617,448	8,506,582	17
44	5,259,665	6,425,175	4,805,929	1,619,246	8,505,052	16
45	5,262,139	6,420,473	4,799,427	1,621,046	8,503,522	15
46	5,264,612	6,415,774	4,792,927	1,622,847	8,501,991	14
47	5,267,085	6,411 078	4,786,429	1,624,649	8,500,459	13
48	5,269,557	6,406,385	4,779,933	1,626,452	8,498,927	12
49	5,272,029	6,401,695	4,773,439	1,628,256	8,497,394	11
50	5,274,501	6,397,008	4,766,947	1,630,061	8,495,860	10
51	5,276,972	6,392,324	4,760,456	1,631,868	8,494,326	9
52	5,279,443	6,387,643	4,753,967	1,633,676	8,492,791	8
53	5,281,913	6,382,965	4,747,480	1,635,485	8,491,255	7
54	5,284,383	6,378,290	4,740,995	1,637,295	8,489,718	6
55	5,286,852	6,373,618	4,734,512	1,639,106	8,488,180	5
56	5,299,321	6,368,949	4,728,031	1,640,918	8,486,641	4
57	5,291,789	6,364,283	4,721,552	1,642,731	8,485,102	3
58	5,294,257	6,359,620	4,715,074	1,644,546	8,483,562	2
59	5,296,725	6,354,96	4,708,599	1,646,362	8,482,022	1
60	5,299,192	6,350,305	4,702,126	1,648,179	8,480,481	0

min. Gr.

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Gr. 32

32 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	5,299,192	6,350,305	4,702,126	1,648,179	8,480,481	60
1	5,301,659	6,345,652	4,695,655	1,649,997	8,478,939	59
2	5,304,125	6,341,002	4,689,186	1,651,816	8,477,297	58
3	5,306,591	6,336,354	4,682,717	1,653,637	8,475,854	57
4	5,309,056	6,331,709	4,676,240	1,655,459	8,474,310	56
5	5,311,521	6,327,067	4,669,785	1,657,282	8,472,765	55
6	5,313,985	6,322,428	4,663,322	1,659,106	8,471,219	54
7	5,316,449	6,317,792	4,656,861	1,660,931	8,469,673	53
8	5,318,913	6,313,159	4,650,402	1,662,757	8,468,126	52
9	5,321,376	6,308,529	4,643,944	1,664,585	8,466,579	51
10	5,323,839	6,303,902	4,637,488	1,666,414	8,465,031	50
11	5,326,301	6,299,278	4,631,034	1,668,244	8,463,482	49
12	5,328,763	6,294,657	4,624,582	1,670,075	8,461,932	48
13	5,331,224	6,290,039	4,618,131	1,671,908	8,460,381	47
14	5,333,685	6,285,424	4,611,682	1,673,742	8,458,830	46
15	5,336,145	6,280,812	4,605,235	1,675,577	8,457,278	45
16	5,338,605	6,276,203	4,598,790	1,677,413	8,455,725	44
17	5,341,065	6,271,597	4,592,347	1,679,250	8,454,172	43
18	5,343,524	6,266,994	4,585,906	1,681,088	8,452,618	42
19	5,345,983	6,262,394	4,579,467	1,682,927	8,451,064	41
20	5,348,441	6,257,797	4,573,030	1,684,767	8,449,509	40
21	5,350,898	6,253,203	4,566,594	1,686,609	8,447,953	39
22	5,353,355	6,248,612	4,560,160	1,688,452	8,446,396	38
23	5,355,812	6,244,024	4,553,728	1,690,296	8,444,838	37
24	5,358,268	6,239,439	4,547,298	1,692,141	8,443,280	36
25	5,360,724	6,234,857	4,540,859	1,693,988	8,441,721	35
26	5,363,179	6,230,278	4,534,442	1,695,836	8,440,161	34
27	5,365,634	6,225,702	4,528,017	1,697,685	8,438,600	33
28	5,368,088	6,221,129	4,521,594	1,699,535	8,437,039	32
29	5,370,542	6,216,559	4,515,172	1,701,387	8,435,477	31
30	5,372,996	6,211,992	4,508,752	1,703,240	8,423,915	30

Gr. 32

32 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	5,372,996	6,211,992	4,508,752	1,703,240	8,433,915	30
31	5,375,449	6,207,427	4,502,333	1,705,094	8,432,352	29
32	5,377,902	6,202,865	4,495,916	1,706,949	8,430,788	28
33	5,380,354	6,198,306	4,489,501	1,708,805	8,429,223	27
34	5,382,806	6,193,750	4,483,088	1,710,662	8,427,658	26
35	5,385,258	6,189,197	4,476,676	1,712,521	8,426,092	25
36	5,387,709	6,184,647	4,470,266	1,714,381	8,424,525	24
37	5,390,159	6,180,100	4,463,858	1,716,242	8,422,957	23
38	5,392,609	6,175,556	4,457,452	1,718,104	8,421,389	22
39	5,395,058	6,171,015	4,451,048	1,719,967	8,419,820	21
40	5,397,507	6,166,477	4,444,646	1,721,831	8,418,250	20
41	5,399,855	6,161,942	4,438,245	1,723,697	8,416,679	19
42	5,402,403	6,157,409	4,431,845	1,725,564	8,415,108	18
43	5,404,851	6,152,879	4,425,447	1,727,432	8,413,536	17
44	5,407,298	6,148,352	4,419,051	1,729,301	8,411,963	16
45	5,409,745	6,143,828	4,412,656	1,731,172	8,410,390	15
46	5,412,191	6,139,307	4,406,263	1,733.044	8,418,816	14
47	5,414,637	6,134,789	4,399,872	1,734,917	8,407,241	13
48	5,417,082	6,130,274	4,393,483	1,736,791	8,405,666	12
49	5,419,527	6,125,762	4,387,096	1,738,666	8,404,090	11
50	5,421,972	6,121,253	4,380,711	1,740,542	8,402,513	10
51	5,424,416	6,116,747	4,374,327	1,742,420	8,400,935	9
52	5,426,859	6,112,244	4,367,945	1,744,299	8,399,357	8
53	5,429,302	6,107,744	4,361,565	1,746,179	8,397,778	7
54	5,431,745	6,103,246	4,355,186	1,748,060	8,396,199	6
55	5,434,187	6,098,751	4,348,809	1,749,942	8,394,619	5
56	5,436,629	6,094,259	4,342,433	1,751,826	8,393,038	4
57	5,439,070	6,089,770	4,336,059	1,753,711	8,391,456	3
58	5,441,510	6,085,284	4,329,687	1,755,597	8,389,873	2
59	5,443,950	6,080,800	4,323,316	1,757,484	8,388,290	1
60	5,446,390	6,076,319	4,316,947	1,759,372	8,386,706	0

min.
Gr.
57
57

Gr. 33

33 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	5,446,390	6,076,319	4,316,947	1,759,372	8,386,706	60
1	5,448,829	6,071,841	4,310,579	1,761,262	8,385,121	59
2	5,451,268	6,067,366	4,304,213	1,763,153	8,383,536	58
3	5,453,707	6,062,894	4,297,849	1,765,045	8,381,950	57
4	5,456,145	6,058,425	4,291,487	1,766,938	8,380,363	56
5	5,458,583	6,053,958	4,285,126	1,768,832	8,378,776	55
6	5,461,020	6,049,494	4,278,766	1,770,728	8,377,188	54
7	5,463,456	6,045,033	4,272,408	1,772,625	8,375,599	53
8	5,465,802	6,040,575	4,266,052	1,774,523	8,374,009	52
9	5,468,328	6,036,120	4,259,698	1,776,422	8,372,419	51
10	5,470,763	6,031,668	4,253,346	1,778,322	8,370,828	50
11	5,473,198	6,027,218	4,246,994	1,780,224	8,369,236	49
12	5,475,632	6,022,771	4,240,644	1,782,127	8,367,644	48
13	5,478,066	6,018,327	4,234,296	1,784,031	8,366,051	47
14	5,480,499	6,013,886	4,227,950	1,785,936	8,364,457	46
15	5,482,932	6,009,448	4,221,605	1,787,843	8,362,862	45
16	5,485,364	6,005,013	4,215,262	1,789,751	8,361,266	44
17	5,487,796	6,000,580	4,208,920	1,791,660	8,359,670	43
18	5,490,228	6,996,150	4,202,580	1,793,570	8,358,073	42
19	5,492,659	6,991,723	4,196,241	1,795,482	8,356,476	41
20	5,495,090	6,987,299	4,189,904	1,797,395	8,354,878	40
21	5,497,520	6,982,878	4,183,569	1,799,309	8,353,279	39
22	5,499,950	6,978,460	4,177,236	1,801,224	8,351,680	38
23	5,502,379	6,974,044	4,170,904	1,803,140	8,350,080	37
24	5,504,808	6,969,631	4,164,573	1,805,058	8,348,479	36
25	5,507,236	6,965,221	4,156,244	1,806,977	8,346,877	35
26	5,509,664	6,960,814	4,151,917	1,808,897	8,345,274	34
27	5,512,091	6,956,409	4,145,591	1,810,818	8,343,671	33
28	5,514,518	6,952,007	4,139,267	1,812,740	8,342,067	32
29	5,516,944	6,947,608	4,132,944	1,814,664	8,340,463	31
30	5,519,370	6,943,212	4,126,623	1,816,589	8,338,858	30

Gr. 33

33 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	5,519,370	5,943,212	4,126,623	1,816,589	8,338,858	30
31	5,521,795	5,938,829	4,120,314	1,818,515	8,337,252	29
32	5,524,220	5,934,438	4,113,996	1,820,442	8,335,646	28
33	5,526,645	5,930,050	4,107,680	1,822,370	8,334,039	27
34	5,529,069	5,925,665	4,101,366	1,824,299	8,332,431	26
35	5,531,493	5,921,273	4,095,043	1,826,230	8,330,822	25
36	5,533,916	5,916,893	4,088,731	1,828,162	8,329,212	24
37	5,536,338	5,912,516	4,082,421	1,830,095	8,327,602	23
38	5,538,760	5,908,142	4,076,113	1,832,029	8,325,991	22
39	5,541,182	5,903,771	4,069,807	1,833,964	8,324,380	21
40	5,543,603	5,899,402	4,063,501	1,835,901	8,322,768	20
41	5,546,024	5,895,036	4,057,197	1,837,839	8,321,155	19
42	5,548,444	5,890,673	4,050,895	1,839,778	8,319,541	18
43	5,550,864	5,886,313	4,044,594	1,841,719	8,317,927	17
44	5,553,283	5,881,955	4,038,294	1,843,661	8,316,312	16
45	5,555,702	5,877,600	4,031,996	1,845,604	8,314,696	15
46	5,558,120	5,873,248	4,025,700	1,847,548	8,313,079	14
47	5,560,538	5,868,899	4,019,405	1,849,494	8,311,462	13
48	5,562,956	5,864,552	4,013,111	1,851,441	8,309,844	12
49	5,565,373	5,860,208	4,006,819	1,853,389	8,308,226	11
50	5,567,790	5,855,867	4,000,529	1,855,338	8,306,607	10
51	5,570,206	5,851,529	3,994,241	1,857,288	8,304,987	9
52	5,572,622	5,847,193	3,987,953	1,859,240	8,303,367	8
53	5,575,037	5,842,860	3,981,667	1,861,193	8,301,746	7
54	5,577,452	5,838,530	3,975,383	1,863,147	8,300,124	6
55	5,579,866	5,834,203	3,969,101	1,865,102	8,298,501	5
56	5,582,280	5,829,878	3,962,819	1,867,059	8,296,877	4
57	5,584,693	5,825,556	3,956,539	1,869,017	8,295,253	3
58	5,587,106	5,821,237	3,950,261	1,870,976	8,293,628	2
59	5,589,518	5,816,920	3,943,984	1,872,936	8,292,002	1
60	5,591,929	5,812,606	3,937,709	1,874,897	8,290,376	0

min. Gr.

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Gr. 34

34 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
0	5,591,929	5,812,606	3,937,709	1,874,897	8,290,376	60
1	5,594,340	5,808,295	3,931,435	1,876,860	8,288,794	59
2	5,596,751	5,803,987	3,925,163	1,878,824	8,287,121	58
3	5,599,161	5,799,681	3,918,892	1,880,789	8,285,493	57
4	5,601,571	5,795,378	3,912,623	1,882,755	8,283,864	56
5	5,603,981	5,791,078	3,906,355	1,884,723	8,282,234	55
6	5,606,390	5,786,780	3,900,088	1,886,692	8,280,603	54
7	5,608,798	5,782,485	3,893,823	1,888,662	8,278,972	53
8	5,611,206	5,778,192	3,887,559	1,890,633	8,277,340	52
9	5,613,614	5,773,902	3,881,297	1,892,605	8,275,708	51
10	5,616,021	5,769,615	3,875,036	1,894,579	8,274,075	50
11	5,618,427	5,765,330	3,868,776	1,896,554	8,272,441	49
12	5,620,833	5,761,048	3,862,518	1,898,530	8,270,806	48
13	5,623,239	5,756,769	3,856,261	1,900,508	8,269,170	47
14	5,625,644	5,752,493	3,850,006	1,902,487	8,267,534	46
15	5,628,049	5,748,219	3,843,752	1,904,467	8,265,897	45
16	5,630,453	5,743,948	3,837,500	1,906,448	8,264,259	44
17	5,632,857	5,739,680	3,831,249	1,908,431	8,262,621	43
18	5,635,260	5,735,414	3,824,999	1,910,415	8,260,982	42
19	5,637,663	5,731,151	3,818,751	1,912,400	8,259,343	41
20	5,640,066	5,726,891	3,812,505	1,914,386	8,257,703	40
21	5,642,468	5,722,634	3,806,261	1,916,373	8,256,062	39
22	5,644,869	5,718,379	3,800,017	1,918,362	8,254,421	38
23	5,647,270	5,714,127	3,793,775	1,920,352	8,252,779	37
24	5,649,670	5,709,878	3,787,538	1,922,343	8,251,136	36
25	5,652,070	5,705,631	3,781,296	1,924,335	8,249,492	35
26	5,654,469	5,701,387	3,775,059	1,926,328	8,247,847	34
27	5,656,868	5,697,145	3,768,822	1,928,323	8,246,202	33
28	5,659,266	5,692,906	3,762,587	1,930,319	8,244,556	32
29	5,661,664	5,688,670	3,756,354	1,932,316	8,242,909	31
30	5,664,062	5,684,436	3,750,122	1,934,314	8,241,262	30

Gr. 34

34 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	5,664,062	5,684,436	3,750,122	1,934,314	8,241,262	30
31	5,666,459	5,680,205	3,743,891	1,936,314	8,239,614	29
32	5,668,856	5,675,976	3,737,661	1,938,315	8,237,965	28
33	5,671,252	5,671,750	3,731,433	1,940,317	8,236,316	27
34	5,673,648	5,667,527	3,725,206	1,942,321	8,234,666	26
35	5,676,043	5,663,306	3,718,980	1,944,326	8,233,015	25
36	5,678,438	5,659,088	3,712,756	1,946,332	8,231,363	24
37	5,680,832	5,654,872	3,706,532	1,948,340	8,229,711	23
38	5,683,226	5,650,659	3,700,310	1,950,349	8,228,058	22
39	5,685,619	5,646,449	3,694,090	1,952,359	8,226,405	21
40	5,688,012	5,642,241	3,687,871	1,954,370	8,224,751	20
41	5,690,404	5,638,036	3,681,653	1,956,383	8,223,096	19
42	5,692,796	5,633,834	3,675,437	1,958,397	8,221,440	18
43	5,695,187	5,629,635	3,669,223	1,960,412	8,219,784	17
44	5,697,578	5,625,438	3,663,010	1,962,428	8,218,127	16
45	5,699,968	5,621,244	3,656,799	1,964,445	8,216,469	15
46	5,702,358	5,617,052	3,650,588	1,966,464	8,214,810	14
47	5,704,747	5,612,863	3,644,379	1,968,484	8,213,151	13
48	5,707,136	5,608,676	3,638,171	1,970,505	8,211,491	12
49	5,709,524	5,604,492	3,631,965	1,972,527	8,209,831	11
50	5,711,912	5,600,311	3,625,761	1,974,550	8,208,170	10
51	5,714,289	5,596,132	3,619,557	1,976,575	8,206,508	9
52	5,716,686	5,591,956	3,613,355	1,978,601	8,204,846	8
53	5,719,672	5,587,782	3,607,154	1,980,628	8,203,183	7
54	5,721,458	5,583,611	3,600,954	1,982,657	8,201,519	6
55	5,723,844	5,579,443	3,594,756	1,984,687	8,199,854	5
56	5,726,229	5,575,277	3,588,559	1,986,718	8,198,188	4
57	5,728,613	5,571,114	3,582,364	1,988,750	8,196,522	3
58	5,730,997	5,566,953	3,576,169	1,990,784	8,194,855	2
59	5,733,381	5,562,795	3,569,976	1,992,819	8,193,188	1
60	5,735,764	5,558,639	3,563,784	1,994,855	8,191,520	0

min.
Gr.

Gr. 35

35 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	5,735,764	5,558,639	3,563,784	1,994,855	8,191,520	60
1	5,738,147	5,554,486	3,557,594	1,996,892	8,189,851	59
2	5,740,529	5,550,336	3,551,405	1,998,931	8,188,182	58
3	5,742,911	5,546,188	3,545,217	2,000,971	8,186,512	57
4	5,745,292	5,542,043	3,539,031	2,003,012	8,184,841	56
5	5,747,672	5,537,900	3,531,846	2,005,054	8,183,170	55
6	5,750,052	5,533,760	3,526,662	2,007,098	8,181,498	54
7	5,752,432	5,529,622	3,520,479	2,009,143	8,179,825	53
8	5,754,811	5,525,487	3,514,298	2,011,189	8,178,151	52
9	5,757,190	5,521,354	3,508,118	2,013,236	8,176,477	51
10	5,759,568	5,517,224	3,501,939	2,015,285	8,174,802	50
11	5,761,946	5,513,096	3,495,761	2,017,335	8,173,126	49
12	5,764,323	5,508,971	3,489,585	2,019,386	8,171,449	48
13	5,766,700	5,504,849	3,483,410	2,021,439	8,169,772	47
14	5,769,076	5,500,729	3,477,236	2,023,493	8,168,094	46
15	5,771,452	5,496,612	3,471,064	2,025,548	8,166,416	45
16	5,773,827	5,492,497	3,464,892	2,027,605	8,164,737	44
17	5,776,202	5,488,385	3,458,722	2,029,663	8,163,057	43
18	5,778,576	5,484,275	3,452,553	2,031,722	8,161,376	42
19	5,780,950	5,480,168	3,446,386	2,033,782	8,159,695	41
20	5,783,324	5,476,063	3,440,219	2,035,844	8,158,013	40
21	5,785,697	5,471,961	3,434,054	2,037,907	8,156,330	39
22	5,788,069	5,467,861	3,427,890	2,039,971	8,154,647	38
23	5,790,441	5,463,764	3,421,728	2,042,036	8,152,963	37
24	5,792,812	5,459,669	3,415,566	2,044,103	8,151,278	36
25	5,795,183	5,455,577	3,409,406	2,046,171	8,149,593	35
26	5,797,553	5,451,488	3,403,248	2,048,240	8,147,907	34
27	5,799,923	5,447,401	3,397,090	2,050,311	8,146,220	33
28	5,802,292	5,443,317	3,390,934	2,052,383	8,144,532	32
29	5,804,661	5,439,235	3,384,779	2,054,456	8,142,844	31
30	5,807,030	5,435,156	3,378,626	2,056,530	8,141,155	30

Gr. 35

35 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	5,807,030	5,435,156	3,378,626	2,056,530	8,141,155	30
31	5,809,398	5,431,079	3,372,473	2,058,606	8,139,465	29
32	5,811,766	5,427,005	3,366,422	2,060,683	8,137,775	28
33	5,814,133	5,422,933	3,360,172	2,062,761	8,136,084	27
34	5,816,499	5,418,864	3,354,024	2,064,840	8,134,393	26
35	5,818,865	5,414,797	3,347,877	2,066,920	8,132,701	25
36	5,821,230	5,410,733	3,341,731	2,069,002	8,131,008	24
37	5,823,595	5,406,671	3,335,586	2,071,085	8,129,314	23
38	5,825,959	5,402,612	3,329,443	2,071,369	8,127,620	22
39	5,828,323	5,398,555	3,323,300	2,075,255	8,125,925	21
40	5,830,687	5,394,501	3,317,159	2,077,342	8,124,229	20
41	5,833,050	5,390,449	3,311,019	2,079,430	8,122,532	19
42	5,835,412	5,386,400	3,304,880	2,081,520	8,120,835	18
43	5,837,774	5,382,353	3,298,742	2,083,611	8,119,137	17
44	5,840,136	5,378,308	3,292,605	2,085,703	8,117,439	16
45	5,842,497	5,374,266	3,286,470	2,087,796	8,115,740	15
46	5,844,858	5,370,226	3,280,335	2,089,891	8,114,040	14
47	5,847,218	5,366,189	3,274,202	2,091,987	8,112,339	13
48	5,849,578	5,362,154	3,268,070	2,094,084	8,110,638	12
49	5,851,937	5,358,122	3,261,939	2,096,183	8,108,936	11
50	5,854,295	5,354,093	3,255,810	2,098,283	8,107,234	10
51	5,856,653	5,350,067	3,249,683	2,100,384	8,105,531	9
52	5,859,010	5,346,043	3,243,557	2,102,486	8,104,827	8
53	5,861,367	5,342,021	3,237,431	2,104,590	8,102,122	7
54	5,863,724	5,338,002	3,231,307	2,106,695	8,100,417	6
55	5,866,080	5,333,985	3,225,184	2,108,801	8,098,711	5
56	5,868,436	5,329,970	3,219,061	2,110,909	8,097,004	4
57	5,870,791	5,325,958	3,212,940	2,113,018	8,095,296	3
58	5,873,145	5,321,948	3,206,820	2,115,128	8,093,588	2
59	5,875,499	5,317,940	3,200,700	2,117,240	8,091,879	1
60	5,877,852	5,313,935	3,194,582	2,119,353	8,090,170	0 min. Gr.

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Gr. 36

36 min	Sinus	Logarithmi	Differentiæ	Logarithmi	Sinus	
			+ | −			
0	5,877,852	5,313,935	3,194,582	2,119,353	8,090,170	60
1	5,880,205	5,309,932	3,188,465	2,121,467	8,089,460	59
2	5,882,558	5,305,932	3,182,350	2,123,582	8,086,749	58
3	5,884,910	5,301,935	3,176,236	2,125,699	8,085,038	57
4	5,887,262	5,297,940	3,170,123	2,127,817	8,083,326	56
5	5,889,613	5,293,947	3,164,011	2,129,936	8,081,613	55
6	5,891,964	5,289,957	3,157,900	2,132,057	8,079,899	54
7	5,894,314	5,285,969	3,151,790	2,134,179	8,078,185	53
8	5,896,664	5,281,984	3,145,682	2,136,302	8,076,470	52
9	5,899,013	5,278,001	3,139,575	2,138,426	8,074,754	51
10	5,901,361	5,274,020	3,133,468	2,140,552	8,073,038	50
11	5,903,709	5,270,042	3,127,363	2,142,679	8,071,321	49
12	5,906,056	5,266,066	3,121,259	2,144,807	8,069,603	48
13	5,908,403	5,262,092	3,115,155	2,146,937	8,067,885	47
14	5,910,750	5,258,121	3,109,053	2,149,068	8,066,166	46
15	5,913,096	5,254,152	3,102,952	2,151,200	8,064,446	45
16	5,915,442	5,250,186	3,096,853	2,153,333	8,062,726	44
17	5,917,787	5,246,222	3,090,754	2,155,468	8,061,005	43
18	5,920,132	5,242,261	3,084,657	2,157,604	8,059,283	42
19	5,922,476	5,238,302	3,078,561	2,159,741	8,057,561	41
20	5,924,820	5,234,346	3,072,466	2,161,880	8,055,838	40
21	5,927,163	5,230,392	3,066,372	2,164,020	8,054,114	39
22	5,929,505	5,226,441	3,060,280	2,166,161	8,052,389	38
23	5,931,847	5,222,492	3,054,188	2,168,304	8,050,664	37
24	5,934,189	5,218,545	3,048,097	2,170,448	8,048,938	36
25	5,936,530	5,214,601	3,042,008	2,172,593	8,047,212	35
26	5,938,871	5,210,659	3,035,919	2,174,740	8,045,485	34
27	5,941,211	5,206,720	3,029,832	2,176,888	8,043,757	33
28	5,945,551	5,202,783	3,023,746	2,179,037	8,042,028	32
29	5,945,890	5,198,848	3,017,660	2,181,188	8,040,299	31
30	5,948,228	5,194,916	3,011,576	2,183,340	8,038,569	30

Gr. 36

36 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	5,948,228	5,194,916	3,011,576	2,183,340	8,038,569	30
31	5,950,566	5,190,986	3,005,493	2,185,493	8,036,838	29
32	5,952,904	5,187,059	2,999,412	2,187,647	8,035,107	28
33	5,955,241	5,183,134	2,993,331	2,189,803	8,033,375	27
34	5,957,578	5,179,211	2,987,251	2,191,960	8,031,642	26
35	5,959,914	5,175,291	2,981,173	2,194,118	8,029,909	25
36	5,962,250	5,171,373	2,975,095	2,196,278	8,028,175	24
37	5,964,585	5,167,457	2,969,018	2,198,439	8,026,440	23
38	5,966,919	5,163,544	2,962,943	2,200,601	8,024,705	22
39	5,969,253	5,159,633	2,956,868	2,202,765	8,022,969	21
40	5,971,586	5,155,724	2,950,794	2,204,930	8,021,232	20
41	5,973,919	5,151,818	2,944,722	2,207,096	8,019,494	19
42	5,976,251	5,147,914	2,938,650	2,209,264	8,017,756	18
43	5,978,583	5,144,012	2,932,579	2,211,433	8,016,017	17
44	5,980,915	5,140,113	2,926,510	2,213,603	8,014,278	16
45	5,983,246	5,136,216	2,920,442	2,215,774	8,012,538	15
46	5,985,577	5,132,322	2,914,375	2,217,947	8,010,797	14
47	5,987,907	5,128,430	2,908,309	2,220,121	8,009,056	13
48	5,990,237	5,124,540	2,902,244	2,222,296	8,007,314	12
49	5,992,566	5,120,653	2,896,180	2,224,473	8,005,571	11
50	5,994,894	5,116,768	2,890,117	2,226,651	8,003,828	10
51	5,997,222	5,112,886	2,884,056	2,228,830	8,002,084	9
52	5,999,549	5,109,006	2,877,995	2,231,011	8,000,339	8
53	6,001,876	5,105,128	2,871,935	2,233,193	7,998,593	7
54	6,004,202	5,101,253	2,865,877	2,235,376	7,996,847	6
55	6,006,528	5,097,380	2,859,819	2,237,561	7,995,100	5
56	6,008,853	5,093,509	2,853,762	2,239,747	7,993,352	4
57	6,011,178	5,089,641	2,847,706	2,241,935	7,991,604	3
58	6,013,502	5,085,775	2,841,651	2,244,124	7,989,855	2
59	6,015,826	5,081,911	2,835,597	2,246,314	7,988,105	1
60	6,018,150	5,078,050	2,829,544	2,248,506	7,986,355	0

min.
Gr.

Gr. 37

37 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	6,018,150	5,078,050	2,829,544	2,248,506	7,986,355	60
1	6,020,473	5,074,191	2,823,492	2,250,699	7,984,604	59
2	6,022,796	5,070,334	2,817,441	2,252,893	7,982,852	58
3	6,025,118	5,066,479	2,811,391	2,255,088	7,981,100	57
4	6,027,439	5,062,627	2,805,342	2,257,285	7,979,347	56
5	6,029,760	5,058,777	2,799,294	2,259,483	7,977,593	55
6	6,032,080	5,054,929	2,793,247	2,261,682	7,975,838	54
7	6,034,400	5,051,084	2,787,201	2,263,883	7,974,084	53
8	6,036,719	5,047,241	2,781,156	2,266,085	7,972,328	52
9	6,039,038	5,043,401	2,775,113	2,268,288	7,970,572	51
10	6,041,357	5,039,563	2,769,071	2,270,492	7,968,815	50
11	6,043,675	5,035,727	2,763,029	2,272,698	7,967,057	49
12	6,045,992	5,031,894	2,756,989	2,274,905	7,965,299	48
13	6,048,309	5,028,063	2,750,949	2,277,114	7,963,540	47
14	6,050,625	5,024,234	2,744,910	2,279,324	7,961,780	46
15	6,052,940	5,020,408	2,738,873	2,281,535	7,960,020	45
16	6,055,255	5,016,584	2,732,836	2,283,748	7,958,259	44
17	6,057,570	5,012,762	2,726,800	2,285,962	7,956,497	43
18	6,059,884	5,008,942	2,720,764	2,288,178	7,954,735	42
19	6,062,198	5,005,125	2,714 730	2,290,395	7,952,972	41
20	6,064,511	5,001,310	2,708,696	2,292,614	7,951,208	40
21	6,066,824	4,997,497	2,702,663	2,294,834	7,949,443	39
22	6,069,136	4,993,687	2,696,632	2,297,055	7,947,678	38
23	6,071,448	4,989,879	2,690,602	2,299,277	7,945,912	37
24	6,073,759	4,986,073	2,684,573	2,301,500	7,944,146	36
25	6,076,069	4,982,270	2,678,545	2,303,725	7,942,379	35
26	6,078,379	4,978,469	2,672,518	2,305,951	7,940,611	34
27	6,080,688	4,974,670	2,666,492	2,308,178	7,938,842	33
28	6,082,997	4,970,873	2,660,467	2,310,406	7,937,073	32
29	6,085,306	4,967,079	2,654,444	2,312,635	7,935,303	31
30	6,087,614	4,963,287	2,648,421	2,314,866	7,933,533	30

Gr. 37

37 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
30	6,087,614	4,963,287	2,648,421	2,314,866	7,933,533	30
31	6,089,922	4,959,497	2,642,399	2,317,098	7,931,762	29
32	6,092,229	4,955,710	2,636,378	2,319,332	7,929,990	28
33	6,094,536	4,951,925	2,630,358	2,321,567	7,928,218	27
34	6,096,842	4,948,142	2,624,338	2,323,804	7,926,445	26
35	6,099,147	4,944,361	2,618,319	2,326,042	7,924,671	25
36	6,101,452	4,940,582	2,612,301	2,328,281	7,922,896	24
37	6,103,756	4,936,806	2,606,284	2,330,522	7,921,121	23
38	6,106,060	4,933,032	2,600,268	2,332,764	7,919,345	22
39	6,108,364	4,929,260	2,594,252	2,335,008	7,917,569	21
40	6,110,667	4,925,490	2,588,237	2,337,253	7,915,792	20
41	6,112,970	4,921,723	2,582,224	2,339,499	7,914,014	19
42	6,115,272	4,917,958	2,576,211	2,341,747	7,912,235	18
43	6,117,573	4,914,195	2,570,199	2,343,996	7,910,456	17
44	6,119,873	4,910,435	2,564,189	2,346,246	7,908,676	16
45	6,122,173	4,906,677	2,558,179	2,348,498	7,906,896	15
46	6,124,473	4,902,921	2,552,170	2,350,751	7,905,114	14
47	6,126,772	4,899,168	2,546,163	2,353,005	7,903,332	13
48	6,129,071	4,895,417	2,540,156	2,355,261	7,901,550	12
49	6,131,369	4,891,668	2,534,150	2,357,518	7,899,767	11
50	6,133,667	4,887,921	2,528,145	2,359,776	7,897,983	10
51	6,135,964	4,884,177	2,522,141	2,362,036	7,896,198	9
52	6,138,261	4,880,435	2,516,138	2,364,297	7,894,413	8
53	6,140,557	4,876,695	2,510,136	2,366,559	7,892,627	7
54	6,142,853	4,872,957	2,504,134	2,368,823	7,890,841	6
55	6,145,148	4,869,222	2,498,134	2,371,088	7,889,054	5
56	6,147,442	4,865,489	2,492,135	2,373,354	7,887,266	4
57	6,149,746	4,861,758	2,486,136	2,375,622	7,885,477	3
58	6,152,030	4,858,029	2,480,138	2,377,891	7,883,688	2
59	6,154,323	4,854,302	2,474,140	2,380,162	7,881,898	1
60	6,156,615	4,850,578	2,468,144	2,382,434	7,880,108	0

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38 min	Sinus	Logarithmi	Differentiæ	Logarithmi	Sinus	
			+ | −			
0	6,156,615	4,850,578	2,468,144	2,382,434	7,880,108	60
1	6,158,907	4,846,856	2,462,149	2,384,707	7,878,317	59
2	6,161,198	4,843,136	2,456,154	2,386,982	7,876,525	58
3	6,163,489	4,839,418	2,450,160	2,389,258	7,874,732	57
4	6,165,781	4 835,702	2,444,167	2,391,535	7,872,939	56
5	6,168,070	4,831,989	2,438,175	2,393,814	7,871,145	55
6	6,170,259	4,828,278	2,432,184	2,396,094	7,869,350	54
7	6,172,648	4,824,569	2,426,193	2,398,376	7,867,555	53
8	6,174,936	4,820,862	2,420,203	2,400,659	7,865,759	52
9	6,177,224	4,817,158	2,414,215	2,402,943	7,863,963	51
10	6,179,512	4,813,456	2,408,227	2,405,229	7,862,166	50
11	6,181,799	4,809,756	2,402,240	2,407,516	7,860,368	49
12	6,184,085	4,806,058	2,396,254	2,409,804	7,858,569	48
13	6,186,371	4,802,363	2,390,269	2,412,094	7,856,770	47
14	6,188,656	4,798,670	2,384,285	2,414,385	7,854,970	46
15	6,190,940	4,794,979	2,378,301	2,416,678	7,853,169	45
16	6,193,224	4,791,290	2,372 318	2,418,972	7,851,368	44
17	6,195,508	4,787,603	2,366,336	2,421,267	7,849,566	43
18	6,197,791	4,783,919	2,360,355	2,423,564	7,847,764	42
19	6,200,074	4,780,237	2,354,375	2,425,862	7,845,961	41
20	6,202,356	4,776,557	2,348,396	2,428,161	7,844,157	40
21	6,204,638	4,772,880	2,342,418	2,430,462	7,842,352	39
22	6,206,919	4,769,205	2,336,441	2,432,764	7,840,547	38
23	6,209,199	4,765,532	2,330,465	2,435,067	7,838,741	37
24	6,211,479	4,761,861	2,324,489	2,437,372	7,836,935	36
25	6,213,758	4,758,192	2,318,514	2,439,678	7,835,128	35
26	6,216,037	4,754,525	2,312,539	2,441,986	7,833,320	34
27	6,218,315	4,750,860	2,306,565	2,444,295	7,831,511	33
28	6,220,593	4,747,198	2,300,593	2,446,605	7,829,702	32
29	6,222,870	4,743,538	2,294,621	2,448,917	7,827,892	31
30	6,225,146	4,739,880	2,288,650	2,451,230	7,826,082	30

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Gr. 38

38 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	6,225,146	4,739,880	2,288,650	2,451,230	7,826,082	30
31	6,227,422	4,736,224	2,282,680	2,453,544	7,824,271	29
32	6,229,698	4,732,571	2,276,711	2,455,860	7,822,459	28
33	6,231,973	4,728,920	2,270,743	2,458,177	7,820,647	27
34	6,234,248	4,725,271	2,264,775	2,460,496	7,818,834	26
35	6,236,522	4,721,624	2,258,808	2,462,816	7,817,020	25
36	6,238,796	4,717,979	2,252,842	2,465,137	7,815,205	24
37	6,241,069	4,714,336	2,246,876	2,467,460	7,813,390	23
38	6,243,342	4,710,695	2,240,911	2,469,784	7,811,574	22
39	6,245,614	4,707,056	2,234,946	2,472,110	7,809,758	21
40	6,247,885	4,703,419	2,228,982	2,474,437	7,807,941	20
41	6,250,156	4,699,785	2,223,020	2,476,765	7,806,123	19
42	6,252,426	4,696,153	2,217,058	2,479,095	7,804,304	18
43	6,254,696	4,692,523	2,211,097	2,481,426	7,802,485	17
44	6,256,966	4,688,895	2,205,136	2,483,759	7,800,665	16
45	6,259,235	4,685,269	2,199,176	2,486,093	7,798,845	15
46	6,261,503	4,681,645	2,193,217	2,488,428	7,797,024	14
47	6,263,771	4,678,024	2,187,259	2,490,765	7,795,202	13
48	6,266,038	4,674,405	2,181,302	2,493,103	7,793,380	12
49	6,268,305	4,670,788	2,175,345	2,495,443	7,791,557	11
50	6,270,572	4,667,173	2,169,389	2,497,784	7,789,733	10
51	6,272,838	4,663,561	2,163,435	2,500,126	7,787,909	9
52	6,275,103	4,659,951	2,157,481	2,502,470	7,786,084	8
53	6,277,368	4,656,343	2,151,528	2,504,815	7,784,258	7
54	6,279,632	4,652,737	2,145,575	2,507,162	7,782,432	6
55	6,281,895	4,649,133	2,139,623	2,509,510	7,780,605	5
56	6,284,158	4,645,531	2,133,672	2,511,859	7,778,777	4
57	6,286,420	4,641,931	2,127,721	2,514,210	7,776,949	3
58	6,288,682	4,638,334	2,121,772	2,516,562	7,775,120	2
59	6,290,943	4,634,739	2,115,824	2,518,915	7,773,290	1
60	6,293,204	4,631,146	2,109,876	2,521,270	7,771,460	0

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51	51

Gr. 39

39 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	6,293,204	4,631,146	2,109,876	2,521,270	7,771,460	60
1	6,295,464	4,627,555	2,103,929	2,523,626	7,769,629	59
2	6,297,724	4,623,966	2,097,982	2,525,984	7,767,797	58
3	6,299,983	4,620,379	2,092,036	2,528,343	7,765,965	57
4	6,302,242	4,616,794	2,086,091	2,530,703	7,764,132	56
5	6,304,501	4,613,211	2,080,146	2,533,065	7,762,299	55
6	6,306,759	4,609,630	2,074,202	2,535,428	7,760,465	54
7	6,309,016	4,606,052	2,068,259	2,537,793	7,758,630	53
8	6,311,273	4,602,476	2,062,317	2,540,159	7,756,794	52
9	6,313,529	4,598,902	2,056,376	2,542,526	7,754,958	51
10	6,315,784	4,595,330	2,050,435	2,544,895	7,753,121	50
11	6,318,039	4,591,760	2,044,495	2,547,265	7,751,283	49
12	6,320,293	4,588,192	2,038,555	2,549,637	7,749,445	48
13	6,322,547	4,584,627	2,032,617	2,552,010	7,747,606	47
14	6,324,800	4,581,064	2,026,679	2,554,385	7,745,766	46
15	6,327,053	4,577,503	2,020,742	2,556,761	7,743,926	45
16	6,329,305	4,573,944	2,014,806	2,559,138	7,742,085	44
17	6,331,557	4,570,387	2,008,870	2,561,517	7,740,244	43
18	6,333,808	4,566,832	2,002,935	2,563,897	7,738,402	42
19	6,336,059	4,563,279	1,997,000	2,566,279	7,736,559	41
20	6,338,310	4,559,728	1,991,066	2,568,662	7,734,716	40
21	6,340,560	4,556,179	1,985,133	2,571,046	7,732,872	39
22	6,342,809	4,552,632	1,979,200	2,573,432	7,731,028	38
23	6,345,058	4,549,088	1,973,269	2,575,819	7,729,183	37
24	6,347,309	4,545,546	1,967,338	2,578,208	7,727,337	36
25	6,349,553	4,542,006	1,961,408	2,580,598	7,725,490	35
26	6,351,800	4,538,468	1,955,478	2,582,990	7,723,642	34
27	6,354,046	4,534,932	1,949,549	2,585,383	7,721,794	33
28	6,356,292	4,531,398	1,943,621	2,587,777	7,719,945	32
29	6,358,537	4,527,866	1,937,693	2,590,173	7,718,096	31
30	6,360,782	4,524,336	1,931,766	2,592,570	7,716,246	30

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Gr. 39

39 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	6,360,782	4,524,336	1,931,766	2,592,570	7,716,246	30
31	6,363,026	4,520,808	1,925,839	2,594,969	7,714,395	29
32	6,365,270	4,517,282	1,919,913	2,597,369	7,712,544	28
33	6,367,513	4,513,758	1,913,988	2,599,770	7,710,692	27
34	6,369,756	4,510,236	1,908,063	2,602,173	7,708,839	26
35	6,371,999	4,506,717	1,902,140	2,604,577	7,706,986	25
36	6,374,241	4,503,200	1,896,217	2,606,983	7,705,132	24
37	6,376,482	4,499,685	1,890,295	2,609,390	7,703,277	23
38	6,378,722	4,496,172	1,884,373	2,611,799	7,701,422	22
39	6,380,962	4,492,661	1,878,452	2,614 209	7,699,566	21
40	6,383,201	4,489,152	1,872,531	2,616,621	7,697,710	20
41	6,385,440	4,485,645	1,866,611	2,619,034	7,695,853	19
42	6,387,678	4,482,140	1,860,692	2,621,448	7,693,995	18
43	6,389,916	4,478,637	1,854,773	2,623,864	7,692,137	17
44	6,392,153	4,475,136	1,848,855	2,626,281	7,690,278	16
45	6,394,390	4,471,637	1,842,937	2,628,700	7,688,418	15
46	6,396,626	4,468,140	1,837,020	2,631,120	7,686,549	14
47	6,398,862	4,464,646	1,831,105	2,633,541	7,684,687	13
48	6,401,097	4,461,154	1,825,190	2,635,964	7,682,835	12
49	6,403,332	4,457,664	1,819,276	2,638,388	7,680,973	11
50	6,405,566	4,454,176	1,813,363	2,640,813	7,679,110	10
51	6,407,799	4,450,690	1,807,450	2,643,240	7,677,246	9
52	6,410,032	4,447,206	1,801,537	2,645,669	7,675,382	8
53	6,412,264	4,443,724	1,795,625	2,648,099	7,673,517	7
54	6,414,496	4,440,244	1,789,714	2,650,530	7,671,652	6
55	6,416,728	4,436,766	1,783,803	2,652,963	7,669,786	5
56	6,418,959	4,433,290	1,777,893	2,655,397	7,667,919	4
57	6,421,189	4,429,816	1,771,983	2,657,833	7,666,051	3
58	6,423,419	4,426,344	1,766,074	2,660,270	7,664,183	2
59	6,425,648	4,422,875	1,760,166	2,662,709	7,662,314	1
60	6,427,876	4,419,408	1,754,259	2,665,149	7,660,445	0

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Gr. 40

40 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	6,427,876	4,419,408	1,754,259	2,665,149	7,660,445	60
1	6,430,104	4,415,943	1,748,353	2,667,590	7,658,575	59
2	6,432,331	4,412,480	1,742,447	2,670,033	7,656,704	58
3	6,434,558	4,409,019	1,736,542	2,672,477	7,654,833	57
4	6,436,785	4,405,560	1,730,637	2,674,923	7,652,961	56
5	6,439,011	4,402,103	1,724,733	2,677,370	7,651,088	55
6	6,441,236	4,398,648	1,718,829	2,679,819	7,649,215	54
7	6,443,461	4,395,195	1,712,926	2,682,269	7,647,341	53
8	6,445,685	4,391,743	1,707,022	2,684,721	7,645,466	52
9	6,447,909	4,388,293	1,701,119	2,687,174	7,643,591	51
10	6,450,132	4,384,845	1,695,216	2,689,629	7,641,715	50
11	6,452,355	4,381,399	1,689,314	2,692,085	7,639,838	49
12	6,454,577	4,377,955	1,683,412	2,694,543	7,637,960	48
13	6,456,799	4,374,514	1,677,512	2,697,002	7,636,082	47
14	6,459,020	4,371,075	1,671,613	2,699,462	7,634,204	46
15	6,461,240	4,367,638	1,665,714	2,701,924	7,632,325	45
16	6,463,460	4,364,203	1,659,816	2,704,387	7,630,445	44
17	6,465,679	4,360,770	1,653,918	2,706,852	7,628,564	43
18	6,467,898	4,357,339	1,648,021	2,709,318	7,626,683	42
19	6,470,116	4,353,910	1,642,124	2,711,786	7,624,802	41
20	6,472,333	4,350,483	1,636,228	2,714,255	7,622,920	40
21	6,474,550	4,347,058	1,630,332	2,716,726	7,621,037	39
22	6,476,766	4,343,635	1,624,437	2,719,198	7,619,153	38
23	6,478,982	4,340,214	1,618,542	2,721,672	7,617,269	37
24	6,481,198	4,336,795	1,612,648	2,724,147	7,615,384	36
25	6,483,413	4,333,378	1,606,755	2,726,623	7,613,498	35
26	6,485,628	4,329,963	1,600,862	2,729,101	7,611,612	34
27	6,487,842	4,326,550	1,594,970	2,731,580	7,609,725	33
28	6,490,055	4,323,139	1,589,078	2,734,061	7,607,837	32
29	6,492,268	4,319,730	1,583,187	2,736,543	7,605,949	31
30	6,494,480	4,316,323	1,577,296	2,739,027	7,604,060	30

Gr. 40

40 min	Sinus	Logarithmi	Differentiæ	Logarithmi	Sinus	
			+ | −			
30	6,494,480	4,316,323	1,577,296	2,739,027	7,604,060	30
31	6,496,69	4,312,919	1,571,407	2,741,512	7,602,170	29
32	6,498,903	4,309,517	1,565,518	2,743,999	7,600,280	28
33	6,501,114	4,306,116	1,559,629	2,746,487	7,598,389	27
34	6,503,324	4,302,717	1,553,740	2,748,977	7,596,498	26
35	6,505,533	4,299,320	1,547,852	2,751,468	7,594,606	25
36	6,507,742	4,295,925	1,541,964	2,753,961	7,592,713	24
37	6,509,950	4,292,532	1,536,077	2,756,455	7,590,819	23
38	6,512,158	4,289,141	1,530,191	2,758,950	7,588,925	22
39	6,514,365	4,285,752	1,524,305	2,761,447	7,587,031	21
40	6,516,572	4,282,365	1,518,420	2,763,945	7,589,136	20
41	6,518,778	4,278,980	1,512,535	2,766,445	7,583,240	19
42	6,520,984	4,275,597	1,506,651	2,768,946	7,581,343	18
43	6,523,189	4,272,216	1,500,767	2,771,449	7,579,446	17
44	6,525,394	4,268,837	1,494,884	2,773,953	7,577,548	16
45	6,527,598	4,265,460	1,489,001	2,776,459	7,575,650	15
46	6,529,801	4,262,085	1,483,119	2,778,966	7,573,751	14
47	6,532,004	4,258,712	1,477,237	2,781,475	7,571,851	13
48	6,534,206	4,255,341	1,471,356	2,783,985	7,569,951	12
49	6,536,408	4,251,972	1,465,476	2,786,496	7,568,050	11
50	6,538,609	4,248,605	1,459,596	2,789,009	7,566,148	10
51	6,540,809	4,245,240	1,453,717	2,791,523	7,564,246	9
52	6,543,009	4,241,877	1,447,838	2,794,039	7,562,343	8
53	6,545,208	4,238,516	1,441,960	2,796,556	7,560,439	7
54	6,547,407	4,235,157	1,436,082	2,799,075	7,558,535	6
55	6,549,606	4,231,800	1,430,205	2,801,595	7,556,630	5
56	6,551,804	4,228,445	1,424,328	2,804,117	7,554,724	4
57	6,554,001	4,225,092	1,418,451	2,806,641	7,552,818	3
58	6,556,198	4,221,741	1,412,575	2,809,166	7,550,911	2
59	6,558,394	4,218,392	1,406,699	2,811,693	7,549,004	1
60	6,560,590	4,215,044	1,400,823	2,814,221	7,547,096	0
			|			min.
						Gr.
			49			49

Gr. 41

41 min	Sinus	Logarithmi	+ Differentiæ −	Logarithmi	Sinus	
0	6,560,590	4,215,044	1,400,823	2,814,221	7,547,096	60
1	6,562,785	4,211,698	1,394,947	2,816,751	7,545,187	59
2	6,564,979	4,208,354	1,389,072	2,819,282	7,543,277	58
3	6,567,173	4,205,012	1,383,197	2,821,815	7,541,367	57
4	6,569,367	4,201,672	1,377,323	2,824,349	7,539,457	56
5	6,571,560	4,198,334	1,371,450	2,826,884	7,537,546	55
6	6,573,753	4,194,999	1,365,578	2,829,421	7,535,634	54
7	6,575,945	4,191,666	1,359,707	2,831,959	7,533,721	53
8	6,578,136	4,188,335	1,353,836	2,834,499	7,531,808	52
9	6,580,326	4,185,006	1,347,966	2,837,040	7,529,894	51
10	6,582,516	4,181,679	1,342,097	2,839,582	7,527,980	50
11	6,584,705	4,178,354	1,336,228	2,842,126	7,526,065	49
12	6,586,894	4,175,030	1,330,358	2,844,672	7,524,149	48
13	6,589,082	4,171,708	1,324,489	2,847,219	7,522,233	47
14	6,591,270	4,168,388	1,318,620	2,849,768	7,520,316	46
15	6,593,458	4,165,070	1,312,752	2,852,318	7,518,398	45
16	6,595,645	4,161,754	1,306,884	2,854,870	7,516,840	44
17	6,597,831	4,158,440	1,301,017	2,857,423	7,514,561	43
18	6,600,016	4,155,128	1,295,150	2,859,978	7,512,642	42
19	6,602,201	4,151,818	1,289,284	2,862,534	7,510,722	41
20	6,604,386	4,148,510	1,283,418	2,865,092	7,508,801	40
21	6,606,570	4,145,204	1,277,553	2,867,651	7,506,879	39
22	6,608,753	4,141,900	1,271,688	2,870,212	7,504,957	38
23	6,610,936	4,138,598	1,265,824	2,872,774	7,503,034	37
24	6,613,118	4,135,298	1,259,960	2,875,338	7,501,111	36
25	6,615,300	4,132,000	1,254,097	2,877,903	7,499,187	35
26	6,617,481	4,128,703	1,248,233	2,880,470	7,497,262	34
27	6,619,661	4,125,408	1,242,370	2,883,038	7,495,336	33
28	6,621,841	4,122,115	1,236,507	2,885,608	7,493,410	32
29	6,624,021	4,118,824	1,230,645	2,888,179	7,491,484	31
30	6,626,200	4,115,535	1,224,783	2,890,752	7,489,557	30

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41 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	6,626,200	4,115,535	1,224,783	2,890,752	7,489,557	30
31	6,628,379	4,112,248	1,218,922	2,893,326	7,487,629	29
32	6,630,557	4,108,963	1,213,061	2,895,902	7,485,700	28
33	6,632,734	4,105,680	1,207,201	2,898,479	7,483,771	27
34	6,634,911	4,102,399	1,201,341	2,901,058	7,481,842	26
35	6,637,087	4,099,120	1,195,482	2,903,638	7,479,912	25
36	6,639,263	4,095,843	1,189,623	2,906,220	7,477,981	24
37	6,641,438	4,092,567	1,183,763	2,908,804	7,476,049	23
38	6,643,612	4,089,293	1,177,904	2,911,389	7,474,117	22
39	6,645,786	4,086,021	1,172,045	2,913,976	7,472,184	21
40	6,647,959	4,082,751	1,166,187	2,916,564	7,470,251	20
41	6,650,132	4,079,483	1,160,329	2,919,154	7,468,317	19
42	6,652,304	4,076,217	1,154,472	2,921,745	7,466,382	18
43	6,654,476	4,072,953	1,148,615	2,924,338	7,464,447	17
44	6,656,647	4,069,691	1,142,759	2,926,932	7,462,511	16
45	6,658,817	4,066,431	1,136,904	2,929,527	7,460,574	15
46	6,660,987	4,063,173	1,131,049	2,932,124	7,458,637	14
47	6,663,156	4,059,917	1,125,195	2,934,722	7,456,699	13
48	6,665,325	4,056,663	1,119,341	2,937,322	7,454,761	12
49	6,667,493	4,053,410	1,113,487	2,939,923	7,452,822	11
50	6,669,661	4,050,159	1,107,633	2,942,526	7,450,882	10
51	6,671,828	4,046,910	1,101,780	2,945,130	7,448,941	9
52	6,673,994	4,043,663	1,095,927	2,947,736	7,447,000	8
53	6,676,160	4,040,418	1,090,074	2,950,344	7,445,058	7
54	6,678,326	4,037,175	1,084,222	2,952,953	7,443,116	6
55	6,680,491	4,033,934	1,078,370	2,955,564	7,441,173	5
56	6,682,655	4,030,695	1,072,518	2,958,177	7,439,229	4
57	6,684,818	4,027,458	1,066,667	2,960,791	7,437,284	3
58	6,686,981	4,024,223	1,060,816	2,963,407	7,435,339	2
59	6,689,144	4,020,990	1,054,966	2,966,024	7,433,394	1
60	6,691,306	4,017,759	1,049,116	2,968,643	7,431,448	0

min. Gr.

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Gr. 42

42 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
0	6,691,306	4,017,759	1,049,116	2,968,643	7,431,448	60
1	6,693,468	4,014,529	1,043,266	2,971,263	7,429,501	59
2	6,695,629	4,011,301	1,037,416	2,973,885	7,427,553	58
3	6,697,789	4,008,075	1,031,567	2,976,508	7,425,605	57
4	6,699,949	4,004,851	1,025,718	2,979,133	7,423,657	56
5	6,702,108	4,001,629	1,019,870	2,981,759	7,421,708	55
6	6,704,267	3,998,409	1,014,023	2,984,386	7,419,758	54
7	6,706,425	3,995,191	1,008,176	2,987,015	7,417,807	53
8	6,708,582	3,991,974	1,002,329	2,989,645	7,415,856	52
9	6,710,739	3,988,759	996,482	2,992,277	7,413,905	51
10	6,712,895	3,985,546	990,636	2,994,910	7,411,953	50
11	6,715,051	3,982,335	984,790	2,997,545	7,410,000	49
12	6,717,206	3,979,126	978,944	3,000,182	7,408,046	48
13	6,719,361	3,975,919	973,099	3,002,820	7,406,092	47
14	6,721,515	3,972,714	967,254	3,005,460	7,404,137	46
15	6,723,668	3,969,511	961,409	3,008,102	7,402,181	45
16	6,725,821	3,966,310	955,565	3,010,745	7,400,225	44
17	6,727,973	3,963,110	949,720	3,013,390	7,398,268	43
18	6,730,125	3,959,912	943,876	3,016,036	7,396,311	42
19	6,732,276	3,956,716	938,032	3,018,684	7,394,353	41
20	6,734,427	3,953,522	932,189	3,021,333	7,392,394	40
21	6,736,577	3,950,330	926,346	3,023,984	7,390,435	39
22	6,738,726	3,947,140	920,504	3,026,636	7,388,475	38
23	6,740,875	3,943,951	914,661	3,029,290	7,386,515	37
24	6,743,024	3,940,764	908,819	3,031,945	7,384,554	36
25	6,745,172	3,937,579	902,977	3,034,602	7,382,592	35
26	6,747,319	3,934,396	897,135	3,037,261	7,380,629	34
27	6,749,465	3,931,215	891,294	3,039,921	7,378,666	33
28	6,751,611	3,928,036	885,453	3,042,583	7,376,702	32
29	6,753,757	3,924,859	879,613	3,045,246	7,374,738	31
30	6,755,902	3,921,684	873,773	3,047,911	7,372,773	30

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42 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	6,755,902	3,921,684	873,773	3,047,911	7,372,773	30
31	6,758,047	3,918,511	867,934	3,050,577	7,370,807	29
32	6,760,191	3,915,339	862,094	3,053,245	7,368,841	28
33	6,762,334	3,912,169	856,255	3,055,914	7,366,874	27
34	6,764,477	3,909,001	850,416	3,058,585	7,364,907	26
35	6,766,619	3,905,835	844,577	3,061,258	7,362,939	25
36	6,768,760	3,902,671	838,739	3,063,932	7,360,970	24
37	6,770,901	3,899,509	832,901	3,066,608	7,359,001	23
38	6,773,041	3,896,348	827,063	3,069,285	7,357,031	22
39	6,775,181	3,893,189	821,225	3,071,964	7,355,061	21
40	6,777,320	3,890,032	815,388	3,074,644	7,353,090	20
41	6,779,459	3,886,877	809,551	3,077,326	7,351,118	19
42	6,781,597	3,883,723	803,714	3,080,009	7,349,145	18
43	6,783,734	3,880,571	797,877	3,082,694	7,347,173	17
44	6,785,971	3,877,421	792,041	3,085,380	7,345,199	16
45	6,788,007	3,874,273	786,205	3,088,068	7,343,225	15
46	6,790,143	3,871,127	780,369	3,090,758	7,341,250	14
47	6,792,278	3,867,983	774,534	3,093,449	7,339,274	13
48	6,794,413	3,864,841	768,699	3,096,142	7,337,298	12
49	6,796,547	3,861,701	762,865	3,098,836	7,335,322	11
50	6,798,681	3,858,563	757,031	3,101,532	7,333,345	10
51	6,800,814	3,855,426	751,197	3,104,229	7,331,367	9
52	6,802,946	3,852,291	745,363	3,106,928	7,329,388	8
53	6,805,078	3,849,158	739,529	3,109,629	7,327,409	7
54	6,807,209	3,846,027	733,696	3,112,331	7,325,429	6
55	6,809,340	3,842,898	727,863	3,115,035	7,323,449	5
56	6,811,470	3,839,770	722,029	3,117,741	7,321,468	4
57	6,813,599	3,836,644	716,196	3,120,448	7,319,486	3
58	6,815,728	3,833,520	710,363	3,123,157	7,317,504	2
59	6,817,856	3,830,398	704,530	3,125,868	7,315,521	1
60	6,819,984	3,827,278	698,698	3,128,580	7,313,537	0

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43 min	Sinus	Logarithmi	+ Differentiæ −	Logarithmi	Sinus	
0	6,819,984	3,827,278	698,698	3,128,580	7,313,537	60
1	6,822,111	3,824,160	692,866	3,131,294	7,311,553	59
2	6,824,237	3,821,044	687,035	3,134,009	7,309,568	58
3	6,826,363	3,817,929	681,203	3,136,726	7,307,583	57
4	6,828,489	3,814,816	675,372	3,139,444	7,305,597	56
5	6,830,614	3,811,705	669,541	3,142,164	7,303,610	55
6	6,832,738	3,808,596	653,711	3,144,885	7,301,623	54
7	6,834,861	3,805,488	657,880	3,147,608	7,299,635	53
8	6,836,984	3,802,382	652,050	3,150,332	7,297,647	52
9	6,839,107	3,799,278	646,221	3,153,057	7,295,658	51
10	6,841,229	3,796,176	640,392	3,155,784	7,293,668	50
11	6,843,350	3,793,075	634,562	3,158,513	7,291,678	49
12	6,845,471	3,789,976	628,732	3,161,244	7,289,687	48
13	6,847,591	3,786,879	622,903	3,163,976	7,287,695	47
14	6,849,711	3,783,784	617,074	3,166,710	7,285,703	46
15	6,851,830	3,780,691	611,246	3,169,445	7,283,710	45
16	6,853,949	3,777,600	605,418	3,172,182	7,281,716	44
17	6,856,067	3,774,510	599,589	3,174,921	7,279,722	43
18	6,858,184	3,771,422	593,760	3,177,662	7,277,728	42
19	6,860,301	3,768,336	587,932	3,180,404	7,275,733	41
20	6,862,417	3,765,252	582,104	3,183,148	7,273,737	40
21	6,864,533	3,762,170	576,277	3,185,893	7,271,741	39
22	6,866,648	3,759,090	570,450	3,188,640	7,269,744	38
23	6,868,762	3,756,011	564,622	3,191,389	7,267,746	37
24	6,870,876	3,752,934	558,795	3,194,139	7,265,748	36
25	6,872,989	3,749,859	552,968	3,196,891	7,263,749	35
26	6,875,102	3,746,786	547,142	3,199,644	7,261,749	34
27	6,877,214	3,743,714	541,315	3,202,399	7,259,748	33
28	6,879,325	3,740,644	535,489	3,205,155	7,257,747	32
29	6,881,436	3,737,576	529,663	3,207,913	7,255,746	31
30	6,883,546	3,734,510	523,838	3,210,672	7,253,744	30

Gr. 43

43 min	Sinus	Logarithmi	Differentiæ + | −	Logarithmi	Sinus	
30	6,883,546	3,734,510	523,838	3,210,672	7,253,744	30
31	6,885,656	3,731,446	518,013	3,213,433	7,251,741	29
32	6,887,765	3,728,383	512,187	3,216,196	7,249,737	28
33	6,889,874	3,725,322	506,361	3,218,961	7,247,733	27
34	6,891,982	3,722,263	500,536	3,221,727	7,245,729	26
35	6,894,089	3,719,206	494,711	3,224,495	7,243,724	25
36	6,896,196	3,716,150	488,886	3,227,264	7,241,718	24
37	6,898,302	3,713,096	483,061	3,230,035	7,239,711	23
38	6,900,408	3,710,044	477,236	3,232,808	7,237,704	22
39	6,902,513	3,706,994	471,411	3,235,583	7,235,697	21
40	6,904,617	3,703,946	465,587	3,238,359	7,233,689	20
41	6,906,721	3,700,899	459,762	3,241,137	7,231,681	19
42	6,908,824	3,697,854	453,938	3,243,916	7,229,672	18
43	6,910,927	3,694,811	448,114	3,246,697	7,227,662	17
44	6,913,029	3,691,770	442,291	3,249,479	7,225,651	16
45	6,915,131	3,688,730	436,467	3,252,263	7,223,639	15
46	6,917,232	3,685,692	430,644	3,255,048	7,221,627	14
47	6,919,332	3,682,656	424,821	3,257,835	7,219,614	13
48	6,921,432	3,679,622	418,999	3,260,623	7,217,601	12
49	6,923,531	3,676,590	413,177	3,263,413	7,215,588	11
50	6,925,630	3,673,559	407,355	3,266,204	7,213,574	10
51	6,927,728	3,670,530	401,533	3,268,997	7,211,559	9
52	6,929,825	3,667,503	395,711	3,271,792	7,209,543	8
53	6,931,922	3,664,478	389,889	3,274,589	7,207,527	7
54	6,934,018	3,661,454	384,067	3,277,387	7,205,511	6
55	6,936,114	3,658,432	378,245	3,280,187	7,203,494	5
56	6,938,209	3,655,412	372,423	3,282,989	7,201,476	4
57	6,940,303	3,652,394	366,602	3,285,792	7,199,457	3
58	6,942,397	3,649,377	360,780	3,288,597	7,197,438	2
59	6,944,491	3,646,362	354,958	3,291,404	7,195,418	1
60	6,946,584	3,643,349	349,136	3,294,213	7,193,398	0

min.
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Gr. 44

44 min	Sinus	Logarithmi	+ Differentiæ	− Logarithmi	Sinus	
0	6,946,584	3,643,349	349,136	3,294,213	7,193,398	60
1	6,948,676	3,640,338	343,315	3,297,023	7,191,377	59
2	6,950,767	3,637,329	337,494	3,299,835	7,189,355	58
3	6,952,858	3,634,321	331,673	3,302,648	7,187,333	57
4	6,954,949	3,631,315	325,852	3,305,463	7,185,310	56
5	6,957,039	3,628,311	320,032	3,308,279	7,183,287	55
6	6,959,128	3,625,308	314,211	3,311,097	7,181,263	54
7	6,961,216	3,622,307	308,390	3,313,917	7,179,238	53
8	6,963,304	3,619,308	302,570	3,316,738	7,177,213	52
9	6,965,392	3,616,311	296,750	3,319,561	7,175,187	51
10	6,967,479	3,613,315	290,930	3,322,385	7,173,161	50
11	6,969,565	3,610,321	285,110	3,325,211	7,171,134	49
12	6,971,651	3,607,329	279,290	3,328,039	7,169,106	48
13	6,973,736	3,604,338	273,469	3,330,869	7,167,078	47
14	6,975,821	3,601,349	267,649	3,333,700	7,165,049	46
15	6,977,905	3,598,362	261,829	3,336,533	7,163,019	45
16	6,979,988	3,595,377	256,009	3,339,368	7,160,989	44
17	6,982,071	3,592,394	250,190	3,342,204	7,158,958	43
18	6,984,153	3,589,412	244,370	3,345,042	7,156,927	42
19	6,986,235	3,586,432	238,550	3,347,882	7,154,895	41
20	6,988,316	3,583,454	232,731	3,350,723	7,152,863	40
21	6,990,396	3,580,478	226,912	3,353,566	7,150,830	39
22	6,992,476	3,577,504	221,093	3,356,411	7,148,796	38
23	6,994,555	3,574,531	215,274	3,359,257	7,146,762	37
24	6,996,634	3,571,560	209,455	3,362,105	7,144,727	36
25	6,998,712	3,568,590	203,635	3,364,955	7,142,691	35
26	7,000,789	3,565,622	197,816	3,367,806	7,140,655	34
27	7,002,865	3,562,656	191,997	3,370,659	7,138,618	33
28	7,004,942	3,559,691	186,178	3,373,513	7,136,581	32
29	7,007,018	3,556,728	180,359	3,376,369	7,134,543	31
30	7,009,093	3,553,767	174,541	3,379,226	7,132,504	30

Gr. 44

44 min	Sinus	Logarithmi	+ Differentiæ −	Logarithmi	Sinus	
30	7,009,093	3,553,767	174,541	3,379,226	7,132,504	30
31	7,011,167	3,550,808	168,723	3,382,085	7,130,465	29
32	7,013,241	3,547,851	162,905	3,384,946	7,128,425	28
33	7,015,314	3,544,895	157,087	3,387,808	7,126,385	27
34	7,017,387	3,541,941	151,269	3,390,672	7,124,344	26
35	7,019,459	3,538,989	145,451	3,393,538	7,122,303	25
36	7,021,530	3,536,038	139,632	3,396,406	7,120,261	24
37	7,023,601	3,533,089	133,814	3,399,275	7,118,218	23
38	7,025,671	3,530,142	127,996	3,402,146	7,116,175	22
39	7,027,741	3,527 197	122,178	3,405,019	7,114,131	21
40	7,029,810	3,524,253	116,359	3,407,894	7,112,086	20
41	7,031,879	3,521,311	110 541	3,410,770	7,110,041	19
42	7,033,947	3,518,371	104,723	3,413,648	7,107,995	18
43	7,036,014	3,515,432	98,904	3,416,528	7,105,949	17
44	7,038,081	3,512,495	93,086	3,419,409	7,103,902	16
45	7,040,147	3,509,560	87,268	3,422,292	7,101,854	15
46	7,042,213	3,506,626	81,450	3,425,176	7,099,806	14
47	7,044,278	3,503,694	75,632	3,428,062	7,097,757	13
48	7,046,342	3,500,764	69,824	3,430,940	7,095,708	12
49	7,048,406	3,497,835	64,006	3,433,829	7,093,658	11
50	7,050,469	3,494,908	58,178	3,436,730	7,091,607	10
51	7,052,532	3,491,983	52,360	3,439,623	7,089,556	9
52	7,054,594	3,489,060	46,543	3,442,517	7,087,504	8
53	7,056,655	3,486,139	40,726	3,445,413	7,085,452	7
54	7,058,716	3,483,219	34,908	3,448,311	7,083,399	6
55	7,060,776	3,488,301	29,090	3,451,211	7,081,345	5
56	7,062,836	3,477,385	23,273	3,454,112	7,079,291	4
57	7,064,895	3,474,470	17,455	3,457,015	7,077,236	3
58	7,066,953	3,471,557	11,637	3,459,920	7,075,181	2
59	7,069,011	3,468,645	5,818	3,462,827	7,073,125	1
60	7,071,068	3,465,735	0	3,465,735	7,071,068	0

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45 45

Admonitio.

QUum hujus Tabulæ calculus, qui plurimorum Logiſtarum ope & diligentiâ perfici debuiſſet, unius tantùm operâ & induſtriâ abſolutus ſit, non mirum eſt ſi plurimi errores in eam irrepſerint. Hiſce igitùr, ſive à Logiſtæ laſſitudine, ſive Typographi incuriâ profectis, ignoſcant, obſecro, benevoli Lectores: me enim tum infirma valetudo, tum rerum graviorum cura præpedivit, quo minùs ſecundam his curam adhiberem. Verùm, ſi hujus inventi uſum eruditis gratum fore intellexero, dabo fortaſſè brevì (Deo aſpirante) rationem ac methodum aut hunc canonem emendandi, aut emendatiorem de novo condendi, ut ita plurium Logiſtarum diligentiâ, limatior tandèm & accuratior, quàm unius operâ fieri potuit, in lucem prodeat.

Nihil in ortu perfectum.

F I N I S.

OBSERVATIONS

ON THE FOREGOING

TRACT OF LORD NAPIER,

INTITLED,

Mirifici Logarithmorum Canonis Descriptio.

BY FRANCIS MASERES, Esq. F.R.S.

CURSITOR BARON OF THE COURT OF EXCHEQUER.

Article I.

LORD Napier's design in inventing and computing this Canon, or Table, of artificial numbers, called *Logarithms*, was (as he himself informs us in the Preface to the foregoing Tract,) to facilitate the laborious arithmetical computations that occur in the various Problems of Plane and Spherical Trigonometry. For in the solution of these Problems it is very frequently necessary to employ the Rule of Three, or the Rule of Proportion, in order to find a fourth proportional to three given, or known, quantities; which could only be done by multiplying the second and third quantities together, and dividing the product by the first quantity; and these operations, when the quantities, or numbers, to be multiplied, or divided, consist of six, or seven, or more, decimal figures, are very tedious and troublesome. The magnitudes of the Sines, Tangents, and Secants of all the arches contained in the arch of a Quadrant of a circle, or an arch of 90 degrees, beginning with an arch of 1 minute of a degree, and proceeding upwards from that small arch to 90 degrees, or the arch of a whole Quadrant, by the common difference of the said small

ſmall arch of 1 minute, had been already computed to ſeven places of decimal figures by former writers on Trigonometry, and publiſhed in Tables for the uſe of Mathematicians. But the application of the numbers contained in theſe Tables to the Solution of Trigonometrical Problems, by multiplying two of theſe numbers, conſiſting of ſeven, or more, figures, into each other, and then dividing the product by another number, conſiſting likewiſe of ſeven, or more places of figures, was found to be very inconvenient. And this gave riſe to a general wiſh amongſt Mathematicians, that ſome Expedient could be found-out for leſſening the Labour of theſe operations. With this object in view Lord Napier conceived that it might be poſſible to find a ſet of new quantities, equal in number to the ſeveral Sines, Tangents, and Secants of circular arches that were ſet-down in the Trigonometrical Canon, that ſhould be ſo related to the ſaid Sines, Tangents, and Secants, that, if two of the ſaid new quantities that belonged, or correſponded, to any two of the ſaid Sines, Tangents, and Secants, ſhould be added together, the ſum thence ariſing would be equal to the new quantity that belonged, or correſponded, to the product of the multiplication of the ſaid two Sines, Tangents, or Secants, into each other; it being evident that, if ſuch new quantities could be found, and were to be expreſſed in numbers, and ſet-down in a Table annexed to the Trigonometrical Canon, we might then find the product of the multiplication of the ſaid two Sines, or Tangents, or Secants, into each other, without performing the operation of multiplying them together, by only looking-out, in the ſaid Table of new numbers annexed to the Sines, Tangents, and Secants, contained in the Trigonometrical Canon, the new number which is equal to the ſum of the two former new numbers that belonged to the two Sines, or other quantities contained in the Trigonometrical Canon, which were to have been multiplied together, and then taking, from amongſt the old numbers contained in the Trigonometrical Canon, the number to which the ſaid ſum belonged, or correſponded: and thus a laborious operation of multiplying two long numbers together would be avoided by performing the much eaſier operation of adding two ſuch numbers to each other. And, after much meditation on the ſubject, the learned and very ingenious author found, that it *was* poſſible for a ſet of new quantities, or numbers, to exiſt, that would be related to the Sines, Tangents, and Secants ſet-down in the Trigonometrical Canon, in the manner above-mentioned; and he likewiſe found-out a Method of diſcovering and computing ſuch new numbers. And, having made theſe diſcoveries, he, at firſt, diſtinguiſhed theſe new numbers, which were to be annexed to the ſeveral old numbers contained in the Trigonometrical Canon, by the name of *artificial numbers*, belonging, or annexed, to the ſaid old numbers, but afterwards called them the *Logarithms* of the ſaid old numbers, or, rather, of the ſeveral ratios of the ſaid old numbers, (which expreſſed the lengths of the ſeveral Sines, Tangents, and Secants ſet-down in the Trigonometrical Canon) to the number which expreſſed the magnitude of the radius of the circle in the ſaid Canon; this number, (which expreſſes the length of the radius

dius of the circle,) being the number to which he conſtantly compares, or refers, all the other numbers, that expreſs the lengths of the other Sines, Tangents, and Secants, contained in the Trigonometrical Canon, or which he makes the common *Conſequent*, or ſecond term, of all the ratios of which the ſaid other numbers in the ſaid Canon are *the Antecedents*, or firſt terms. This number, expreſſing the length of the radius of the circle in Lord Napier's Trigonometrical Canon, is 10,000,000, or ten millions; or, in other words, the radius of the circle is ſuppoſed, by Lord Napier, to be divided into 10,000,000, or ten millions, of ſmall and equal parts, each of which will be equal to an unit, or 1.

Lord Napier's Method of generating the Sines of circular Arches, by ſuppoſing the Radius of a Circle to decreaſe proportionally.

Art. 2. Lord Napier's Method of proving that it is poſſible for ſuch a ſet of new numbers, or Logarithms, (that are related to the old numbers, expreſſing the lengths of the Sines, Tangents, and Secants contained in the Trigonometrical Canon, in the manner above-deſcribed) to exiſt, is as follows.

He, firſt, ſuppoſes a right line of any propoſed length to be choſen for the radius of a circle. And, as this radius, when the Circle ſhall be drawn, or when only a Quadrant of it ſhall be drawn, (for all the poſſible Sines, Tangents, and Secants that can belong to any arches in the whole circle, will be found to belong alſo to ſome of the arches contained in the Quadrant of it;) will be the Sine of the whole arch of the Quadrant, or of an arch of 90 degrees, and will be greater than the Sine of any other arch, he calls it the *Sinus totus*, or *the whole Sine*, and ſeems to conſider it as the *Parent* of all the other Sines, or as the Fountain-head from which all the other Sines may be derived by a gradual diminution of it. And this diminution he conceives to be effected by the motion of a move-able point, which begins to move along the radius from one extremity of it towards ſome point that is very near the other extremity with a motion that is not uniform, but continually growing ſlower and ſlower. And he ſuppoſes the velocity with which this point moves along the ſaid radius to decreaſe at ſuch a rate that, if we divide the whole time of its motion, from the firſt beginning of it, to the inſtant at which the move-able point arrives at the other point which is near the other extremity of the radius, into any number of equal portions, or leſſer times, the whole original line, or radius of the circle, ſhall be to it's firſt decrement, or the ſmall line deſcribed by the ſaid move-able point in the firſt of the ſaid ſmall and equal

portions

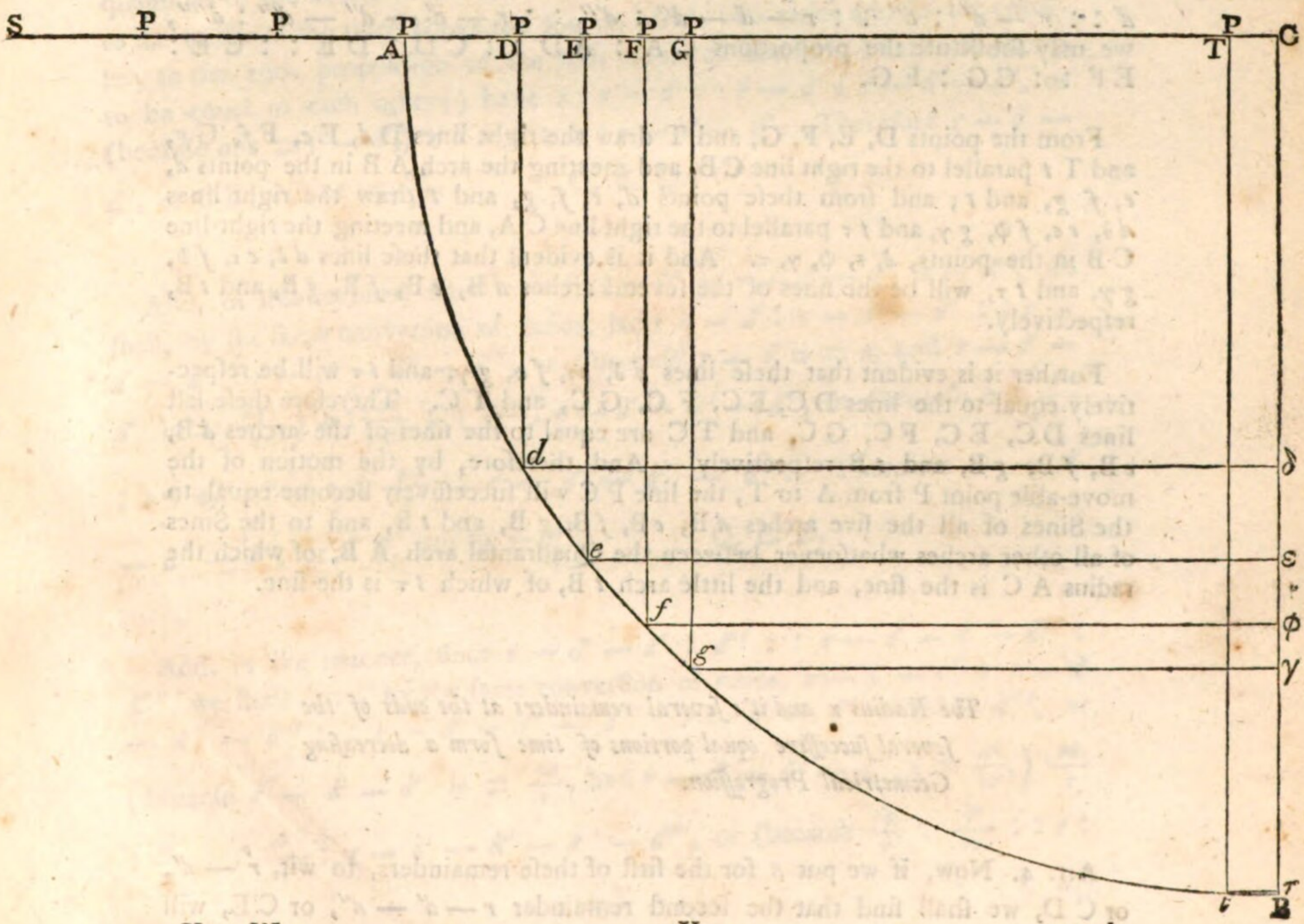
portions of time, in the ſame proportion as the whole remaining line is to the ſecond decrement, or to the ſmall line deſcribed by the ſaid move-able point in the ſecond of the ſaid ſmall and equal portions of time, and as the next remaining line at the end of the ſecond equal portion of time is to the third decrement, or the ſmall line deſcribed by the ſaid move-able point in the third equal portion of time, and as every following remainder of the original line, or radius, is to the next following decrement, or the ſmall line deſcribed by the ſaid move-able point in the next following equal portion of time; which manner of decreaſing Lord Napier calls *decreaſing proportionally*. Thus, if the whole original line, or the radius of the circle, be called r, and the firſt decrement of it be denoted by d', or the Letter d with a ſingle accent placed over it, and the ſecond decrement be denoted by d'', or d with two accents, and the third decrement be denoted by d''', or d with three accents, and the fourth decrement be denoted by d'''', or d with four accents, we ſhall have $r : d' : : r - d' : d'' : : r - d' - d'' : d'' : : r - d' - d'' - d''' : d''''$, &c. This proportional decreaſe of the original line, or radius, r, may be illuſtrated by the following figure.

Art. 3. Let the right line A C be made the radius of a circle, of which it's extremity C is the Center; and let C A B be a Quadrant of the ſaid circle. In the radius C A let the ſmall line C T be taken, which we will ſuppoſe to be only one 10,000,000th, or ten millionth, part of the whole radius C A; which is only about one 2909th part of the ſine of an arch of one minute of a degree, and one 48th part of the ſine of an arch of one ſecond minute of a degree, and very little greater than the ſine of an arch of one third minute of a degree. Then it is evident that all the ſines of the arches in this Quadrant C A B, that are not leſs than one minute of a degree, that is, all the Sines of the arches ſet-down in Lord Napier's Trigonometrical Canon, will be much greater than the line C T. And conſequently, if we ſuppoſe the ſaid line, or radius, C A to decreaſe gradually from C A to C T by the motion of a move-able point, which we will call P, along it from the point A to the point T, it will become ſucceſſively equal to all the Sines of the arches ſet-down in Lord Napier's Trigonometrical Canon, which makes mention of no ſmaller arches than an arch of one minute; and it would even become ſucceſſively equal to all the Sines of the arches that would be ſet-down in a Trigonometrical Canon that ſhould begin with an arch of one ſecond minute of a degree, and proceed upwards, by the common difference of one ſecond minute of a degree, to an arch of 89 degrees, 59 minutes, and 59 ſecond minutes of a degree, if ſuch a copious and accurate Trigonometrical Canon had been computed.

Now let us ſuppoſe the move-able point P not to move from A to T with an uniform velocity, or ſo as to deſcribe equal parts of the line A T in any equal ſucceſſive portions of time, but to move with a velocity that decreaſes continually

continually at ſuch a rate that, if the whole time of it's motion over the line A T be divided into any number of equal parts, and in the firſt of theſe leſſer portions of time the point P ſhould deſcribe the line A D, and in the ſecond of theſe leſſer portions of time it ſhould deſcribe the line D E, and in the third portion of time it ſhould deſcribe the line E F, and in the fourth portion of time it ſhould deſcribe the line F G, the whole radius C A ſhall be to it's firſt decrement A D in the ſame proportion as the firſt remainder C D is to the ſecond decrement D E, and as the ſecond remainder C E is to the third decrement E F, and as the third remainder C F is to the fourth decrement F G,

Fig. 1.



and as the fourth remainder C G is to the fifth decrement, and as every following remainder is to the following, or correſponding, decrement, till the point P co-incides with the point T: which manner of decreaſing in the radius C A from it's firſt magnitude C A to it's laſt magnitude C T, Lord Napier calls *decreaſing proportionally*.

In this figure the little line A D anſwers to d' in the notation of the foregoing article, and D E anſwers to d'', and E F anſwers to d''', and F G anſwers to d''''; and conſequently C A — A D, or C D, anſwers to $r - d'$; and C A — A D — D E — E F, or C F, anſwers to $r - d' - d'' - d'''$; and C A — A D — D E — E F — F G, or C G, anſwers to $r - d' - d'' - d''' - d''''$. And therefore, inſtead of the proportions mentioned in the laſt article, to wit, $r : d' :: r - d' : d'' :: r - d' - d'' : d''' :: r - d' - d'' - d''' : d''''$, we may ſubſtitute the proportions C A : A D : : C D : D E : : C E : E F : : C G : F G.

From the points D, E, F, G, and T draw the right lines D d, E e, F f, G g, and T t parallel to the right line C B, and meeting the arch A B in the points d, e, f, g, and t; and from theſe points d, e, f, g, and t draw the right lines $d\delta$, $e\varepsilon$, $f\varphi$, $g\gamma$, and $t\tau$ parallel to the right line C A, and meeting the right line C B in the points, δ, ε, φ, γ, τ. And it is evident that theſe lines $d\delta$, $e\varepsilon$, $f\varphi$, $g\gamma$, and $t\tau$, will be the ſines of the ſeveral arches dB, eB, fB, gB, and tB, reſpectively.

Further it is evident that theſe lines $d\delta$, $e\varepsilon$, $f\varphi$, $g\gamma$, and $t\tau$ will be reſpectively equal to the lines D C, E C, F C, G C, and T C. Therefore theſe laſt lines D C, E C, F C, G C, and T C are equal to the ſines of the arches dB, eB, fB, gB, and tB reſpectively. And therefore, by the motion of the move-able point P from A to T, the line P C will ſucceſſively become equal to the Sines of all the five arches dB, eB, fB, gB, and tB, and to the Sines of all other arches whatſoever between the Quadrantal arch A B, of which the radius A C is the ſine, and the little arch tB, of which $t\tau$ is the ſine.

The Radius r and it's ſeveral remainders at the ends of the ſeveral ſucceſſive equal portions of time form a decreaſing Geometrical Progreſſion.

Art. 4. Now, if we put a for the firſt of theſe remainders, to wit, $r - d'$, or C D, we ſhall find that the ſecond remainder $r - d' - d''$, or C E, will be

be $= \frac{aa}{r}$, and that the third remainder $r - d' - d'' - d'''$, or C F, will be $= \frac{a^3}{r^2}$, and that the fourth remainder $r - d'' - d''' - d''''$, or C G, will be $= \frac{a^4}{r^3}$, and that the ſeveral following remainders will be equal to $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, &c, and, in a general expreſſion, $\frac{a^n}{r^{n-1}}$.

For, ſince r is to d' as $r - d'$ to d'', we ſhall, by what Euclid calls *the converſion of ratios*, (which is taking the ratio of the firſt of four proportional quantities to the exceſs of the firſt above the ſecond, and the ratio of the third to the exceſs of the third above the fourth, which two ratios are, in the corollary to the 19th propoſition of the fifth book of Euclid's Elements, ſhewn to be equal to each other;) have $r : r - d' : : r - d' : r - d' - d''$, or (becauſe a is $= r - d'$,) $r : a : : a : r - d' - d''$. Therefore $r - d' - d''$, or C E, will be $= \frac{aa}{r}$. Q. E. D.

And, in like manner, ſince $r - d'$ is to d'' as $r - d' - d''$ is to d''', we ſhall, by the ſame converſion of ratios, have $r - d' : r - d' - d'' : : r - d' - d'' : r - d' - d'' - d'''$, or (becauſe $r - d'$ is $= a$, and $r - d' - d''$ is $= \frac{aa}{r}$,) $a : \frac{aa}{r} : : \frac{aa}{r} : r - d' - d'' - d'''$, or (becauſe $a : \frac{aa}{r} : r : a$,) $r : a : : \frac{aa}{r} : r - d' - d'' - d''''$. Therefore $r - d' - d'' - d'''$, or C F, will be $= \frac{a^3}{rr}$. Q. E. D.

And, in like manner, ſince $r - d' - d'' : d''' : : r - d' - d'' - d''' : d''''$, we ſhall have, by the ſame converſion of ratios, $r - d' - d'' : r - d' - d'' - d''' : : r - d' - d'' - d''' : r - d' - d'' - d''' - d''''$, or (becauſe $r - d' - d''$ is $= \frac{aa}{r}$, and $r - d' - d'' - d'''$ is $= \frac{a^3}{rr}$,) $\frac{aa}{r} : \frac{a^3}{rr} : : \frac{a^3}{rr} : r - d' - d'' - d''' - d''''$, or (becauſe $\frac{aa}{r} : \frac{a^3}{rr} : : r :$

a,) $r : a : : \frac{a^3}{rr} : r - d' - d'' - d''' - d''''$. Therefore $r - d' - d'' - d''' - d''''$, or C G, will be $= \frac{a^4}{r^3}$. Q. E. D.

And, in like manner, it may be ſhewn that the following remainders will be equal to $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, &c, to the end of the whole ſeries, or till the line C P is become equal to, or leſs than, the ſmall line C T, or the 10,000,000th part of the radius C A. So that the ſaid ſeries C A, C D, C E, C F, C G, &c, or r, a, $\frac{aa}{r}$, $\frac{a^3}{rr}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, &c, will be a Geometrical progreſſion, the terms of which will decreaſe in the conſtant proportion of r to a, or of C A to C D.

Lord Napier's Method of generating the Logarithms of the Sines of the ſeveral Arches in the Quadrant of a Circle, or the Logarithms of the Ratios of the ſaid Sines to the Radius of the Circle.

Art. 5. After thus ſuppoſing r, or the radius of a circle, to decreaſe proportionally, and thereby, in ſucceſſive equal ſmall portions of time, to become equal to the terms a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, &c, which decreaſe in the conſtant proportion of r to a, Lord Napier ſuppoſes that another right line is deſcribed by another move-able point, which moves with an uniform velocity, or ſo as to deſcribe equal lines in any equal times, how great, or how ſmall, ſo ever. And then he obſerves, that the portions of this ſecond right line which are deſcribed in the ſeveral equal portions of time in which the radius r, or C A, or C P, decreaſes from C A, or r, to C D, or a, and from C D, or a, to C E, or $\frac{aa}{r}$, and from C E, or $\frac{aa}{r}$, to C F, or $\frac{a^3}{rr}$, and from C F, or $\frac{a^3}{r^2}$, to C G, or $\frac{a^4}{r^3}$, &c, will be equal to each other, and conſequently that the portion of this ſecond line that is generated, or deſcribed,

ſcribed, while the radius r, or C A, decreaſes to any one of the terms a, $\frac{aa}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, &c, as, for example, to the term $\frac{a^5}{r^4}$, will be to the greater portion of the ſame ſecond line that is generated, or deſcribed, while the radius decreaſes to a ſecond and ſmaller term of the ſaid decreaſing ſeries, as, for example, to the term $\frac{a^9}{r^8}$, in the ſame proportion as the ratio of minority of the former of thoſe decreaſing terms, to wit, $\frac{a^5}{r^4}$, to the radius r is to the greater ratio of minority of the latter and leſſer of the ſaid decreaſing terms, to wit, $\frac{a^9}{r^8}$, to the ſame radius r; ſo that this ſecond line, which is generated, or deſcribed, by this ſecond move-able point with an uniform velocity, will increaſe exactly in the ſame manner, or at the ſame rate, as the increaſing ratio of minority of the ſucceſſive decreaſing quantities a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, &c. to the radius r, or will be analogous to, or a juſt repreſentative of, the ſaid increaſing ratio of minority, or, in the language of modern Mathematicians for the laſt 150 years (though not in the language of Euclid in his uſe of the expreſſion of *a leſſer quantity's meaſuring a greater*, in the firſt definition of the fifth book of his Elements,) will be *a meaſure* of the ſaid increaſing ratio of minority.

Art. 6. The ſeveral portions of the above-mentioned ſecond right line, that is deſcribed by a point moving with an uniform velocity, will (as Lord Napier has obſerved,) be always proportional to, or meaſures of, the ratios of minority of their correſponding terms in the firſt, or decreaſing, line, to the radius, or firſt magnitude of the ſaid decreaſing line, whatever be the uniform velocity with which the ſaid ſecond line is ſuppoſed to be deſcribed. But, as it was neceſſary to pitch upon ſome particular velocity for the deſcription of this ſecond line, Lord Napier thought fit to chuſe for ſuch uniform velocity, the velocity which the firſt move-able point, by which the decreaſing line is deſcribed, had in the firſt inſtant of it's motion, and made this ſuppoſition the ground-work of his computation of the magnitudes of the ſeveral portions of this ſecond line that are correſpondent to the ſeveral terms C D, C E, C F, C G,

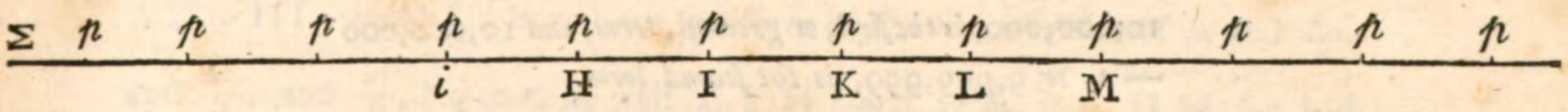
C G, &c, or a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, &c, of the decreasing Geometrical Series r, a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, &c, or are measures, or Logarithms, of their respective ratios to the radius r, or the first term of the said Series. This appears from our author's definition of *the Logarithm of a Sine*, which is his 6th definition, towards the bottom of page 485, and is in these words; *Logarithmus ergò cujusque Sinûs est numerus quàm proximè definiens lineam quæ æqualitèr crevit intereà dum sinûs totius linea proportionalitèr in sinum illum decrevit, existente utroque motu synchrono atque, [in] initio, æquiveloce.*

This second right line, which is described with an uniform velocity that is equal to the first velocity of the point P, by which the decreasing line C P in the foregoing figure was described, may be represented by the following figure.

Let H p be an indefinite right line, along which a move-able point called p moves with an uniform velocity that is equal to the first velocity of the former move-able point P mentioned in Art. 3. And let H I, I K, K L, and L M be the several successive portions of this line that are described by the said point p in the several equal portions of time in which the line C P decreases from C A to C D, or from r to a, and from C D, or a, to C E, or $\frac{a^2}{r}$, and from C E, or $\frac{a^2}{r}$, to C F, or $\frac{a^3}{r^2}$, and from C F, or $\frac{a^3}{r^2}$, to C G, or $\frac{a^4}{r^3}$, &c. Then will the said lines H I, I K, K L, and L M, (being described in equal times by the point p moving with an uniform velocity,) be all equal to each other. And, because the velocity of the point p is equal to the first velocity of the point P, which moves with a decreasing velocity, each of the said equal lines H I, I K, K L, and L M will be nearly equal to, but a little greater than, the little line A D, or d, which is described by the point P in the first of the said equal times. And therefore the number of the equal little lines H I, I K, K L, L M, &c, that will be contained in any portion of the indefinite line H p that is described by the point p in the same time in which the decreasing line C P decreases by the motion of the first point p, from it's first magnitude C A, or r, to any following term of the Geometrical Series r, a, $\frac{a^2}{r}$,

$\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, &c, denoted by the general expreſſion $\frac{a^n}{r^{n-1}}$, will be equal to the number of the *ratiunculæ*, or ſmall ratios of minority equal to the ratio of a to r, that are contained in the greater ratio of minority of $\frac{a^n}{r^{n-1}}$, to r, that is, to the number n, and therefore will be a juſt repreſentative of, or conſtantly proportional to, the ſaid ratio of $\frac{a^n}{r^{n-1}}$, or will be the *Logarithm* of the ſaid ratio; which word is in truth derived from it's having this property of expreſſing the number of the ſaid *ratiunculæ* of a to r (τὸν ἀριθμὸν τῶν λόγων) contained in the ſaid ratio of $\frac{a^n}{r^{n-1}}$ to r.

Fig. 2.



The length of the Sines of all the Arches in the Quadrant of a Circle, and the lengths of all their Logarithms, are expreſſed in whole Numbers in the foregoing Table of Lord Napier.

Art. 7. It has been obſerved above in Art. 1. that Lord Napier, in his Trigonometrical Canon, does not ſuppoſe the radius of the circle, or the Sine of the arch of the whole Quadrant, to be denoted by unity, or 1, (as is often done by other Trigonometrical writers,) but to be denoted by 10,000,000, or ten millions, and conſequently ſuppoſes the 10,000,000nth part of the radius to be denoted by 1. And by this means he avoids all fractions in his Table of Sines and Tangents and their Logarithms; which he carries only to ſeven places of figures, neglecting every quantity that is leſs than 1, or the 10,000,000nth part of the radius, as too ſmall to be worth attending to. This appears from the

the following words of Lord Napier in the foregoing tract, in page 485, to wit, *Ut ſit ſemi-diameter, ſeu Sinus totus, rationalis numerus* 10,000,000 ; *erit Sinus* 45 *graduum radix quadrata* [*numeri*] 200,000,000,000,000, *quæ ſurda, ſeu irrationalis, et numero inexplicabilis eſt, atque inter terminos* 7,071,067, [*ipſâ*] *minorem, et* 7,071,068 [*ipſâ*] *majorem, includitur. Ab horum itaque utrovis non differt unitate. Surdus itaque Sinus ille* 45 *graduum quàm proximè dicitur definiri et explicari quùm per numeros integros* 7,071,067 *et* 7,071,068, *neglectis fractionibus, definitur. In magnis etenim numeris ex fragmentis unitatis ſpretis nullus error ſenſibilis emergit.* In which paſſage he calls the two numbers 7,071,067 and 7,071,068 *numeros integros*, or *whole numbers*, and conſiders either of them as being equal to the Sine of an arch of 45 degrees, becauſe it differs from the true Sine of that arch by leſs than an unit.

The decreaſing Geometrical Progreſſion which Lord Napier has choſen for the foundation of the Logarithms contained in the foregoing Tract, is that of which the high number, 10,000,000, *is the firſt, or greateſt, term, and* 10,000,000 — 1, *or* 9,999,999, *is the ſecond term.*

Art. 8. Lord Napier then ſuppoſes the firſt decrement of the decreaſing line C P in fig. 1, to wit, the decrement A D, or d', to be equal to 1, or the 10,000,000nth part of the radius, and conſequently the ſecond term of the above-mentioned decreaſing Geometrical Progreſſion, to wit, the term C D, or $r - d'$, or a, to be equal to 10,000,000 — 1, or 9,999,999, and the third term of it, or C E, or $r - d' - d''$, or $\frac{aa}{r}$, to be $= \frac{\overline{9,999,999}^{2}}{10,000,000}$, and the fourth term of it, or C F, or $r - d' - d'' - d'''$, or $\frac{a^3}{r^2}$, to be $= \frac{\overline{9,999,999}^{3}}{\overline{10,000,000}^{2}}$, and the fifth term of it, or C G, or $r - d' - d'' - d''' - d''''$, or $\frac{a^4}{r^3}$, to be $= \frac{\overline{9,999,999}^{4}}{\overline{10,000,000}^{3}}$, and, in general, the nth term of the ſaid Geometrical Progreſſion, including the firſt term r, or 10,000,000, to be $= \frac{\overline{9,999,999}^{n-1}}{\overline{10,000,000}^{n-2}}$. And he ſuppoſes H K, or the firſt increment of the increaſ-

ing

ing line H*p*, in fig. 2, (which is described by the point *p* with an uniform velocity equal to the first velocity of the point P, in the same time as the first decrement A D, or *d'*, or 1, of the decreasing line C P, in fig. 1, is described by the point P,) to be equal to the said decrement A D, or *d'*, or 1; though it will always be a little greater than the said decrement, on account of the decrease of the velocity of the point P during the description of the decrement D. But, as the decrease of this velocity is exceedingly small during the description of so small a decrement as 1, or the 10,000,000nth part of the radius of the circle, the value of the little line H I that is grounded on this supposition will be so very nearly equal to it's true value that Lord Napier, in computing the Logarithms set-down in the foregoing Table, thought fit to neglect their difference, and consider them as being equal to each other. And, if the difference had been very considerable, so that A D, or this assumed value of H I, had been less than the true value of H I in the proportion of 9 to 10, or of 8 to 9, or of 7 to 8, or in any greater ratio of minority, the values of H I, H K, H L, H M, &c, derived from such a supposition of the proportion of A D to H I, would still have been measures of the several ratios of *a*, and $\frac{a}{r}$, and $\frac{a^3}{r^2}$, and $\frac{a^4}{r^3}$, &c to *r*, as well as if H I and H K, H L, H M, &c had been taken of their true magnitudes, which would have arisen from their being described by the point *p* with a velocity exactly equal to the first velocity of the point P in describing the line A T; as has been already observed in Art. 6. They would therefore, even in that case, have been as much intitled to the name of *Logarithms* of the said ratios of minority of *a* and $\frac{a^2}{r}$, and $\frac{a^3}{r^2}$, and $\frac{a^4}{r^3}$, &c to *r*, as if the lines H I, H K, H L, H M, &c had been taken of their true magnitudes according to Lord Napier's definition of *the Logarithms of his System*, mentioned above in Art. 6. in the words, *existente utroque motu synchrono atque, [in] initio, æquivoloce*; but they would not, in that case, have been Logarithms of the said ratios *in Lord Napier's System*, but in a system in which all the Logarithms of given ratios would have been less than the Logarithms of the same ratios in Lord Napier's System in a certain given proportion: but Lord Napier thought the decrement A D to be so very nearly equal to the true value of H I that he considered the Logarithms resulting from the supposition of their equality, and which are those which he has

computed and set-down in the Table printed at the end of the foregoing Tract, as being not only true Logarithms of the said ratios of a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, &c to r, or of 9,999,999, $\frac{\overline{9,999,999}^{2}}{10,000,000}$, $\frac{\overline{9,999,999}^{3}}{\overline{10,000,000}^{2}}$, $\frac{\overline{9,999,999}^{4}}{\overline{10,000,000}^{3}}$, &c to 10,000,000, but Logarithms of the same ratios according to his own definition of them just now mentioned, which ends with the words *existente motu synchrono et*, [*in*] *initio, æquivelcce.*

If we were to compute the true magnitude of H I, when the radius C A, or r, is divided into 10,000,000 equal parts, and the first decrement A D is taken equal to one of those parts, or to 1, it would be found that the said true value of H I would be (= A D + 0.000,000,000,000,01 &c × A D = 1 + 0.000,000,100,000,01 &c × 1) = 1.000,000,100,000,01 &c; which exceeds A D, or 1, by so small a difference 0.000,000,100,000,01 &c that the exact Logarithm of any given Sine in the Trigonometrical Canon, derived from this supposition that H I is = 1.000,000,100,000,01 &c, (which is the true magnitude of it required by Lord Napier's Definition, *existente utroque motu synchrono et, in initio, æquiveloce,*) would co-incide with the Logarithm of the same Sine derived from the second supposition that H I is = A D, or 1, (which is the supposition from which the Logarithms set-down in the foregoing Table of Lord Napier are actually derived,) in the first seven places of figures. But Lord Napier informs his readers that, in the foregoing Table of Sines and their Logarithms, all the parts less than 1, or the 10,000,000th part of the radius of the circle, are rejected, as too small to be worth attending-to. And therefore, although the supposition that H I is equal to A D, or 1, is, in strictness, only an approximation to the true value of H I according to Lord Napier's definition, (*existente utroque motu synchrono et, in initio, æquiveloce,*) yet it is so near an approximation to the said true value of H I that it may justly be considered as being equivalent to it, and fit to be substituted for it, when the degree of exactness that is aimed-at in the Computation is only that of seven places of figures, which Lord Napier has adopted for his Table of Sines and Logarithms. And this, I presume, was Lord Napier's reason for considering the Logarithms set-down in his Table (which are derived from the supposition that H I is = A D, or 1,) as being the same with the Logarithms

that

that would have been derived from the ſuppoſition that H I had been taken of it's true magnitude according to his aforeſaid definition, or equal to 1.000, 000,100,000,01 &c.

Art. 9. As Lord Napier makes the line H I in fig. 2, (which is the Logarithm of the ſmall ratio of C D to C A in fig. 1, or of *a* to *r*, or of 9,999,999 to 10,000,000,) equal to the firſt decrement A D, or d', or 1, of the line C P, it is evident that the following increments of the ſecond line H *p*, to wit, I K, K L, L M, &c (which are deſcribed by the point *p* in the ſame equal times as the following decrements of the line C P, to wit, D E, E F, F G, &c, are deſcribed by the point P, and which are, each of them, equal to the firſt increment H I,) will be, each of them, equal alſo to A D, or 1. And conſequently the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c will expreſs the lengths of the ſeveral portions of the ſaid ſecond line H *p*, to wit, H I, H K, H L, H M, &c that correſpond to, or are generated, by the motion of the point *p*, in the ſame times with, the ratios of the 2d, 3d, 4th, 5th, 6th, 7th, 8th, and 9th, &c terms of the aforeſaid decreaſing Geometrical Progreſſion

$$10{,}000{,}000,\ 9{,}999{,}999,\ \frac{9{,}999{,}999^{2}}{10{,}000{,}000},\ \frac{9{,}999{,}999^{3}}{10{,}000{,}000^{2}},\ \frac{9{,}999{,}999^{4}}{10{,}000{,}000^{3}},\ \frac{9{,}999{,}999^{5}}{10{,}000{,}000^{4}},$$

$$\frac{9{,}999{,}999^{6}}{10{,}000{,}000^{5}},\ \frac{9{,}999{,}999^{7}}{10{,}000{,}000^{6}},\ \frac{9{,}999{,}999^{8}}{10{,}000{,}000^{7}},\ \frac{9{,}999{,}999^{9}}{10{,}000{,}000^{8}},$$

&c, to the firſt term 10,000,000, or that are meaſures, or Logarithms, of the ſaid ratios. And theſe natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c are alſo the Indexes of the powers of the ſecond term of the Series, to wit, the number 9,999,999, in the ſaid ſecond term 9,999,999, and in the numerators of the fractions that form the third and fourth and fifth, and other following terms of the ſaid Series. So that the Indexes of the ſeveral powers of the number 9,999,999 in the ſecond and other following terms of this Geometrical Series are Logarithms of the ratios of the ſeveral terms to which they reſpectively belong, to 10,000,000, the firſt term of the ſaid Series. And this is as clear a definition, or deſcription, of Logarithms as can well be given.

The Logarithms of the Sines in the Trigonometrical Canon may be derived from the Logarithms of the terms of the foregoing Geometrical Progreſſion, of which the two firſt terms are 10,000,000, *and* 9,999,999.

Art. 10. The terms of the foregoing Geometrical Progreſſion 10,000,000, 9,999,999, $\frac{9,999,999^{2}}{10,000,000}$, $\frac{9,999,999^{3}}{10,000,000^{2}}$, &c are ſo numerous, and every three, or four, of them that follow each other in immediate ſucceſſion will be ſo nearly equal to each other, that ſome one of the terms of this progreſſion will be very nearly equal to any one of the Sines of all the aforeſaid circular arches, beginning with an arch of 1 minute, and proceeding upwards by the common difference of 1 minute to an arch of 89 degrees and 59 minutes, which are contained in the Trigonometrical Canon adopted by Lord Napier. And conſequently the Logarithm of the ſaid term of the ſaid Geometrical Progreſſion which comes neareſt to an equality with any given Sine in the ſaid Trigonometrical Canon may be conſidered as being alſo the Logarithm of the ſaid given Sine. And thus, when we ſhall have computed the lengths of all the terms in the ſaid decreaſing Geometrical Progreſſion, of which the two firſt terms are 10,000,000 and 9,999,999, and ſhall have ſet-down in a Table annexed to the computed values of the ſaid terms the correſponding terms of the Arithmetical Progreſſion 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, which are the Logarithms of the ratios of the 2nd, 3d, 4th, 5th, 6th, 7th, 8th, 9th, and 10th, &c terms of the ſaid decreaſing progreſſion to it's firſt term 10,000,000, or the radius of the circle, we ſhall thereby have obtained the Logarithms of the ratios of the Sines of all the arches in the Quadrant from 1 minute to 89°, 59′, to the radius of the circle, or to 10,000,000.

Of the Logarithm of the Ratio of 1 *to* 10 *in Lord Napier's Syſtem.*

Art. 11. According to this Syſtem of Logarithms the Logarithm of the ratio of 1 to 10, or of 1,000,000 to 10,000,000, will be found by looking-out in

in the Table of Sines the Sine that is neareſt to 1,000,000; which will be found to be 998,986, which is the Sine of an arch of 5 degrees and 44 minutes. And the Logarithm ſet-down in the foregoing Table of Lord Napier as the Logarithm correſponding to this Sine, or as expreſſing the magnitude of it's ratio to the radius 10,000,000, is 23,035,985. Therefore the number 23,035,985 is nearly equal to the Logarithm of the ratio of 1 to 10. This Sine of 5°, 44′, to wit, 998,986, is, however, ſomewhat leſs than 1,000,000; and conſequently it's ratio to 10,000,000 is a ſomewhat greater ratio of minority than the ratio of 1,000,000 to 10,000,000, and therefore it's Logarithm 23,035,985 will be ſomewhat greater than the Logarithm of the ratio of 1,000,000 to 10,000,000, or of 1 to 10. But the next greater Sine in this Table of Lord Napier (which exhibits the Sines only to every minute of the Quadrant,) is 1,001,881, or the Sine of 5°, 45′, which is greater than 1,000,000; and conſequently it's ratio to the radius 10,000,000 is a leſs ratio of minority than the ratio of 1,000,000 to 10,000,000, or than the ratio of 1 to 10. Therefore the Logarithm ſet-down in this Table as correſponding to 1,001,881, or the Sine of 5°, 45′, which is 23,007,056, will be leſs than the Logarithm of the ratio of 1,000,000 to 10,000,000, or of the ratio of 1 to 10. And conſequently the arch of which the Sine is exactly equal to 1,000,000, or to the 10th part of the radius 10,000,000, will be greater than 5°, 44′, but leſs than 5°, 45′: and it will be found (if I am not miſtaken in my computation of it,) to be very nearly equal to 5°, 44′, 21″. And the Logarithm, in this Syſtem of Lord Napier, that correſponds to it, or that is the Logarithm of the ratio of 1 to 10, is, according to Lord Napier's calculation of it, found to be 23,025,842. See above, page 494.

Of the Logarithm of the Radius of a Circle in Lord Napier's Syſtem of Logarithms.

Art. 12. In every Syſtem of Logarithms, or meaſures, of the ratios of different quantities to each other, the Logarithm of the ratio of equality, or of the ratio of any one quantity to another that is exactly equal to it, muſt be 0; becauſe, if a ratio of equality be either added to, or ſubtracted from, a ratio of inequality, it will make no change in the ſaid ratio of inequality. Thus, for example, if to the ratio of a line called A, containing 3 inches, to a ſecond line called B, containing 2 inches, we add the ratio of the line B, containing 2 inches, to a third line called C, containing alſo 2 inches, the ratio of A, the firſt line to C, the third line, (which is compounded of the two ratios of A to B, and of B to C, or is equal to the ſum of the ſaid two ratios,) will be the ratio of 3 inches to 2 inches, which is the ratio of A, the firſt line, to B, the

ſecond

second line: so that no change is effected in the magnitude of the first ratio of A to B, or of 3 inches to 2 inches, by adding to it the second ratio of B to C, or of 2 inches to 2 inches. Whenever therefore the antecedent term, or first term, of a ratio is equal to it's consequent, or second term, such a ratio has no magnitude, and consequently it's Logarithm will be = 0.

Now Lord Napier, in forming his System of Logarithms, thought fit to make the Logarithm of the radius of a circle, or the great number 10,000,000, equal to 0; or, in other words, he made the radius, or 10,000,000, the common consequent of all the ratios of the Sines, Tangents, and Secants in the Trigonometrical Canon, of the magnitudes of which his Logarithms were intended to be the measures; so that the Logarithm of the radius, or 10,000,000, being the Logarithm of the ratio of the radius to the radius, or of 10,000,000 to 10,000,000, or of a ratio of equality, must be = 0. His reason for making the Logarithm of the radius equal to 0 was to avoid the frequent occasions that would otherwise arise, in solving Trigonometrical Problems, of adding, or subtracting, the Logarithm of the radius, to, or from, the Logarithms of other Sines; which additions and subtractions are operations that cannot take place when the Logarithm that is to be so added, or subtracted, is equal to 0. His own words upon this subject in the last paragraph, intitled, *Admonitio*, of the first chapter of the first book of the foregoing Tract, in page 486, are as follows: *Erat quidem initio liberum cuilibet Sinui, aut quantitati, nullum, seu* 0, *pro Logarithmo attribuisse. Sed præstat id, præ cæteris, Sinui toti accommodâsse: ne unquam in posterum vel minimam molestiam parturiret* [*vel potiùs, pareret*] *nobis additio et subtractio ejus Logarithmi, in omni calculo frequentissimi.*

Of the changes that would take place in Lord Napier's System of Logarithms, if the Radius of the Circle were denoted by unity, or 1, *instead of being denoted by the high number* 10,000,000.

Art. 13. It may, however, be observed that this advantage, "of having 0 for the Logarithm of the radius, and thereby avoiding the additions and subtractions of a number consisting of seven, or eight, figures, every time a Sine, or other quantity in the Trigonometrical Canon, was to be multiplied, or divided, by the radius," might have been obtained as well by denoting the radius by 1 as by denoting it by 10,000,000, and thereby taking the 10,000,000th part of the radius for 1, or the unit of the Table. And then 1 would have been the common consequent, or second term, of all the ratios of the Sines, Tangents, and Secants in the Trigonometrical Canon to the radius,

dius, or of all the ratios of which the ſaid Sines, Tangents, and Secants, would have been the antecedents, or firſt terms. And, in that caſe, the Logarithms of Lord Napier's Syſtem would have borne a greater reſemblance than they do at preſent, (though *not a compleat* reſemblance,) to the Logarithms of Mr. Briggs's Syſtem, in which the Logarithms which are ſet-down near the ſeveral natural numbers 2, 3, 4, 5, 6, 7, 8, 9, &c are conſidered as the Logarithms of the ratios of thoſe ſeveral numbers to unity, or 1. And this change in Lord Napier's Syſtem of Logarithms by denoting the radius of the circle by 1, inſtead of the great number 10,000,000, would, perhaps, have been found very convenient: but then all the Sines in the Quadrant, (being leſs than the radius,) would have been denoted by decimal fractions inſtead of whole numbers; and this he, probably, wiſhed to avoid. But, as Lord Napier (whatever might be his reaſon for it,) thought fit to denote the radius by the high number 10,000,000, which is the ſeventh power of 10, the Logarithm of 1, (being the Logarithm of the ratio of 1 to 10,000,000, or $\overline{10}^7$, muſt be equal to 7 times the Logarithm of the ratio of 1 to 10, and conſequently muſt be equal to 7 times 23,025,842, or to 161,180,894.

Art. 14. If Lord Napier had denoted the radius by unity, or 1, inſtead of the high number 10,000,000, the Sine of 5°, 44′, inſtead of being equal to the whole number 998,986, would have been equal to $\frac{998,986}{10,000,000}$, or the decimal fraction 0.099,898,6; and the Sine of 5°, 45′, inſtead of being equal to the whole number 1,001,881, would have been equal to $\frac{1,001,881}{10,000,000}$, or the decimal fraction 0.100,188,1; and the ſecond term of the Geometrical Progreſſion, deſcribed above in Art. 4, of which the radius 10,000,000 is the firſt term, inſtead of being equal to 10,000,000 — 1, or the high number 9,999,999, would have been equal to $\frac{10,000,000 - 1}{10,000,000}$ $(= 1 - \frac{1}{10,000,000} = 1 - 0.000,000,1)$ or the decimal fraction 0.999,999,9; and the Logarithm of the ratio of the ſaid ſecond term 0.999,999,9 to the firſt term, 1, of the ſaid progreſſion (being equal to the exceſs of the firſt term 1 above the ſaid ſecond term 0.999,999,9,) would have been equal to the ſmall decimal fraction 0.000,000,1, inſtead of being equal to 1; and all the other Logarithms of the ratios

ratios of the ſecond and other following terms of the ſaid Geometrical progreſſion to it's firſt term 1 would have been leſs than they were before in the ſame proportion of 1 to 10,000,000; and conſequently the Logarithm of the ratio of 1,000,000 to 10,000,000, or of 1 to 10, inſtead of being equal to the high number 23,025,842, would have been equal to $\frac{23,025,842}{10,000,000}$, or to the mixt number 2.302,584,2. And ſo, I obſerve, Dr. Halley, in his Diſcourſe on Logarithms (which has been reprinted in the ſecond volume of this Collection of Mathematical Tracts, called *Scriptores Logarithmici*,) and many other writers on Logarithms, have repreſented it to be. But, by the peruſal of this original Tract of Lord Napier upon this ſubject (which I had never examined till lately,) I find that he makes the Logarithm of this ratio of 1 to 10 to be the whole number 23,025,842.

Of the contrariety of the Logarithms of the Secants of circular Arches to the Logarithms of their Sines; the former being meaſures of ratios of majority, and the latter being meaſures of ratios of minority.

Art. 15. As all the Secants of the arches in a Quadrant of a circle, (however ſmall the arches may be,) are greater than the radius of the circle, their ratios to the radius will be of an oppoſite kind to the ratios of the Sines of the ſame arches to the radius; the former being ratios of *majority*, and the latter being ratios of *minority*. And therefore Lord Napier thought fit to diſtinguiſh theſe ratios and their meaſures, or Logarithms, into two oppoſite claſſes, calling the Logarithms of the one claſs *Abundant*, and thoſe of the other claſs *Defective*; which denominations anſwer pretty much to the terms *Affirmative*, or *Poſitive*, and *Negative*, which are uſed for the like purpoſes by modern Algebräiſts. And, as the Sines of circular arches occur more frequently in Trigonometrical calculations than the Secants, he thought it beſt to give the title of *Abundant*, or *Affirmative*, or *Poſitive*, to the Logarithms of the Sines, or of their ratios to the radius, and to prefix to them the ſign +, though theſe ratios are ratios of minority, and to call the Logarithms of the Secants, and likewiſe the Logarithms of ſuch of the Tangents as are greater

than

than the radius, that is, the Logarithms of all the Tangents of arches that are greater than 45 degrees, or half the arch of a Quadrant, *Defective*, or *Negative* Logarithms, and to prefix to them the ſign —, though the ratios of which they are the meaſures, or repreſentatives, are ratios of majority. And he obſerves that this was entirely a matter of choice, as theſe names of *Abundant* or *Affirmative*, or *Poſitive*, and *Defective* or *Negative*, might have been applied to theſe different claſſes of Logarithms in a contrary order, and he declares that he gave the name of *Abundant*, or *Affirmative*, or *Poſitive*, to the Logarithms of the Sines rather than to the Logarithms of the Secants for the reaſon that has been mentioned. But in Mr. Briggs's Syſtem of Logarithms, (which was invented a year, or two, after Lord Napier's, and was conſidered by moſt Mathematicians, and even by Lord Napier himſelf, as an improvement of the former ſyſtem, and which is now in general uſe,) the contrary rule is obſerved, and the Logarithm of a ratio of majority is conſidered as *Affirmative*, or *Poſitive*, and has the ſign + prefixed to it, and the Logarithm of a ratio of minority is conſidered as *Negative*, and has the ſign — prefixed to it.

Of the manner of generating the Secants of circular Arches and the Logarithms of the ſaid Secants, by the motions of two move-able points, which ſeems to be ſuppoſed by Lord Napier in the foregoing Tract, but is not expreſsly deſcribed by him.

Art. 16. Though Lord Napier has told us that the Logarithms of the Secants of circular arches, (being the Logarithms of the ratios of the ſaid Secants to the radius of the circle, which are ratios of majority,) are to be conſidered as of an oppoſite kind to the Logarithms of the Sines of the ſaid arches, (or the Logarithms of the ratios of the ſaid Sines to the radius, which are ratios of minority,) and that he therefore calls the ſaid Logarithms of the Secants *Defective*, or *Negative*, Logarithms, he has not deſcribed to his readers the manner in which he conceives that the ſaid Secants and their Logarithms may be deſcribed by the motion of points along certain right lines, as he has done with reſpect to the Sines and their Logarithms. But this omiſſion may be ſupplied in the following manner.

The Generation of the Secants.

We have already ſuppoſed the radius C A of the circle of which C is the center, and C A B a Quadrant, in Fig. 1. to be diminiſhed gradually from it's firſt magnitude C A to the very ſmall line C T, which is only one 10,000,000nth part of C A, by the motion of the point P from C to T with a velocity that decreaſes at ſuch a rate that, if the whole time of the motion from A to T be divided into any number of ſmall and equal parts, and the portions of the line A T deſcribed in the firſt four of the ſaid equal portions of time be A D, D E, E F, and F G, we ſhall have C A to A D as C D to D E, and as C E to E F, and as C F to F G; and ſo on, as to the following decrements of C A deſcribed by the ſaid point P in the following equal portions of time; which manner of decreaſing of the radius C A by the motion of the point P Lord Napier calls *decreaſing proportionally*. And by this means the remaining line C P after the loſs of the ſeveral decrements A D, A D + D E, A D + D E + E F, and A D + D E + E F + E G, &c, or A D, A E, A F, A G, &c, becomes ſucceſſively equal to the Sines of all the arches in the quadrantal arch A B that are not leſs than the little arch *t* B, of which the Sine *t τ* is equal to C T.

Now let us ſuppoſe the radius C A, in the ſame Fig. 1, to be produced beyond the point A to a very great extent, and ſo as to become equal to the Secant of an arch of 89 degrees and 59 minutes. And let us ſuppoſe the move-able point P to be again placed in the point A, and to move along this new right line, (which we will call the line A S,) in a direction contrary to the direction of it's former motion from A to T, and with an increaſing velocity inſtead of a decreaſing velocity. And let the velocity with which the point P begins it's motion from A in the new direction A S be ſuppoſed to be equal to the velocity with which it began it's motion from A in the former direction A T; and let the rate of it's increaſe be ſuch that, if we divide the time of it's deſcribing the whole line A S, (which, together with the radius C A, is equal to the Secant of 89°, 59′,) into any number of ſmall and equal parts, and denote the portion of the line A S that is deſcribed in the firſt of theſe ſmall and equal times, or the firſt increment of the radius C A generated by this new motion of the point P, by I′, or the Letter I with a ſingle accent placed over it; and the portion of the line A S that is deſcribed in the ſecond of theſe ſmall and equal times, or the ſecond increment of the radius C A generated by this new motion of the point P, by I″, or the Letter I with two accents placed over it; and the third increment of the radius C A generated

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in the third equal portion of time by I'''; and the fourth increment of it by I''''; the radius C A, or r, ſhall be to the firſt increment I' in the ſame proportion as C A + I', or $r + I'$, is to I'', and as $r + I' + I''$ is to I''', and as $r + I' + I'' + I'''$ is to I''''; which manner of increaſing of the radius C A, or r, by this new motion of the point P is called by Mathematicians *increaſing proportionally*. Then it is evident that the line C A + A P, or C A P, will ſucceſſively become equal to all the Secants of the ſeveral arches in the Quadrantal arch A B that are not greater than an arch of 89°, 59′. And this I conceive to be the manner in which Lord Napier ſuppoſed all the Secants of the arches in the the Trigonometrical Canon to be generated, when he was, in this Tract, deſcribing the nature of their Logarithms.

Art. 17. In this manner of generating the ſeveral Secants C A + A P, or C A P, of the arches of the quadrant C A P by the motion of the move-able point P from the point A to the point S, the radius C A and the Secants C A + I', C A + I' + I'', C A + I' + I'' + I''', and C A + I' + I'' + I''' + I'''', &c, that are generated in ſucceſſive equal portions of time, will be to each other in the ſame conſtant proportion of the radius C A to the firſt Secant C A + I', or will form an increaſing Geometrical Progreſſion, of which the common ratio will be that of C A to C A + I', or, if we put b = C A + I', or $r + I'$, will be that of r to b. This may be proved in the following manner.

Since C A is to I' as C A + I' is to I'', or r is to I' as $r + I'$ is to I'', it follows, *invertendo*, that I' will be to r as I'' is to $r + I'$, and therefore, *componendo*, that $r + I'$ will be to r as $r + I' + I''$ is to $r + I'$, and, *invertendo*, that r will be to $r + I'$ as $r + I'$ is to $r + I' + I''$; that is, r will be to b as b is to $r + I' + I''$. Therefore $r + I' + I''$ will be $= \frac{bb}{r}$. Q. E D.

Secondly, ſince C A + I' is to I'' as C A + I' + I'' is to I''', or $r + I'$ is to I'' as $r + I' + I''$ is to I''', we ſhall have, *invertendo*, $I'' : r + I' : : I''' : r + I' + I''$, and, *componendo*, $r + I' + I'' : r + I' : : r + I' + I'' + I''' : r + I' + I''$, and, *invertendo*, $r + I' : r + I' + I'' : : r + I' + I'' : r + I' + I'' + I'''$. But $r + I'$ is $= b$, and $r + I' + I''$ is $= \frac{bb}{r}$. There-

fore we ſhall have $b : \frac{bb}{r} : : \frac{bb}{r} ; r + I' + I'' + I'''$, or (becauſe $b : \frac{bb}{r} : : r : b$,) $r : b : : \frac{bb}{r} : r + I' + I'' + I'''$. Therefore $r + I' + I'' + I'''$ will be $= \frac{b^3}{r^2}$. Q. E. D.

Thirdly, ſince $r + I' + I'' : I''' : : r + I' + I'' + I''' : I''''$, we ſhall have, *invertendo*, $I''' : r + I' + I'' : : I'''' : r + I' + I'' + I'''$, and, *componendo*, $r + I' + I'' + I''' : r + I' + I'' : : r + I' + I'' + I''' + I'''' : r + I' + I'' + I'''$, and, *invertendo*, $r + I' + I'' : r + I' + I'' + I''' : : r + I' + I'' + I''' : r + I' + I'' + I''' + I''''$. But $r + I' + I''$ is $= \frac{bb}{r}$, and $r + I' + I'' + I'''$ is $\frac{b^3}{r^2}$. Therefore $\frac{bb}{r} : \frac{b^3}{r^2} : : \frac{b^3}{r^2} : r + I' + I'' + I''' + I''''$, or (becauſe $\frac{bb}{r} : \frac{b^3}{r^2} : : r : b$,) $r : b : : \frac{b^3}{r^2} : r + I' + I'' + I''' + I''''$. Therefore $r + I' + I'' + I''' + I''''$ will be $= \frac{b^4}{r^3}$. Q. E. D.

Therefore the five firſt terms of the above-mentioned increaſing Series, to wit, the terms r, $r + I'$, $r + I + I''$, $r + I' + I'' + I'''$, and $r + I' + I'' + I''' + I''''$, will be r, b, $\frac{b^2}{r}$, $\frac{b^3}{r^2}$, and $\frac{b^4}{r^3}$, which form an increaſing Geometrical Progreſſion, of which the common ratio is that of r to b, or to $r + I'$. And, in like manner, the following terms of the increaſing Series of which the two firſt terms are r and $r + I'$, or b, that are generated in the following equal portions of time by the motion of the ſaid point P in it's new direction A S, will be equal to $\frac{b^5}{r^4}$, $\frac{b^6}{r^5}$, $\frac{b^7}{r^6}$, $\frac{b^8}{r^7}$, $\frac{b^9}{r^8}$, &c, or will be members of the ſame Geomertrical Progreſſion of which the common ratio by which the terms increaſe is that of r to b, or to $r + I'$. Q. E. D.

Having thus ſhewn how all the Secants of the circular arches ſet-down in the Trigonometrical Canon may be generated by the motion of the point P in the new direction A S in a manner that is ſimilar, or analogous, to the generation

neration of the Sines of the ſaid arches that has been deſcribed by Lord Napier in the foregoing Tract, it remains that we ſhew how the Logarithms of theſe Secants, or of their ratios to the radius of the circle, may be generated likewiſe by the motion of a move-able point moving along a right line in a manner that is ſimilar, or analogous, to the generation of the Logarithms of the Sines of the ſame arches, or of their ratios to the radius of the Circle, deſcribed by Lord Napier in the foregoing Tract.

The Generation of the Logarithms of the foregoing Secants.

Art. 18. Let the ſecond right line, mentioned above in Art. 5, to wit, the line H I K L M *p*, in Fig. 2, (which is there ſuppoſed to be generated by the move-able point *p* that moves with an uniform velocity equal to the firſt velocity of the move-able point P in Fig. 1,) be ſuppoſed to be produced indefinitely on the oppoſite ſide of the point H, from which the former generation of it by the motion of the point *p* from H towards K began, ſo that this new line (which we will call H Σ,) ſhall lie *in directum* with the ſaid former line H K; and let us ſuppoſe the move-able point *p* to be brought-back to the point H, from which it's former motion towards K began, and then to move with an uniform velocity along the indefinite line H Σ in a direction contrary to it's former direction when it moved from H to K, but with the ſame velocity as before, that is, with the firſt velocity of the former move-able point P, in the point A, in deſcribing the line A T in Fig. 1, or it's firſt velocity in the ſame point A, in deſcribing the increaſing line A S. And let the little line H *i*, in Fig. 2, be equal to H I, and ſeveral other little lines, each equal to H *i* or H I, be ſuppoſed to be taken in the indefinite line A Σ. Then it is evident that, while the line C A + A P, or C A P, increaſes from *r* to *r* + I′, or *b*, by the motion of the point P from A towards S, the little line H *i* will be deſcribed, or generated, by the motion of the point *p* from H towards Σ; and, while C A P increaſes further from *b* to $\frac{bb}{r}$, another portion of the indefinite line A Σ equal to H *i* will be deſcribed by the point *p*; and

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in the ſeveral ſucceſſive equal portions of time in which the line C A P increaſes to $\frac{b^1}{r^0}$, $\frac{b^2}{r^1}$, $\frac{b^3}{r^2}$, $\frac{b^4}{r^3}$, $\frac{b^5}{r^4}$, $\frac{b^6}{r^5}$, $\frac{b^7}{r^6}$, $\frac{b^8}{r^7}$, &c, the line H Σ will receive the ſame number of increments, by the motion of the point *p*, that will be all equal to each other and to the firſt increment H *i*; ſo that the portion of the line H Σ that is generated by the motion of the point *p* from H towards Σ will increaſe in the ſame manner, or at the ſame rate, as the ratio of the ſecond term *b* of the increaſing Geometrical Progreſſion, *r*, *b*, $\frac{b^2}{r}$, $\frac{b^3}{r^2}$, $\frac{b^4}{r^3}$, $\frac{b^5}{r^4}$, $\frac{b^6}{r^5}$, $\frac{b^7}{r^6}$, $\frac{b^8}{r^7}$, $\frac{b^9}{r^8}$, &c, to the firſt term *r* increaſes by the ſaid ſecond term's increaſing from *b* to $\frac{b^2}{r}$, and from $\frac{b^2}{r}$ to $\frac{b^3}{r^2}$, and from $\frac{b^3}{r^2}$ to $\frac{b^4}{r^3}$, and from $\frac{b^4}{r^3}$ to $\frac{b^5}{r^4}$, and from $\frac{b^5}{r^4}$ to $\frac{b^6}{r^5}$, and from $\frac{b^6}{r^5}$ to $\frac{b^7}{r^6}$, and from $\frac{b^7}{r^6}$ to $\frac{b^8}{r^7}$, and from $\frac{b^8}{r^7}$ to $\frac{b^9}{r^8}$, &c; and conſequently the portion of the indefinite line H Σ generated by the motion of the point *p*, in the ſame time in which the line C A P, by the motion of the point P, becomes equal to any particular term of the increaſing Geometrical Progreſſion *r*, *b*, $\frac{b^2}{r}$, $\frac{b^3}{r^2}$, $\frac{b^4}{r^3}$, $\frac{b^5}{r^4}$, $\frac{b^6}{r^5}$, $\frac{b^7}{r^6}$, $\frac{b^8}{r^7}$, $\frac{b^9}{r^8}$, &c, or to any particular Secant in the Trigonometrical Canon, will be the Logarithm of the ratio of the ſaid term to the firſt term, *r*, of the ſaid progreſſion, or of the ratio of the ſaid Secant to the radius, or, in a more conciſe way of ſpeaking, will be the Logarithm of the ſaid term, or of the ſaid Secant. Or, if we take the Secants of two different arches of the Quadrant, and theſe Secants are equal to two different terms of the ſaid increaſing Geometrical Progreſſion, the portion of the line H Σ which is generated by the point *p* in the ſame time in which the portion of the line C A P, or the term of the ſaid increaſing Geometrical Series, which is equal to the greater Secant, is generated by the point P, will be to the portion of the line H Σ which is generated by the point *p* in the ſame time in which the portion of the line C A P, or the term of the ſaid Geometrical Series, which is equal to the leſſer Secant, is generated by the point P, in the ſame proportion as the ratio of the former term of the ſaid Geometrical Series, (which is equal to the greater Secant,) to the radius *r* is to the ratio of the latter and leſſer term of the ſaid Geometrical

Geometrical Series, (which is equal to the leſſer Secant,) to the ſame quantity, or the radius *r*.

Art. 19. And it appears from this generation of the portions of the indefinite line H Σ which are generated by the point *p*, and are the Logarithms of the Secants of circular arches, or of the ratios of the ſaid Secants to the radius of the Circle, that they are generated by the point *p* when it moves in a direction contrary to that in which it moved before, when it generated the Logarithms of the Sines of the ſaid arches, or of the ratios of the ſaid Sines to the radius: which agrees with the nature of the two ſorts of ratios of which they are the repreſentatives, or meaſures; to wit, the ratios of the Secants to the radius, which are *ratios of majority*, and the ratios of the Sines to the radius, which are *ratios of minority*.

The manner of finding the Logarithm of the Secant of a given arch of a Circle in the foregoing Table of Lord Napier.

Art. 20. The ratio of the Secant of every arch of a circle to the radius is equal to the ratio of the radius to the co-ſine of the ſame arch, or to the ſine of the complement of the ſaid arch to an arch of 90 degrees, or the arch of a quadrant. Therefore the ratio of the Secant to the radius is equal, but contrary, to the ratio of the co-ſine to the radius, the former being a ratio of majority, and the latter a ratio of minority. Therefore the Logarithm of the ratio of the Secant to the radius will be equal to, but drawn in a direction contrary to the direction of, the Logarithm of the ratio of the co-ſine to the radius; that is, the Logarithm of the ratio of the Secant to the radius will be a portion of the right line H Σ, in Fig. 2, that is equal to the portion of the right line H I K L M which is the Logarithm of the ratio of the co-ſine to the radius. Now in the foregoing Table of Lord Napier the Logarithms of the co-ſines of the arches are ſet-down in the fifth column. Therefore the Logarithm of the ratio of the Secant of any circular arch to the radius may be found

found by looking-out in the ſaid fifth column of the Table the Logarithm of the ratio of the co-ſine of the ſaid arch to the radius, and prefixing to the ſaid Logarithm the ſign —, in order to ſhew that it is the Logarithm of the ratio of the Secant to the radius, or of a ratio of majority that is equal and contrary to the ratio of minority of the co-ſine to the radius; to the Logarithms of which latter ratios Lord Napier has thought fit to give the name of *Abundant*, or *Affirmative*, or *Poſitive*, and to prefix to them the ſign +, and to aſſign the denomination of *Defective*, or *Negative*, to the Logarithms of ratios of majority, and to prefix to them the ſign —; as has been obſerved above in Art. 15.

The manner of finding the Logarithm of the Tangent of a given Arch of a Circle in the foregoing Table of Lord Napier.

Art. 21. And the Logarithms of the Tangents of circular arches, or of the ratios of the ſaid Tangents to the radius of the circle, may likewiſe be found in Lord Napier's foregoing Table of Logarithms by means of the fourth column of the ſaid Table, or the Column of the Differences between the Logarithms of the Sines, which are ſet-down in the Third Column, and the Logarithms of the Co-ſines, which are ſet-down in the Fifth Column.

For every Tangent of a circular arch is to the radius as the Sine is to the Co-ſine. Therefore the Logarithm of the ratio of the Tangent to the radius will be equal to the Logarithm of the ratio of the Sine to the Co-ſine. Now let us ſuppoſe the arch to be leſs than 45 degrees, or half the arch of a Quadrant. Then will the Tangent be leſs than the radius, and the Sine leſs than the Co-ſine; and conſequently the ratio of the Sine to the radius will be a greater ratio of minority than the ratio of the Co-ſine to the radius, and will be equal to the ſum of the ratio of the Sine to the Co-ſine and of the ratio of the Co-ſine to the radius. Therefore the ratio of the Sine to the Co-ſine will be equal to the exceſs of the ratio of the Sine to the radius above the ratio of the Co-ſine to the radius. Therefore the ratio of the Tangent to the radius (which is equal to the ratio of the Sine to the Co-ſine,) will alſo be equal to the

the excefs of the ratio of the Sine to the radius above the ratio of the Co-fine to the radius. Therefore the Logarithm of the ratio of the Tangent to the radius will be equal to the excefs of the Logarithm of the ratio of the Sine to the radius above the Logarithm of the ratio of the Co-fine to the radius; that is, the Logarithm of the Tangent will be equal to the excefs of the Logarithm of the Sine above the Logarithm of the Co-fine: which excefs, or difference of the faid two Logarithms, is fet-down under the title of *Differentiæ*, in the Fourth Column of Lord Napier's faid Table of Logarithms. And in this manner we may find in this Table of Lord Napier the Logarithm of the Tangent of any arch that is lefs than 45 degrees, or half the arch of a Quadrant.

If the arch is greater than 45 degrees, and confequently it's Tangent is greater than the radius, it's Logarithm may be found by prefixing the fign — to the Logarithm of it's Co-tangent, or the tangent of it's complement to 90 degrees, which tangent (being the tangent of an arch lefs than 45 degrees,) may be found in Lord Napier's Column of Differences in the manner above-explained. This follows from the known property of the tangents of circular arches, to wit, that every tangent is to the radius of the circle in the fame proportion as the radius is to the Co-tangent; whence it follows that the Logarithm of the tangent muft be equal and contrary to the Logarithm of the Co-tangent.

And thus it appears that Lord Napier's foregoing Table of Logarithms will enable us to find the Logarithms of the Secants and Tangents of circular arches as well as thofe of their Sines and Co-fines.

A SCHOLIUM.

Art. 22. Though, when Lord Napier's Invention of Logarithms was firft publifhed in the foregoing Tract, in the year 1614, it was much admired and applauded on account of it's great utility in facilitating the computations that occur in Plane and Spherical Trigonometry, yet his manner of explaining

it by introducing the above-mentioned ſuppoſitions of different right lines generated by different kinds of motion, (ſome of which were made with uniform velocities, and others with velocities that continually decreaſed, or increaſed, in a certain manner, and ſo as to generate a ſet of right lines that would form a continual Geometrical Progreſſion, while the uniform motions generated other ſets of right lines that formed Arithmetical Progreſſions,) was not univerſally approved-of by the Mathematicians of his own time. For we are informed by the celebrated Kepler (who was the greateſt Aſtronomer, and one of the greateſt Mathematicians, of his time,) that in the year 1621 (which was ſeven years after the publication of the foregoing Tract of Lord Napier,) ſeveral of the Profeſſors of Mathematicks in Upper Germany, and more eſpecially thoſe of them who were ſomewhat advanced in years, and were grown averſe to new methods of reaſoning that carried them out of the old doctrines and principles with which Time and Habit had made them familiar, doubted in ſome degree, whether Lord Napier's demonſtration of the properties of Logarithms was perfectly true, and whether the application of them to the abridgement of Trigonometrical calculations might not be unſafe, and lead the calculator, who ſhould truſt in them, into erroneous reſults; and, in either caſe, whether the doctrine were true, or not, they thought that Lord Napier's demonſtration of it was illegitimate and unſatisfactory. This we are told by the famous Kepler, in the ſecond of the two excellent Tracts that he wrote upon this ſubject of Logarithms, and which are reprinted in the firſt volume of this Collection of Mathematical Tracts, called *Scriptores Logarithmici*, in pages 1, 2, 3, 4, 5, &c, - - - - 166. For in the beginning of the ſecond of theſe two Tracts, which is intitled *Supplementum Chiliadis*, in page 95, he gives us the following account of the ſentiments of the German Mathematicians upon this ſubject. *Cum anno* 1621 *veniſſem in Germaniam ſuperiorem, paſſimque cum peritis rerum Mathematicarum de Logarithmis Neperianis contuliſſem; deprehendi eos quibus ætas prudentiam addebat, promptitudinem minuebat, ſupèr hoc genere numerorum, loco Canonis Sinuum, in uſum recipiendo, cunctari: Quòd dicerent turpe eſſe Profeſſori Mathematico ſupèr compendio aliquo calculi puerilitèr exultare, interímque ſine demonſtratione legitimâ formam calculi in uſum recipere, quæ olim, cum minimè metueres, in erroris inſidias te pertrahere poſſit. Querebantur Neperi demonſtrationem niti figmento motûs cujuſdam Geometrici, cujus lubricitas et fluxibilitas inepta eſſet in quâ ſolidus ille ſtylus rationis demonſtrationúmque firmum poneret veſtigium. Hæc mihi cauſa fuit ſtatìm tunc concipiendi rudimentum aliquod demonſtrationis legitimæ*, &c, as in pages 95 et 96.

The

The words *figmento motûs cujuſdam Geometrici, cujus lubricitas et flux'bilitas inepta eſſet,* &c relate to the motion by which Lord Napier ſuppoſed the ſeveral Sines in the arch of a Quadrant of a circle to be generated, and which has been deſcribed above in Art. 2, and illuſtrated by Fig. 1, in Art. 3; and they ſeem to be very happily choſen to expreſs the objection of theſe German Mathematicians to this motion, "as being a motion of a very ſubtle and ſlippery kind, of which it is not eaſy to form a clear conception, and which differs widely from the plain ideas of fixed and determinate magnitudes that occur in the elementary parts of Geometry, and more eſpecially in the works of Euclid." And for this objection I muſt needs confeſs I think there is ſome foundation.

But Lord Napier might have deſcribed his noble Invention of Logarithms, and given his readers a clear idea of the Table of them which is contained in the latter part of the foregoing Tract, without having recourſe to the ſuppoſition of this ſubtle generation of the Sines of circular arches by motion, or departing in any degree from the ſimple conceptions of antient Geometry, and the known properties of the terms of a Geometrical Progreſſion, by proceeding in the following manner.

A Deſcription of the Logarithms of the Sines of the circular arches contained in the Trigonometrical Canon, or of their ratios to the radius of the Circle, invented by Lord Napier, and ſet-down in the Table at the end of the foregoing Tract, without ſuppoſing either the ſaid Sines or their Logarithms to be generated by the motion of points, but merely by means of the known properties of a Geometrical Progreſſion of which the terms decreaſe in ſome known, or given, ratio, or proportion.

Art. 23. Let the radius of a circle be called 10,000,000, or be ſuppoſed to be divided into 10,000,000, or ten millions, of ſmall and equal parts. And let 1, or one of theſe very ſmall parts of the radius, be ſubtracted from the whole radius, or 10,000,000; and the remainder will be 10,000,000 — 1,

or 9,999,999. Thirdly, let a decreaſing Geometrical Progreſſion be formed, of which 10,000,000, or the whole radius, ſhall be the firſt term, and 9,999,999 ſhall be the ſecond term. This Progreſſion will be as follows, to wit, $10{,}000{,}000,\ 9{,}999{,}999,\ \frac{9{,}999{,}999^{2}}{10{,}000{,}000},\ \frac{9{,}999{,}999^{3}}{10{,}000{,}000^{2}},\ \frac{9{,}999{,}999^{4}}{10{,}000{,}000^{3}},\ \frac{9{,}999{,}999^{5}}{10{,}000{,}000^{4}},$ &c; in which the Index of the power of 9,999,999 that forms the numerator of any of the terms is always greater by an unit than the Index of the power of 10,000,000, that forms the denominator of the ſame term. Fourthly, let this Geometrical Progreſſion be continued till the laſt term of it becomes equal to, or leſs than, 1, or the 10,000,000th part of the radius. This ſmall part of the radius is much leſs than the Sine of 1 minute of a degree, which contains 2909 ſuch parts; and it is alſo much leſs than the Sine of a ſecond, or ſecond minute of a degree, which is very nearly equal to the 60th part of the Sine of a firſt minute, that is, to $\frac{2909}{60}$, or 48 ſuch parts; and it is very little greater than the Sine of a third, or third minute of a degree, which is equal to $\frac{48}{60}$, or 0.8, or $\frac{8}{10}$, of ſuch a ſmall part of the radius. Therefore all Sines, or parts of Sines, that are leſs than 1, or one 10,000,000th part of the radius, may be juſtly conſidered as being too ſmall to be worth attending-to in forming a Trigonometrical Canon of Sines and Tangents and Secants of circular arches, in which the arches proceed by the common difference of 1 minute of a degree, as is the caſe with the foregoing Table of Lord Napier. Fifthly, let us, for the ſake of brevity, ſubſtitute the Letter r inſtead of the number 10,000,000, (or the number uſed in the foregoing Geometrical Progreſſion to expreſs the radius of the circle, or the firſt term of the ſaid progreſſion,) and the Letter a for the number 9,999,999, or the ſecond term of the ſaid progreſſion. And then the ſaid progreſſion will be changed into the following progeſſion, to wit, $r,\ a,\ \frac{a^2}{r},\ \frac{a^3}{r^2},\ \frac{a^4}{r^3},\ \frac{a^5}{r^4},\ \frac{a^6}{r^5},\ \frac{a^7}{r^6},\ \frac{a^8}{r^7},\ \frac{a^9}{r^8}$, &c.

Now in this Geometrical Progreſſion it is evident that the Index of the power of a in the ſecond term a, or a^1, and in the third term $\frac{a^2}{r}$, and in the fourth term $\frac{a^3}{r^2}$, and in all the following terms of it, expreſſes the diſtance of the

the term to which it belongs from the firſt term *r*, or the number of intervals between the ſaid term and the firſt term *r*, or the number of equal *ratiunculæ*, or ſmall ratios of minority, (each of which is equal to the ratio of *a* to *r*, or of 9,999,999 to 10,000,000,) that are contained in the ratio of the ſaid term to the firſt term *r*. Thus, for example, the Index of the power of *a* in the ſixth term $\frac{a^5}{r^4}$, to wit, the number 5, expreſſes the diſtance of the ſaid term $\frac{a^5}{r^4}$ from the firſt term *r*, or the number of intervals between the ſaid term $\frac{a^5}{r^4}$ and the firſt term *r*, or the number of equal *ratiunculæ*, or ſmall ratios of minority equal to the ratio of *a* to *r*, or of 9,999,999 to 10,000,000, that are contained in the ratio of $\frac{a^5}{r^4}$ to *r*. And, in like manner, the Index of the power of *a* in the tenth term, $\frac{a^9}{r^8}$, of the ſaid Geometrical Progreſſion, to wit, the number 9, expreſſes the diſtance of the ſaid tenth term, $\frac{a^9}{r^8}$, from the firſt term *r*, or the number of intervals between the ſaid term $\frac{a^9}{r^8}$ and firſt term *r*, or the number of equal *ratiunculæ*, or ſmall ratios of minority, equal to the ratio of *a* to *r*, or of 9,999,999 to 10,000,000, contained in the ratio of $\frac{a^9}{r^8}$ to *r*. Therefore the Index 9 will be to the Index 5 in the ſame proportion as the ratio of minority of the term $\frac{a^9}{r^8}$ to *r* is to the leſſer ratio of minority of the term $\frac{a^5}{r^4}$ to *r*. And the ſame thing, it is evident, will be true of any other two terms whatſoever in this Geometrical Progreſſion, to wit, that the Index of the power of *a* in the ſmaller, or more remote, term in the progreſſion, will be to the Index of the power of *a* in the greater, or leſs remote, term in the progreſſion, in the ſame proportion as the greater ratio of minority of the ſmaller, or more remote, term to the firſt term *r* is to the leſſer ratio of minority of the greater, or leſs remote, term to the ſaid firſt term *r*. Theſe Indexes of the powers of *a* in the ſecond and other following terms of the Geometrical Progreſſion *r*, *a*, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, &c are therefore proportional to, or meaſures of, or *Logarithms* of, the ratios of the ſeveral terms of the ſaid progreſſion to it's firſt term *r*. And theſe Indexes of the powers of *a* in the ſaid Geometrical Progreſſion may with great pro-

priety

priety be called the *Logarithms* of the ratios of the ſeveral terms to which they belong to the firſt term r, becauſe they are ἀριθμοὶ τῶν λο'γων, or the numbers of the *ratiunculæ*, or of the ſmall and equal ratios of a to r, or of 9,999,999 to 10,000,000, which are contained in the ſeveral ratios of the terms to which they belong, to the firſt term r, or 10,000,000, of the ſaid progreſſion. And this was Lord Napier's reaſon for giving them this name of *Logarithms*.

And thus it appears that we may form a clear idea of Lord Napier's ſyſtem of Logarithms, without having recourſe to thoſe motions of points over right lines with different kinds of velocities, which he has introduced for the purpoſe into the foregoing Tract, and to which ſome of the German Mathematicians made objections.

Of the Derivation of the Logarithms of the Sines of circular arches ſet-down in the Trigonometrical Canon from the Logarithms of the terms of the foregoing decreaſing Geometrical Progreſſion.

Art. 24. Having now obtained a clear idea of the Logarithms of the ſecond and other following terms of the Geometrical Progreſſion r, a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, &c, we may apply it to the finding of the Logarithms of the Sines of the ſeveral circular arches contained in the Trigonometrical Canon in the following manner.

The Differences of the ſeveral terms of this Geometrical Progreſſion will decreaſe continually in the ſame proportion as the terms themſelves. And therefore the difference between every two contiguous terms of it, after the two firſt terms r and a, or 10,000,000 and 9,999,999, will be leſs than the difference between the ſaid two firſt terms, that is, than 1, or the 10,000,000th part of the radius r. Therefore every Sine in the Trigonometrical Canon muſt either be equal to ſome one of the terms of this Geometrical Progreſſion, or muſt be ſo nearly equal to it as to differ from it by leſs than 1, or the 10,000,000nth part of the radius; which, for practical purpoſes, may be conſidered

ſidered as perfect equality. Therefore the ratio of any Sine in the Trigonometrical Canon to the radius will be the ſame as the ratio of that term of the foregoing Geometrical Progreſſion which is equal to the ſaid Sine, to the radius, or the firſt term of the ſaid progreſſion. And conſequently the Logarithm of the ratio of the ſaid Sine to the radius will be equal to the Logarithm of the ratio of that term of the ſaid progreſſion which is equal to the ſaid Sine, to the radius, or the firſt term of the ſaid progreſſion; or, in other words, the Logarithm of the ſaid Sine will be equal to the Logarithm of that term of the ſaid progreſſion which is equal to the ſaid Sine. Therefore, if all the terms of the ſaid Geometrical Progreſſion of which 10,000,000 is the firſt term, and 9,999,999 is the ſecond term, were once computed, and the Indexes of the powers of *a*, or 9,999,999, in it's ſecond, third, fourth, and other following terms, to wit, the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c, were placed near the ſeveral terms to which they reſpectively belong, we need only look-out in the Table containing the ſeveral terms of ſuch Geometrical Progreſſion reduced from their firſt form of fractions (ſuch as $\frac{\overline{9,999,999}^{9}}{\overline{10,000,000}^{8}}$) to the form of ſingle whole numbers, ſimilar to the whole numbers that expreſs the lengths of the Sines of the arches in Lord Napier's Trigonometrical Canon,—I ſay, we need only look-out in the Table containing the terms of ſuch Geometrical Progreſſion, ſo reduced, the term that was equal to any given Sine, of which we were ſeeking the Logarithm; and the Logarithm, or Index of the power of *a*, or 9,999,999, that correſponded to the ſaid term, would be the Logarithm ſought.

Lord Napier's Logarithms of the Sines of the circular arches contained in his Trigonometrical Canon, which are ſet-down in the third and fifth Columns of the Table printed at the end of the foregoing Tract, are the Logarithms, or the Indexes of the powers of *a*, or 9,999,999, belonging to ſo many of the terms of the foregoing Geometrical Progreſſion r, a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, &c as are equal to the Sines of the ſeveral arches in a Quadrant of a Circle, proceeding from 1 minute of a Degree to 89 Degrees and 59 minutes, by the common difference of one minute; which are all the Sines ſet-down in Lord Napier's Trigonometrical Canon.

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The Computation of the third and fourth and the six following terms of the foregoing Geometrical Progression r, a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, *&c, in which* r *is supposed to be* $= 10{,}000{,}000$, *and* a *is supposed to be* $= 9{,}999{,}999$, *which is the foundation of Lord Napier's Logarithms of the Sines of the several arches in a Quadrant of a Circle that are set-down in the Table printed above at the end of the foregoing Tract.*

Art. 25. The third term, $\frac{a^2}{r}$, of the foregoing Geometrical Progression is $= a \times \frac{a}{r} = 9{,}999{,}999 \times \frac{9{,}999{,}999}{10{,}000{,}000}$ $(= 9{,}999{,}999 \times \frac{10{,}000{,}000 - 1}{10{,}000{,}000}$ $= \frac{99{,}999{,}990{,}000{,}000 - 9{,}999{,}999}{10{,}000{,}000} = \frac{99{,}999{,}980{,}000{,}001}{10{,}000{,}000}$ $= 9{,}999{,}998.000{,}000{,}1)$ $= 9{,}999{,}998$.

The fourth term $\frac{a^3}{r^2}$ is $= \frac{a^2}{r} \times \frac{a}{r} = 9{,}999{,}998 \times \frac{9{,}999{,}999}{10{,}000{,}000}$ $(= 9{,}999{,}998 \times \frac{10{,}000{,}000 - 1}{10{,}000{,}000} = \frac{99{,}999{,}980{,}000{,}000 - 9{,}999{,}998}{10{,}000{,}000} = \frac{99{,}999{,}970{,}000{,}002}{10{,}000{,}000}$ $= 9{,}999{,}997.000{,}000{,}2)$ $= 9{,}999{,}997$.

The fifth term $\frac{a^4}{r^3}$ is $= \frac{a^3}{r^2} \times \frac{a}{r} = 9{,}999{,}997 \times \frac{9{,}999{,}999}{10{,}000{,}000}$ $(= 9{,}999{,}997 \times \frac{10{,}000{,}000 - 1}{10{,}000{,}000} = \frac{99{,}999{,}970{,}000{,}000 - 9{,}999{,}997,}{10{,}000{,}000} = \frac{99{,}999{,}960{,}000{,}003}{10{,}000{,}000}$ $= 9{,}999{,}996.000{,}000{,}3)$ $= 9{,}999{,}996$.

The

The fixth term $\frac{a^5}{r^4}$ is $= \frac{a^4}{r^3} \times \frac{a}{r} = 9{,}999{,}996 \times \frac{9{,}999{,}999}{10{,}000{,}000}$ ($=$

$9{,}999{,}996 \times \frac{10{,}000{,}000 - 1}{10{,}000{,}000} = \frac{99{,}999{,}960{,}000{,}000 - 9{,}999{,}996}{10{,}000{,}000} =$

$\frac{99{,}999{,}950{,}000{,}004}{10{,}000{,}000} = 9{,}999{,}995.000{,}004) = 9{,}999{,}995.$

The feventh term $\frac{a^6}{r^5}$ is $= \frac{a^5}{r^4} \times \frac{a}{r} = 9{,}999{,}995 \times \frac{9{,}999{,}999}{10{,}000{,}000}$ ($=$

$9{,}999{,}995 \times \frac{10{,}000{,}000 - 1}{10{,}000{,}000} = \frac{99{,}999{,}950{,}000{,}000 - 9{,}999{,}995}{10{,}000{,}000} =$

$\frac{9{,}999{,}940{,}000{,}005}{10{,}000{,}000} = 9{,}999{,}994.000{,}000{,}4) = 9{,}999{,}994.$

The eighth term $\frac{a^7}{r^6}$ is $= \frac{a^6}{r^5} \times \frac{a}{r} = 9{,}999{,}994 \times \frac{9{,}999{,}999}{10{,}000{,}000}$ ($=$

$9{,}999{,}994 \times \frac{10{,}000{,}000 - 1}{10{,}000{,}000} = \frac{99{,}999{,}940{,}000{,}000 - 9{,}999{,}994}{10{,}000{,}000} =$

$\frac{99{,}999{,}930{,}000{,}006}{10{,}000{,}000} = 9{,}999{,}993.000{,}000{,}6) = 9{,}999{,}993.$

The ninth term $\frac{a^8}{r^7}$ is $= \frac{a^7}{r^6} \times \frac{a}{r} = 9{,}999{,}993 \times \frac{999{,}999{,}9}{10{,}000{,}000}$ ($=$

$9{,}999{,}993 \times \frac{10{,}000{,}000 - 1}{10{,}000{,}000} = \frac{99{,}999{,}930{,}000{,}000 - 9{,}999{,}993}{10{,}000{,}000} =$

$\frac{99{,}999{,}920{,}000{,}007}{10{,}000{,}000} = 9{,}999{,}992.000{,}000{,}7) = 9{,}999{,}992.$

And the tenth term $\frac{a^9}{r^8}$ is $= \frac{a^8}{r^7} \times \frac{a}{r} = 9{,}999{,}992 \times \frac{9{,}999{,}999}{10{,}000{,}000}$ ($=$

$9{,}999{,}992 \times \frac{10{,}000{,}000 - 1}{10{,}000{,}000} = \frac{99{,}999{,}920{,}000{,}000 - 9{,}999{,}992,}{10{,}000{,}000} =$

$\frac{99{,}999{,}910{,}000{,}008}{10{,}000{,}000} = 9{,}999{,}991.000{,}000{,}8) = 9{,}999{,}991.$

And, in general, if any term in this ſeries is denoted by the Letter T, the next term will be $\left(= T \times \frac{a}{r} = T \times \frac{9\,999,999}{10,000,000} = T \times \frac{10,000,000-1}{10,000,000} \right)$ $= \frac{10,000,000\,T - T}{10,000,000}$; which is an expreſſion that is eaſily computed, the ſubtraction of T from 10,000,000 T being the moſt difficult of the three arithmetical operations that are to be performed in computing it, and yet being a ſhort and eaſy operation. And I conjecture that it was in this way that Lord Napier computed the terms of this Geometrical Progreſſion, or at leaſt ſome of it's firſt terms.

Theſe firſt ten terms of the foregoing Geometrical Progreſſion, if ſet-down in order, with the Logarithms of the ratios of all the terms, after the firſt term 10,000,000, to the ſaid firſt term, ſet-down over-againſt them, will be as follows.

The terms of the foregoing Geometrical Progreſſion.	The corresponding Logarithms.
r = 10,000,000.	
a = 9,999,999. - - - -	1.
$\frac{a^2}{r}$ = 9,999,998 - - - -	2.
$\frac{a^3}{r^2}$ = 9,999,997 - - - -	3.
$\frac{a^4}{r^3}$ = 9,999,996 - - - -	4.
$\frac{a^5}{r^4}$ = 9,999,995 - - - -	5
$\frac{a^6}{r^5}$ = 9,999,994 - - - -	6
$\frac{a^7}{r^6}$ = 9,999,993 - - - -	7
$\frac{a^8}{r^7}$ = 9,999,992 - - - -	8.
$\frac{a^9}{r^8}$ = 9,999,991 - - - -	9.

The third term of this Geometrical Progreſſion, to wit, $\frac{a^2}{r}$, or 9,999,998, is equal to the Sine of an arch of 89°, 58′ ſet-down in the ſixth column of the firſt page of Lord Napier's Table of Sines and their Logarithms printed at the end of the foregoing Tract. And accordingly the Logarithm of this Sine in the fifth column of the ſame page is the number 2, as it is in the foregoing ſhort Table of ten terms.

And the fifth term of this Geometrical Progreſſion, to wit, $\frac{a^4}{r^3}$, or 9,999,996, is equal to the Sine of an arch of 89°, 57′ in Lord Napier's Table. And according Lord Napier's Logarithm of this Sine is the number 4, as it is in the foregoing ſhort Table.

And the eighth term of this Geometrical Progreſſion, to wit, $\frac{a^7}{r^6}$, or 9,999,993, is equal to the Sine of 89°, 56′ in Lord Napier's Table. And accordingly Lord Napier's Logarithm of this Sine is the number 7, as it is in the foregoing ſhort Table.

Art. 26. It has been ſhewn above, in Art. 23, that the Index of the power of a in the ſecond and every following term of the decreaſing Geometrical Progreſſion r, a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, &c is the Logarithm of the ſaid term, or of it's ratio to the firſt term r. And therefore, if the numeral values of the ſeveral terms of this Progreſſion were once computed, and their Logarithms, (which would be the natural numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, &c,) were ſet down over-againſt them, all further inquiries concerning the method of finding the Logarithm of any given number leſs than r, or 10,000,000, but greater than the leaſt term of the Series, (which is ſuppoſed to be not greater than 1, or the 10,000,000th part of the firſt term r, or 10,000,000,) would become unneceſſary. But, though the method of computing the third and fourth and other following terms of the foregoing Geometrical Progreſſion from the two firſt terms r and a, and from each other, which has been deſcribed in the laſt article and exemplified by the computation of $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, and $\frac{a^9}{r^8}$, is a very clear and eaſy one, yet,

perhaps, from the prodigious number of the terms of which this progreſſion would conſiſt, it might be found a very laborious and almoſt impracticable taſk to compute them all in this manner. And therefore it would probably be convenient to have a method of finding the Logarithm of any given Sine in the Trigonometrical Canon without having ſuch a Table of the values of the terms of this Geometrical Progreſſion ready-computed to our hands. Now this might be done by the help of Sir Iſacc Newton's Binomial Theorem in the following manner.

A Method of computing Lord Napier's Logarithm of any given Sine in the Trigonometrical Canon, without computing all the very numerous terms of the Geometrical Progreſſion r, a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, *&c, &c, (continued to the term that is equal to, or leſs than,* $r - a$,) *in which* r *is* $=$ 10,000,000, *and* a *is* $=$ 10,000,000 $-$ 1, *or* 9,999,999.

Art. 27. Let the given, or known, Sine in the Trigonometrical Canon be called S, and the Logarithm of it, (which is unknown, and which we wiſh to find,) be denoted by the Letter x. Then, becauſe the Index of the power of a in every term of the Geometrical Progreſſion r, a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, $\frac{a^7}{r^6}$, $\frac{a^8}{r^7}$, $\frac{a^9}{r^8}$, &c is the Logarithm of the term to which ſuch power of a belongs, (as is ſhewn above in Art. 23,) it follows that the term of the ſaid progreſſion which is equal to the given Sine S (of which we are ſeeking the Logarithm x,) will be the term $\frac{a^x}{r^{x-1}}$. In order therefore to find the Logarithm x of the given Sine S, we muſt reſolve the equation $\frac{a^x}{r^{x-1}} = S$. Now this

equation

equation has been resolved by my learned and ingenious friend Mr. James Ivory, M. A., of the Royal Military College at Great Marlow in Buckinghamshire, in the following manner.

The Resolution of the equation $\frac{a^x}{r^{x-1}} = S.$

By Mr. James Ivory, M. A., of the Royal Military College at Great Marlow in Buckinghamshire.

The second term a of the Geometrical Progression r, a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, &c, or 10,000,000, 9,999,999, $\frac{\overline{9,999,999}^2}{10,000,000}$, $\frac{\overline{9,999,999}^3}{\overline{10,000,000}^2}$, $\frac{\overline{9,999,999}^4}{\overline{10,000,000}^3}$, $\frac{\overline{9,999,999}^5}{\overline{10,000,000}^4}$, $\frac{\overline{9,999,999}^6}{\overline{10,000,000}^5}$, &c, is $= r - 1$. Therefore the term $\frac{a^x}{r^{x-1}}$ will be $= \frac{\overline{r-1}^x}{r^{x-1}}$; and consequently the equation $\frac{a^x}{r^{x-1}} = S$ will be changed into the equation $\frac{\overline{r-1}^x}{r^{x-1}} = S.$

But $\frac{\overline{r-1}^x}{r^{x-1}}$ is $\left(= \frac{r}{r} \times \frac{\overline{r-1}^x}{r^{x-1}} = \frac{r \times \overline{r-1}^x}{r \times r^{x-1}} = \frac{r \times \overline{r-1}^x}{r^x} = r \times \frac{\overline{r-1}^x}{r^x} = r \times \overline{\frac{r-1}{r}}^x\right) = r \times \overline{1 - \frac{1}{r}}^x$. Therefore $r \times \overline{1 - \frac{1}{r}}^x$ will be $= S.$

Further, since S denotes the Sine of a circular arch, it must be less than the radius r, and therefore may be subtracted from it. Let it be so subtracted; and let the remainder, or difference, $r - S$ be called v. Then will r be $= S + v$, and S will be $= r - v$. But $r - v$ is $= r \times \overline{1 - \frac{v}{r}}$. Therefore S will be $= r \times \overline{1 - \frac{v}{r}}$.

But

But S has been ſhewn to be $= r \times \overline{1 - \frac{1}{r}}^{x}$.

Therefore $r \times \overline{1 - \frac{v}{r}}$ will be $= r \times \overline{1 - \frac{1}{r}}^{x}$, and conſequently (dividing both ſides of the equation by r,) we ſhall have $1 - \frac{v}{r} = \overline{1 - \frac{1}{r}}^{x}$.

Now let us ſuppoſe that n is put to denote ſome very great number, as, for example, a nonillion, or the ninth power of a million, or the 54th power of 10; and let the nth roots of the two oppoſite ſides of the laſt equation $1 - \frac{v}{r} = \overline{1 - \frac{1}{r}}^{x}$ be extracted. And we ſhall then have $\overline{1 - \frac{v}{r}}^{\frac{1}{n}} = \overline{1 - \frac{1}{r}}^{\frac{x}{n}}$.

But, by Sir Iſaac Newton's Binomial Theorm, $\overline{1 - \frac{v}{r}}^{\frac{1}{n}}$ is $=$ the Series $1 - \frac{1}{n} \times \frac{v}{r} - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{v^2}{r^2} - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{v^3}{r^3} -$ &c, and $\overline{1 - \frac{1}{r}}^{\frac{x}{n}}$ is $=$ the Series $1 - \frac{x}{n} \times \frac{1}{r} - \frac{x}{n} \times \frac{n-x}{2n} \times \frac{1}{r^2} - \frac{x}{n} \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^3} -$ &c; as is ſhewn in the ſecond volume of this Collection of Tracts called *Scriptores Logarithmici*, page 346, &c.

Therefore the Series $1 - \frac{1}{n} \times \frac{v}{r} - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{v^2}{r^2} - \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{v^3}{r^3} -$ &c will be $=$ the Series $1 - \frac{x}{n} \times \frac{1}{r} - \frac{x}{n} \times \frac{n-x}{2n} \times \frac{1}{r^2} - \frac{x}{n} \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^3} -$ &c.

Now each of theſe ſerieſes is evidently leſs than it's firſt term 1, becauſe the ſecond and other following terms of each of them are continually decreaſing quantities, and are all marked with the ſign —, or the ſign of Subtraction,

traction, and consequently are subtracted from the first term 1. Therefore each of these serieses may be subtracted from it's first term, which in both serieses is the same quantity, to wit, 1. Let them be so subtracted. And the remainders will be equal to each other. But these remainders are the Serieses $\frac{1}{n} \times \frac{v}{r} + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{v^2}{r^2} + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{v^3}{r^3} +$ &c and $\frac{x}{n} \times \frac{1}{r} + \frac{x}{n} \times \frac{n-x}{2n} \times \frac{1}{r^2} + \frac{x}{n} \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^3} +$ &c. Therefore the Series $\frac{1}{n} \times \frac{v}{r} + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{v^2}{r^2} + \frac{1}{n} \times \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{v^3}{r^3} +$ &c will be = the Series $\frac{x}{n} \times \frac{1}{r} + \frac{x}{n} \times \frac{n-x}{2n} \times \frac{1}{r^2} + \frac{x}{n} \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^3} +$ &c; and consequently, if we multiply both these Serieses into n, we shall have the series $\frac{v}{r} + \frac{n-1}{2n} \times \frac{v^2}{r^2} + \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{v^3}{r^3} +$ &c = the series $\frac{x}{r} + x \times \frac{n-x}{2n} \times \frac{1}{r^2} + x \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^3} +$ &c.

Now, because n is supposed to be an exceeding great number, such as a nonillion, or the 9th power of a million, or the 54th power of 10, it is evident that the fractions $\frac{n-1}{2n}$ and $\frac{2n-1}{3n}$, and $\frac{3n-1}{4n}$ and $\frac{4n-1}{5n}$, &c (which enter the several terms of the first of the two serieses contained in this last equation,) will be very nearly equal to the fractions $\frac{n-0}{2n}$ and $\frac{2n-0}{3n}$ and $\frac{3n-0}{4n}$ and $\frac{4n-0}{5n}$, &c, or to the fractions $\frac{n}{2n}$, $\frac{2n}{3n}$, $\frac{3n}{4n}$, $\frac{4n}{5n}$, &c, or to the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, &c. And consequently the said Series $\frac{v}{r} + \frac{n-1}{2n} \times \frac{v^2}{r^2} + \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{v^3}{r^3} +$ &c will be very nearly equal to the series $\frac{v}{r} + \frac{1}{2} \times \frac{v^2}{r^2} + \frac{1}{2} \times \frac{2}{3} \times \frac{v^3}{r^3} + \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{v^4}{r^4} + \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{v^5}{r^5} +$ &c, or to the series $\frac{v}{r} + \frac{1}{2} \times \frac{v^2}{r^2} + \frac{1}{3} \times \frac{v^3}{r^3} + \frac{1}{4} \times \frac{v^4}{r^4} + \frac{1}{5} \times \frac{v^5}{r^5} +$ &c, or to the series $\frac{v}{r} + \frac{v^2}{2r^2} + \frac{v^3}{3r^3} + \frac{v^4}{4r^4} + \frac{v^5}{5r^5}$

$\frac{v^5}{5r^5}$ + &c. And by increaſing the number n to a ſtill greater number than the 9th power of a million, we might make the ſeries $\frac{v}{r} + \frac{n-1}{2n} \times \frac{v^2}{r^2} + \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{v^3}{r^3}$ + &c come to be as nearly equal to the ſeries $\frac{v}{r} + \frac{v^2}{2r^2} + \frac{v^3}{3r^3} + \frac{v^4}{4r^4} + \frac{v^5}{5r^5}$ + &c as we pleaſe; ſo that this latter Series is *the Limit of the magnitude* of the Series $\frac{v}{r} + \frac{n-1}{2n} \times \frac{v^2}{r^2} + \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{v^3}{r^3}$ + &c, which forms the firſt, or left-hand, ſide of the laſt equation.

And in like manner it may be ſhewn that the Limit of the magnitude of the ſeries $\frac{x}{r} + x \times \frac{n-x}{2n} \times \frac{1}{r^2} + x \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^3}$ + &c (which forms the right-hand ſide of the laſt equation,) will be the Series $\frac{x}{r} + \frac{x}{2r^2} + \frac{x}{3r^3} + \frac{x}{4r^4} + \frac{x}{5r^5}$ + &c. For the Logarithm x (which is the Index of the power of a in the fraction $\frac{a^x}{r^{x-1}}$ which is equal to the given Sine S) muſt be ſome whole number that is leſs than the whole number of terms in the Geometrical Progreſſion r, a, $\frac{a^2}{r}$, $\frac{a^3}{r^2}$, $\frac{a^4}{r^3}$, $\frac{a^5}{r^4}$, $\frac{a^6}{r^5}$, &c, (which is continued only till it's laſt term is equal to, or leſs than 1, or the 10,000,000th part of the radius r, or 10,000,000,) and therefore will be vaſtly leſs than the great number n, or a nonillion; and conſequently the ſeveral fractions $\frac{n-x}{2n}$, $\frac{2n-x}{3n}$, $\frac{3n-x}{4n}$, $\frac{4n-x}{5n}$, &c, (which enter the terms of this ſecond Series,) will be very nearly equal to the fractions $\frac{n-0}{2n}$, $\frac{2n-0}{3n}$, $\frac{3n-0}{4n}$, $\frac{4n-0}{5n}$, &c, reſpectively, or to the fractions $\frac{n}{2n}$, $\frac{2n}{3n}$, $\frac{3n}{4n}$, $\frac{4n}{5n}$, &c, or to the fractions $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, &c. And conſequently the ſaid Series $\frac{x}{r} + x \times \frac{n-x}{2n} \times \frac{1}{r^2} + x \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^3}$ + &c will be very nearly equal

equal to the Series $\frac{x}{r} + x \times \frac{1}{2} \times \frac{1}{r^2} + x \times \frac{1}{2} \times \frac{2}{3} \times \frac{1}{r^3} + x \times \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{1}{r^4} + x \times \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{4}{5} \times \frac{1}{r^5}$ + &c, or to the Series $\frac{x}{r} + \frac{x}{2r^2} + \frac{x}{3r^3} + \frac{x}{4r^4} + \frac{x}{5r^5}$ + &c. And by increaſing the number n continually, (which may be taken of as great a magnitude as we think fit,) the ſaid Series $\frac{x}{r} + x \times \frac{n-x}{2n} \times \frac{1}{r^2} + x \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^3}$ + &c might be made to be as nearly equal to the ſeries $\frac{x}{r} + \frac{x}{2r^2} + \frac{x}{3r^3} + \frac{x}{4r^4} + \frac{x}{5r^5}$ + &c as we pleaſe; ſo that this latter Series $\frac{x}{r} + \frac{x}{2r^2} + \frac{x}{3r^3} + \frac{x}{4r^4} + \frac{x}{5r^5}$ + &c is *the Limit of the magnitude* of the ſeries $\frac{x}{r} + x \times \frac{n-x}{2n} \times \frac{1}{r^2} + x \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^3}$ + &c, which forms the ſecond, or right-hand, ſide of the laſt equation.

Now, ſince, by the ſaid laſt equation, the Series $\frac{v}{r} + \sqrt{\frac{n-1}{2n}} \times \frac{v^2}{r^2} + \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{v^3}{r^3}$ + &c is equal to the ſeries $\frac{x}{r} + x \times \frac{n-x}{2n} \times \frac{1}{r^2} + x \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^3}$ + &c in all poſſible magnitudes of n, however great thoſe magnitudes may be taken, it follows that the two ſerieſes which are the Limits of the magnitudes of thoſe two ſerieſes, or the quantities to which they may be made to approach as near as we pleaſe by increaſing continually the magnitude of the number n, muſt alſo be equal to each other. And therefore the ſeries $\frac{v}{r} + \frac{v^2}{2r^2} + \frac{v^3}{3r^3} + \frac{v^4}{4r^4} + \frac{v^5}{5r^5}$ + &c, which is the Limit of the magnitude of the former Series $\frac{v}{r} + \sqrt{\frac{n-1}{2n}} \times \frac{v^2}{r^2} + \frac{n-1}{2n} \times \frac{2n-1}{3n} \times \frac{v^3}{r^3}$ + &c, will be equal to the Series $\frac{x}{r} + \frac{x}{2r^2} + \frac{x}{3r^3} + \frac{x}{4r^4} + \frac{x}{5r^5}$ + &c, which is the Limit of the magnitude of the latter ſeries $\frac{x}{r} + x \times \frac{n-x}{2n} \times \frac{1}{r^2} + x \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^3}$ + &c.

But the Series $\frac{x}{r} + \frac{x}{2r^2} + \frac{x}{3r^3} + \frac{x}{4r^4} + \frac{x}{5r^5}$ + &c is $= x \times$ the Series $\frac{1}{r} + \frac{1}{2r^2} + \frac{1}{3r^3} + \frac{1}{4r^4} + \frac{1}{5r^5}$ + &c.

Therefore the Series $\frac{v}{r} + \frac{v^2}{2r^2} + \frac{v^3}{3r^3} + \frac{v^4}{4r^4} + \frac{v^5}{5r^5}$ + &c will be $=$ $x \times$ the Series $\frac{1}{r} + \frac{1}{2r^2} + \frac{1}{3r^3} + \frac{1}{4r^4} + \frac{1}{5r^5}$ + &c; and consequently x, or the Logarithm sought, will be equal to the fraction

$$\frac{\frac{v}{r} + \frac{v^2}{2r^2} + \frac{v^3}{3r^3} + \frac{v^4}{4r^4} + \frac{v^5}{5r^5} + \&c}{\frac{1}{r} + \frac{1}{2r^2} + \frac{1}{3r^3} + \frac{1}{4r^4} + \frac{1}{5r^5} + \&c},$$

or (if we multiply both the numerator and the denominator of this fraction by r, which will make no change in it's magnitude,) the said Logarithm x will be equal to the fraction

$$\frac{v + \frac{v^2}{2r} + \frac{v^3}{3r^2} + \frac{v^4}{4r^3} + \frac{v^5}{5r^4} + \&c}{1 + \frac{1}{2r} + \frac{1}{3r^2} + \frac{1}{4r^3} + \frac{1}{5r^4} + \&c},$$

or to $v \times$ the fraction

$$\frac{1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \&c}{1 + \frac{1}{2r} + \frac{1}{3r^2} + \frac{1}{4r^3} + \frac{1}{5r^4} + \&c}.$$ Q. E. I.

Art. 28. Coroll. The Series $1 + \frac{1}{2r} + \frac{1}{3r^2} + \frac{1}{4r^3} + \frac{1}{5r^4} + \frac{1}{6r^5} +$ &c (which forms the denominator of the foregoing fraction,) may be considered as equal to it's first term 1, on account of the extreme smallness of the second term $\frac{1}{2r}$, or $\frac{1}{2 \times 10,000,000}$, or $\frac{0.000,000,1}{2}$, or 0.000,000,05, and, *à fortiori*, of all the following terms of it, in comparison of the first term 1; and therefore the said fraction $\frac{1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \&c}{1 + \frac{1}{2r} + \frac{1}{3r^2} + \frac{1}{4r^3} + \frac{1}{5r^4} + \&c}$ may be considered as be-

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ing equal to the fraction $\frac{1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \&c}{1}$ or as equal to the numerator of that fraction, or to the Series $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \&c$. And therefore $v \times$ the fraction

$\frac{1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \&c}{1 + \frac{1}{2r} + \frac{1}{3r^2} + \frac{1}{4r^3} + \frac{1}{5r^4} + \&c}$ may be considered as equal to $v \times$ the Series $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \&c$. Therefore x, or the Logarithm of the given Sine S, may be considered as being equal to $v \times$ the Series $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \&c$.

An Example of the Computation of the Logarithm of a given Sine by means of the foregoing expreſſion.

Art. 29. Let the given Sine S be $= 9{,}990{,}000$, which is a little leſs than $9{,}990{,}098$, or the Sine of an arch of $87°, 27'$ according to the foregoing Trigonometrical Canon of Lord Napier. Then will v, or $r - S$, be $(= 10{,}000{,}000 - 9{,}990{,}000) = 10{,}000$, and $\frac{v}{r}$ will be $(= \frac{10{,}000}{10{,}000{,}000} = \frac{1}{1000}) = 0.001$, and conſequently $\frac{v^2}{r^2}$ will be $= 0.000{,}001$, and $\frac{v^3}{r^3}$ will be $= 0.000{,}000{,}001$, and $\frac{v^4}{r^4}$ will be $= 0.000{,}000{,}000{,}001$. Therefore the ſeries $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \&c$ will be $= 1 + \frac{0.001}{2} + \frac{0.000{,}001}{3} + \frac{0.000{,}000{,}001}{4}$ $(= 1 + 0.000{,}500{,}000 + 0.000{,}000{,}111 + 0.000{,}000{,}000) = 1.000{,}500{,}111$; and $v \times$ the ſeries $1 + \frac{v}{2r} + \frac{v^2}{3r^2}$

$\frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4}$ + &c will be = 10,000 × 1.000,500,111 = 10,005.001,11; that is, the Logarithm of the Sine 9,990,000, or of it's ratio to the radius 10,000,000, will be equal to 10,005.001,11, or (neglecting the fractional part .001,11, as being leſs than 1, or the 10,000,000th part of the radius,) to 10,005.

As the Sine 9,990,000 is leſs than 9,990,098, or the Sine of 87°, 27′, it's ratio to the radius 10,000,000 will be a greater ratio of minority than the ratio of 9,990,098 to the radius, and therefore it's Logarithm muſt be greater than the Logarithm of 9,990,098, or the Sine of 87°, 27′. And ſo we find it is: for Lord Napier's Logarithm of the Sine of 87°, 27′ is 9907, which is leſs than 10,005, or the Logarithm juſt now found.

Further, the Sine of an arch of 87°, 26′, in Lord Napier's Trigonometrical Canon, is 9,989,968; which is leſs than the Sine 9,990,000. Therefore the given Sine 9,990,000 is of an intermediate magnitude between the two Sines 9,989,968 and 9,990,098, and conſequently it's Logarithm 10,005 ought to be of an intermediate magnitude between the Logarithms of the ſaid two Sines, that is, greater than the Logarithm of the Sine 9,990,098 or the Sine of an arch of 87°, 27′, but leſs than the Logarithm of the Sine of 9,989,968, or the Sine of an arch of 87°, 26′. And ſo we find it is: for Lord Napier's Logarithm of the Sine 9,989,968, or the Sine of 87°, 26′, is 10,037, which is greater than 10,005; and we have already obſerved that Lord Napier's Logarithm of 9,990,098, or the Sine of 87°, 27′, is 9,907, which is leſs than 10,005.

And, if we employ the Differential Method of Approximation, (or the Method of Trial and Errour, or the Double Rule of Falſe Poſition, as it is often called,) to derive from Lord Napier's two Logarithms of the Sines of 87°, 26′ and 87°, 27′, to wit, 10,037 and 9,907, the Logarithm of the intermediate Sine 9,990,000, we ſhall find it to be equal to 10,005, which

is

is exactly equal to the value of the said Logarithm found by the foregoing expression. The Process of this application will be as follows.

The Sine of 87°, 26′ is = 9,989,968, and it's Logarithm is 10,037;
The given Sine S is = 9,990,000, and it's Logarithm is x;
The Sine of 87°, 27′ is = 9,990,098, and it's Logarithm is 9,907.

Therefore the difference of the first and second Sines will be to the difference of the second and third Sines in, nearly, the same proportion as the difference of the first and second Logarithms is to the difference of the second and third Logarithms; that is, 9,990,000 — 9,989,068 will be to 9,990,098 — 9,990,000 as 10,037 — x is to x — 9,907, or 32 will be to 98 as 10,037 — x is to x — 9,907. Therefore, *componendo*, 32 + 98 will be to 98 as 10,037 — x + x — 9,907 is to x — 9,907, or 130 will be to 98 as 10,037 — 9,907 is to x — 9,907, or 130 will be to 98 as 130 is to x — 9,907. Therefore 98 will be = x — 9,907, and x will be (= 98 + 9,907) = 10,005. Therefore the Logarithm, x, of the given Sine 9,990,000, when derived from the Logarithms 10,037, and 9,907 of the two contiguous Sines 9,989,968 and 9,990,098 by means of this Differential Method of Approximation, will be = 10,005; which is exactly equal to the former value of the same Logarithm, which was obtained by Mr. Ivory's expression.

Another Example of a like Computation of the Logarithm of a given Sine.

Art. 30. Let the given Sine S be 9,900,000, which is a little less than 9,900,237, or the Sine of 81°, 54′, and a little greater than 9,899,827, or the Sine of 81°, 53′, according to the foregoing Trigonometrical Canon of Lord Napier. Then will v, or r — S, be (= 10,000,000 — 9,900,000) = 100,000, and $\frac{v}{r}$ will be $\left(= \frac{100,000}{10,000,000} = \frac{1}{100}\right) = 0.01$, and consequently

$\frac{v^2}{r^2}$

$\frac{v^2}{r^2}$ will be = 0.000,1, and $\frac{v^3}{r^3}$ will be = 0.000,001, and $\frac{v^4}{r^4}$ will be = 0.000,000,01, and $\frac{v^5}{r^5}$ will be = 0.000,000,000,1. Therefore the Series $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4}$ + &c will be = $1 + \frac{0.010,000,000}{2} + \frac{0.000,001,000}{3} + \frac{0.000,000,010}{4}$ + &c (= 1 + 0.005,000,000 + 0.000,000,333 + 0.000,000,002 + &c) = 1.005,000,335; and $v \times$ the Series $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4}$ + &c will be (= 100,000 × 1.005,000,335) = 100,500.0335; that is, the Logarithm x of the Sine 9,900,000, or of it's ratio to the radius 10,000,000, will be = 100,500.0335, or (neglecting the fraction .0335, as being leſs than 1, or the 10,000,000th part of the radius 10,000,000,) will be = 100,500.

Q. E. I.

Lord Napier's Logarithm of 9,900,237, or the Sine of 81°, 54′, is 100,265; and his Logarithm of 9,899,827, or the Sine of 81°, 53′, is 100,679. And, ſince the given Sine S, or 9,900,000, is of an intermediate magnitude between the Sines 9,900,237 and 9,899,827, it follows that the Logarithm of the Sine S, or 9,900,000, ought to be of an intermediate magnitude between the Logarithms of the two other Sines, or between the Logarithms 100,265 and 100,679. And ſo the Logarithm juſt-now obtained by Mr. Ivory's Series, to wit, 100,500, is found to be, being greater than the Logarithm 100,265, but leſs than the Logarithm 100,679. And, if we derive the Logarithm of the given Sine 9,900,000, (which does not occur in Lord Napier's Trigonometrical Canon,) from the Logarithms of the two contiguous Sines 9,899,827 and 9,900,237 in that Canon, to wit, the two Logarithms 100,679 and 100,265, by the Differential Method of Approximation, we ſhall find that the Logarithm of the Sine 9,900,000 that will be thereby obtained, will differ very little from the foregoing value of the ſame Logarithm,

to

to wit, the number 100,500, which was obtained by Mr. Ivory's expreſſion. This computation of the ſaid Logarithm by the Differential Method of Approximation will be as follows.

The Sine of 81°, 53′ is = 9,899,827, and it's Logarithm is 100,679;
The given Sine S is = 9,900,000, and it's Logarithm is x;
The Sine of 81°, 54′ is = 9,900,237, and it's Logarithm is 100,265.

Therefore the difference of the firſt and ſecond Sines will be to the difference of the ſecond and third Sines in, nearly, the ſame proportion as the difference of the firſt and ſecond Logarithms is to the difference of the ſecond and third Logarithms; that is, 9,900,000 — 9,899,827 will be to 9,900,237 — 9,900,000 in, nearly, the ſame proportion as 100,679 — x is to x — 100,265, or 173 will be to 237 in, nearly, the ſame proportion as 100,679 — x is to x — 100,265. Therefore, *componendo*, 173 + 237, or 410, will be to 237, very nearly, in the ſame proportion as 100,679 — x + x — 100,265, (or 100,679 — 100,265, or 679 — 265,) or 414, is to x — 100,265. Therefore $410 \times \overline{x - 100{,}265}$ will be, very nearly, = 237 × 414, or $410x - 41{,}108{,}650$ will be, very nearly, = 98,118, and conſequently $410x$ will be (= 98,118 + 41,108,650) = 41,206,768, and x will be $\left(= \frac{41{,}206{,}768}{410}\right)$ = 100,504; that is, the Logarithm of the given Sine S, or 9,900,000, when derived from the two contiguous Logarithms 100,679 and 100,265 by this Differential Method of Approximation, will be, very nearly, = 100,504; which number differs only in the ſixth, or laſt, figure, 4, from the former value of the ſame Logarithm, to wit, the number 100,500, which was obtained by Mr. Ivory's expreſſion

A Third Example of the like Computation of the Logarithm of a given Sine by the foregoing expreſſion of Mr. Ivory.

Art. 31. Let the given Sine S be 9,000,000, which is a little leſs than 9,000,654, or the Sine of 64°, 10′, and a little greater than 8,999,386, or the Sine of 64°, 9′, according to Lord Napier's Trigonometrical Canon. And let it be required to find the Logarithm, x, of this Sine 9,000,000, by means of the foregoing expreſſion of Mr. Ivory, to wit, $x = v \times$ the Series $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \frac{v^5}{6r^5} + \frac{v^6}{7r^6} + \frac{v^7}{8r^7} + \frac{v^8}{9r^8} + \frac{v^9}{10r^9}$ + &c, in which v is $= r -$ S, or 10,000,000 — S.

Now, ſince S is = 9,000,000, we ſhall have $r -$ S, or v, (= 10,000,000 — 9,000,000) = 1,000,000, and conſequently $\frac{v}{r}$ ($= \frac{1,000,000}{10,000,000} = \frac{1}{10}$) = 0.10. Therefore $\frac{v^2}{r^2}$ will be = 0.01, and $\frac{v^3}{r^3}$ will be = 0.001, and $\frac{v^4}{r^4}$ will be = 0.0001, and $\frac{v^5}{r^5}$ will be = 0.000,01, and $\frac{v^6}{r^6}$ will be = 0.000,001, and $\frac{v^7}{r^7}$ will be = 0.000,000,1, and $\frac{v^8}{r^8}$ will be = 0.000,000,01, and $\frac{v^9}{r^9}$ will be = 0.000,000,001; and conſequently the Series $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \frac{v^5}{6r^5} + \frac{v^6}{7r^6} + \frac{v^7}{8r^7} + \frac{v^8}{9r^8} + \frac{v^9}{10r^9}$ + &c will be $= 1 + \frac{0.10}{2} + \frac{0.010}{3} + \frac{0.001,00}{4} + \frac{0.000,100}{5} + \frac{0.000,010,000}{6} + \frac{0.000,001,000}{7} + \frac{0.000,000,100}{8} + \frac{0.000,000,010}{9} + \frac{0.000,000,001}{10}$ + &c = 1 + 0.05 + 0.003,333,333 + 0.000,250,000 + 0.000,020,000 + 0.000,001,666 + 0.000,000,142 + 0.000,000,012 + 0.000,000,001 + 0.000,000,000 + &c = 1.053,605,154. Therefore x,

or

or $v \times$ the Series $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \frac{v^5}{6r^5} + \frac{v^6}{7r^6} + \frac{v^7}{8r^7} + \frac{v^8}{9r^8} + \frac{v^9}{10r^9} +$ &c, will be $=$ 1,000,000 $\times$ 1,053,605,154 $=$ 1,053,605.154; that is, the Logarithm of the Sine S, or 9,000,000, or of it's ratio to the radius r, or 10,000,000, will be $=$ 1,053,605.154, or (omitting the fraction .154, becauſe it is leſs than 1, or the 10,000,000th part of the radius r, or 10,000,000) it will be $=$ 1,053,605. Q. E. I.

This value, 1,053,605, of the Logarithm of the Sine S, or 9,000,000, agrees very well with Lord Napier's Logarithm of the two contiguous Sines 9,000,654 and 8,999,386, which are 1,052,878 and 1,054,287; being of an intermediate magnitude between thoſe two Logarithms, juſt as the given Sine S, or 9,000,000, to which it belongs, is of an intermediate magnitude between the ſaid two Sines 9,000,654 and 8,999,386. And, if we derive the Logarithm of the given Sine 9,000,000 from the Logarithms of the ſaid two contiguous Sines 9,000,654 and 8,999,386, to wit, the Logarithms 1,052,878 and 1,054,287, by means of the Differential Method of Approximation, we ſhall find that the value of the Logarithm of the given Sine 9,000,000, that will be thereby obtained, will differ very little from the value of it found above by means of Mr. Ivory's expreſſion, to wit, 1,053,605. This computation will be as follows.

Since the three Sines 9,000,654, 9,000,000, and 8,999,386, differ but little from each other, their three Logarithms, to wit, 1,052,878, the unknown Logarithm x, and 1,054,287, will conſequently alſo differ but little from each other. Therefore the differences of the former three quantities will be to each other in, nearly, the ſame proportion as the differences of the three latter quantities; that is, 9,000,654 — 9,000,000 will be to 9,000,000 — 8,999,386 in, nearly, the ſame proportion as x — 1,052,878 is to 1,054,287 — x, or 654 will be to 614 in, nearly, the ſame proportion as x — 1,052,878 is to 1,054,287 — x. Therefore, *componendo*, 654 + 614 will be to 614 in, nearly, the ſame proportion as x — 1,052,878 + 1,054,287 — x is to 1,054,287 — x; that is, 1268 will be to 614 in, nearly, the ſame proportion

tion as 1,054,287 — 1,052,878, or 1409, is to 1,054,287 — x. Therefore $1268 \times \overline{1{,}054{,}287 - x}$ will be, nearly, = 614 × 1409, that is, 1268 × 1,054,287 − 1268 × x will be, nearly, = 865,126, or 1,336,835,916 — 1268 × x will be, nearly, = 865,126. Therefore 1,336,835,916 will be, nearly, = 865,126 + 1268 × x, and 1268 × x will be, nearly, (= 1,336,835,916 — 865,126) = 1,335,970,790, and consequently x will be, nearly, $\left(= \frac{1{,}335{,}970{,}790}{1268}\right)$ = 1,053,604.7; which differs by less than an unit from the value of the same Logarithm that was found above by Mr. Ivory's expression, which was 1,053,605. This very near agreement of the two values of this Logarithm of the given Sine 9,000,000, (of which the first is obtained by Mr. Ivory's expression $x = v \times$ the infinite Series $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \frac{v^5}{6r^5}$ + &c, and the second is derived from the Logarithms of the two contiguous Sines 9,000,654 and 8,999,386, by the Differential Method of Approximation,) is a confirmation both of the accuracy of this expression of Mr. Ivory and of the exactness of the two Logarithms of the said two contiguous Sines, to wit, the Logarithms 1,052,878 and 1,054,287, that are set-down in Lord Napier's Table.

A Fourth Example of the like Computation of the Logarithm of a given Sine by the foregoing expression of Mr. Ivory.

Art. 32. Let the given Sine S be 5,000,000, which is the Sine of an arch of 30 degrees in Lord Napier's Trigonometrical Canon. And let it be required to find the Logarithm of this Sine, or of it's ratio to the radius 10,000,000, in Lord Napier's System of Logarithms, by means of the foregoing expression.

Since S is = 5,000,000, we ſhall have $r - S$, or v, (= 10,000,000 − 5,000,000) = 5,000,000, and $\frac{v}{r}$ ($= \frac{5,000,000}{10,000,000} = \frac{5}{10}$) $= \frac{1}{2}$, or 0.5. Therefore $\frac{v^2}{r^2}$ will be ($= \frac{1}{4}$) = 0.25, and $\frac{v^3}{r^3}$ will be ($= \frac{1}{8}$) = 0.125, and $\frac{v^4}{r^4}$ will be ($= \frac{0.125}{2}$) = 0.0625, and $\frac{v^5}{r^5}$ will be ($= \frac{0.0625}{2}$) = 0.031,25, and $\frac{v^6}{r^6}$ will be ($= \frac{0.031,25}{2}$) = 0.015,625, and $\frac{v^7}{r^7}$ will be ($= \frac{0.015,625}{2}$) = 0.007,812,5, and $\frac{v^8}{r^8}$ will be ($= \frac{0.007,812,5}{2}$) = 0.003, 906,25, and $\frac{v^9}{r^9}$ will be ($= \frac{0.003,906,25}{2}$) = 0.001,953,125, and $\frac{v^{10}}{r^{10}}$ will be ($= \frac{0.001,953,125}{2}$) = 0.000,976,562, and $\frac{v^{11}}{r^{11}}$ will be ($= \frac{0.000,976,562}{2}$) = 0.000,488,281, and $\frac{v^{12}}{r^{12}}$ will be ($= \frac{0.000,488,281}{2}$) = 0.000,244,140, and $\frac{v^{13}}{r^{13}}$ will be ($= \frac{0.000,244,140}{2}$) = 0.000,122,070, and $\frac{v^{14}}{r^{14}}$ will be ($= \frac{0.000,122,070}{2}$) = 0.000,061,035, and $\frac{v^{15}}{r^{15}}$ will be ($= \frac{0.000,061,035}{2}$) = 0.000,030,517, and $\frac{v^{16}}{r^{16}}$ will be ($= \frac{0.000,030,517}{2}$) = 0.000,015,258, and $\frac{v^{17}}{r^{17}}$ will be ($= \frac{0.000,015,258}{2}$) = 0.000,007,629, and $\frac{v^{18}}{r^{18}}$ will be ($= \frac{0.000,007,629}{2}$) = 0.000,003,814, and $\frac{v^{19}}{r^{19}}$ will be ($= \frac{0.000,003,814}{2}$ = 0.000,001,907. Therefore the Series $1 + \frac{v}{2r} + \frac{v^2}{3r^2} + \frac{v^3}{4r^3} + \frac{v^4}{5r^4} + \frac{v^5}{6r^5} + \frac{v^6}{7r^6} + \frac{v^7}{8r^7} + \frac{v^8}{9r^8} + \frac{v^9}{10r^9} + \frac{v^{10}}{11r^{10}} + \frac{v^{11}}{12r^{11}} + \frac{v^{12}}{13r^{12}} + \frac{v^{13}}{14r^{13}} + \frac{v^{14}}{15r^{14}} + \frac{v^{15}}{16r^{15}} + \frac{v^{16}}{17r^{16}} + \frac{v^{17}}{18r^{17}} + \frac{v^{18}}{19r^{18}} + \frac{v^{19}}{20r^{19}}$ + &c will be = $1.000,000,000 + \frac{0.50}{2} + \frac{0.250}{3} + \frac{0.125}{4} + \frac{0.0625}{5} + \frac{0.031,25}{6} + \frac{0.015,625}{7} + \frac{0.007,812,5}{8} + \frac{0.003,906,25}{9} + \frac{0.001,953,125}{10} + \frac{0.000,976,562}{11} + \frac{0.000,488,281}{12} + \frac{0.000,244,140}{13} + \frac{0.000,122,070}{14}$ +

$$\frac{0.000,122,070}{14} + \frac{0.000,061,035}{15} + \frac{0.000,030,517}{16} + \frac{0.000,015,258}{17} + \frac{0.000,007,629}{18} + \frac{0.000,003,814}{19} + \&c = 1.386,294,172;$$

	1.000,000,000	+ 0.000,088,778
+	.250,000,000	+ 0.000,040,690
+	.083,333,333	+ 0.000,018,780
+	.031,250,000	+ 0.000,008,719
+	.012,500,000	+ 0.000,004,069
+	.005,208,333	+ 0.000,001,907
+	.002,232,142	+ 0.000,000,897
+	.000,976,562	+ 0.000,000,423
+	.000,434,027	+ 0.000,000,200
+	.000,195,312	
	1.386,129,709	0.000,164,463
	.000,164,463	
	1.386,294,172	

and consequently $v \times$ the said series will be $= v \times$ 1.386,294,172 = 5,000,000 × 1.386,294,172 = 6,931,470.86. Therefore x, or the Logarithm of the Sine S, or 5,000,000, of an arch of 30 degrees, in Lord Napier's System of Logarithms, will be = 6,931,470.86, or (omitting the fraction .86, because it is less than 1, or the 10,000,000th part of the radius 10,000,000,) will be = 6,931,470. Q. E. I.

This value of Lord Napier's Logarithm of the Sine S, or 5,000,000, of an arch of 30°, agrees with the value of it set-down in Lord Napier's Table, (which is 6,931,469,) in the five highest figures 6,931,4, and exceeds that value by only an unit in the lowest place of figures; and it is rather nearer to the true value of that Logarithm than the said value of it set-down in Lord Napier's Table. For the more accurate value of this Logarithm, (which is the

Logarithm

Logarithm of the ratio of 1 to 2, or of the ratio of 2 to 1,) is 6,931,471.805, 599,453,04, as appears from the computation of the Logarithm of 2, in the first volume of this Collection of Tracts called *Scriptores Logarithmici*, page 267.

These four examples are sufficient to illustrate this method of computing Lord Napier's Logarithms of the Sines of circular arches set-down in the foregoing Trigonometrical Canon, or of the ratios of those Sines to the radius, by means of the foregoing expression communicated to me by Mr. Ivory. But it may sometimes be also convenient to be able to derive from a given Logarithm in the foregoing system of Lord Napier the value of the Sine to which such given Logarithm belongs, or of the ratio of which to the radius r, or 10,000,000, it is the measure. Now this Problem has likewise been solved by Mr. Ivory in a very clear and satisfactory manner, and he has communicated the solution of it to me with permission to publish it on this occasion. This I shall therefore now proceed to do.

A Method of deriving from Lord Napier's Logarithm of a Sine of which the magnitude is unknown, the value of the Sine to which such given Logarithm belongs, or of the ratio of which to the radius r, or 10,000,000, such Logarithm is the measure.

By James Ivory, M. A. of the New Military College at Great Marlow in Buckinghamshire.

Art. 33. It has been shown above, in Art. 27, that, if the radius 10,000,000 be denoted by r, and $r-1$, or 10,000,000 — 1, or 9,999,999, be denoted by a, and the Sine S be equal to $\frac{a^x}{r^{x-1}}$, or to $\frac{\overline{r-1}^x}{r^{x-1}}$, that x will be the Logarithm

of

of S, or the quantity $\frac{a^x}{r^{x-1}}$, or $\frac{\overline{r-1}^x}{r^{x-1}}$, or of the ratio of S, or $\frac{a^x}{r^{x-1}}$, or $\frac{\overline{r-1}^x}{r^{x-1}}$, to the radius r, or 10,000,000. Therefore when the Logarithm x is known, and the Sine S is unknown, (as we have here supposed them to be,) the Solution of the present Problem will depend on the resolution of the equation $S = \frac{\overline{r-1}^x}{r^x-1}$, which Mr. Ivory performs in the following manner.

Mr. Ivory's Resolution of the Equation $S = \frac{\overline{r-1}^x}{r^x-1}$ *when* r *and* x *are known, and* S *is unknown.*

Art. 34. The fraction $\frac{\overline{r-1}^x}{r^x-1}$ is $\left(= \frac{r \times \overline{r-1}^x}{r \times r^x-1} = \frac{r \times \overline{r-1}^x}{r^x} = r \times \frac{\overline{r-1}^x}{r^x} = r \times \overline{\frac{r-1}{r}}^x \right) = r \times \overline{1 - \frac{1}{r}}^x$. Therefore S, which is equal to $\frac{\overline{r-1}^x}{r^x-1}$, will also be equal to $r \times \overline{1 - \frac{1}{r}}^x$; and consequently $\frac{S}{r}$ will be $= \overline{1 - \frac{1}{r}}^x$.

Now let n be any whole number whatsoever that is greater than x; and let the nth roots of these two equal quantities $\frac{S}{r}$ and $\overline{1 - \frac{1}{r}}^x$ be taken, to wit, $\surd^n \overline{\frac{S}{r}}$ and $\surd \overline{1 - \frac{1}{r}}^n$. Then it is evident that these roots will be equal to each other; that is, $\surd^n \overline{\frac{S}{r}}$ will be $= \surd^n \overline{1 - \frac{1}{r}}^x$, or $\overline{\frac{S}{r}}^{\frac{1}{n}}$ will be $= \overline{\overline{1 - \frac{1}{r}}^x}^{\frac{1}{n}} \left(= \overline{1 - \frac{1}{r}}^{x \times \frac{1}{n}} \right) = \overline{1 - \frac{1}{r}}^{\frac{x}{n}}$.

But, because x, the numerator of the fraction $\frac{x}{n}$ (which is the Index of the power of the binomial quantity $1 - \frac{1}{r}$,) is less than n the denominator, it follows

lows from Sir Isaac Newton's binomial theorem (in the case of the quantity $\overline{1-x}\Big|^{\frac{m}{n}}$ when m is less than n,) that the quantity $\overline{1-\frac{1}{r}}\Big|^{\frac{x}{n}}$ will be equal to the infinite series $1-\frac{x}{n}\times\frac{1}{r}-\frac{x}{n}\times\frac{n-x}{2n}\times\frac{1}{r^2}-\frac{x}{n}\times\frac{n-x}{2n}\times\frac{2n-x}{3n}\times\frac{1}{r^3}$ $-\frac{x}{n}\times\frac{n-x}{2n}\times\frac{2n-x}{3n}\times\frac{3n-x}{4n}\times\frac{1}{r^4}-\frac{x}{n}\times\frac{n-x}{2n}\times\frac{2n-x}{3n}\times\frac{3n-x}{4n}\times\frac{4n-x}{5n}$ $\times\frac{1}{r^5}-$ &c, in which all the terms after the first term 1 are marked with the sign $-$, or are subtracted from the said first term; as is shown in Volume IInd of the *Scriptores Logarithmici*, pages 362, 363, 364, and 365.

Therefore $\overline{\frac{S}{r}}\Big|^{\frac{1}{n}}$ will be equal to this Series.

Now let the sum of all the terms of this Series, except the first term 1, from which all the rest are subtracted, be called p; that is, let the Series $\frac{x}{n}\times\frac{1}{r}+\frac{x}{n}\times\frac{n-x}{2n}\times\frac{1}{r^2}+\frac{x}{n}\times\frac{n-x}{2n}\times\frac{2n-x}{3n}\times\frac{1}{r^3}+\frac{x}{n}\times$ $\frac{n-x}{2n}\times\frac{2n-x}{3n}\times\frac{3n-x}{4n}\times\frac{1}{r^4}+\frac{x}{n}\times\frac{n-x}{2n}\times\frac{2n-x}{3n}\times\frac{3n-x}{4n}\times\frac{4n-x}{5n}\times$ $\frac{1}{r^5}+$ &c be called p. Then will the former series be equal to $1-p$.

And therefore the quantity $\overline{\frac{S}{r}}\Big|^{\frac{1}{n}}$ (to which it has been shown that the said former series is equal,) will also be equal to $1-p$.

But the Series $\frac{x}{n}\times\frac{1}{r}+\frac{x}{n}\times\frac{n-x}{2n}\times\frac{1}{r^2}+\frac{x}{n}\times\frac{n-x}{2n}\times\frac{2n-x}{3n}\times$ $\frac{1}{r^3}+\frac{x}{n}\times\frac{n-x}{2n}\times\frac{2n-x}{3n}\times\frac{3n-x}{4n}\times\frac{1}{r^4}+\frac{x}{n}\times\frac{n-x}{2n}\times\frac{2n-x}{3n}\times\frac{3n-x}{4n}$ $\times\frac{4n-x}{5n}\times\frac{1}{r^5}+$ &c is (= the Series $\frac{x}{n}\times\frac{1}{r}\times 1+\frac{x}{n}\times\frac{1}{r}\times\frac{n-x}{2n}$ $\times\frac{1}{r}+\frac{x}{n}\times\frac{1}{r}\times\frac{n-x}{2n}\times\frac{2n-x}{3n}\times\frac{1}{r^2}+\frac{x}{n}\times\frac{1}{r}\times\frac{n-x}{2n}\times\frac{2n-x}{3n}\times\frac{3n-x}{4n}$ $\times\frac{1}{r}$

$\times \frac{1}{r^3} + \frac{x}{n} \times \frac{1}{r} \times \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{3n-x}{4n} \times \frac{4n-x}{5n} \times \frac{1}{r^4} + \&c\Big) = \frac{x}{n}$ $\times \frac{1}{r} \times$ the Series $1 + \frac{n-x}{2n} \times \frac{1}{r} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^2} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{3n-x}{4n} \times \frac{1}{r^3} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{3n-x}{4n} \times \frac{4n-x}{5n} \times \frac{1}{r^4} + \&c,$ which laſt Series we will denote by the letter y. And then we ſhall have $p = \frac{x}{n} \times \frac{1}{r} \times y \left(= \frac{x}{r} \times \frac{1}{n} \times y\right) = \frac{x}{r} \times \frac{y}{n}$, and $1 - p = 1 - \frac{x}{r} \times \frac{y}{n}$, and conſequently $\overline{\frac{S}{r}}^{\frac{1}{n}} (= 1 - p) = 1 - \frac{x}{r} \times \frac{y}{n}$.

Art. 35. Having thus obtained the equation $\frac{S}{r}^{\frac{1}{n}} = 1 - \frac{x}{r} \times \frac{y}{n}$, or $\overline{\frac{S}{r}}^{\frac{1}{n}} = 1 - \frac{xy}{rn}$, let both ſides of it be raiſed to the nth power. And we ſhall then have $\frac{S}{r} =$ (by the binomial Theorem,) to the Series $1 - \frac{n}{1} \times \frac{xy}{rn} + \frac{n}{1} \times \frac{n-1}{2} \times \frac{x^2 y^2}{r^2 n^2} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{x^3 y^3}{r^3 n^3} \times \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{x^4 y^4}{r^4 n^4} - \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{x^5 y^5}{r^5 n^5} + \&c$ $= 1 - \frac{xy}{r} + \frac{n-1}{2n} \times \frac{x^2 y^2}{r^2} - \overline{\frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{x^3 y^3}{r^3}} + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{x^4 y^4}{r^4} - \overline{\frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{n-4}{5n} \times \frac{x^5 y^5}{r^5}} + \&c = 1 - \frac{x}{r} \times y + \frac{n-1}{2n} \times \frac{x^2}{r^2} \times y^2 - \overline{\frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{x^3}{r^3}} \times y^3 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{x^4}{r^4} \times y^4 - \overline{\frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{n-4}{5n} \times \frac{x^5}{r^5}} \times y^5 + \&c.$ And this equation will be accurately true, of whatſoever magnitude greater than x we may ſuppoſe the number n to be taken.

Art. 36. Now, when the number n is of any magnitude greater than x, the

the Series denoted by the letter y, to wit, the Series $1 + \frac{n-x}{2n} \times \frac{1}{r} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^2} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{3n-x}{4n} \times \frac{1}{r^3} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{3n-x}{4n} \times \frac{4n-x}{5n} \times \frac{1}{r^4}$ + &c will be leſs than the Series $1 + \frac{n}{2n} \times \frac{1}{r} + \frac{n}{2n} \times \frac{2n}{3n} \times \frac{1}{r^2} + \frac{n}{2n} \times \frac{2n}{3n} \times \frac{3n}{4n} \times \frac{1}{r^3} + \frac{n}{2n} \times \frac{2n}{3n} \times \frac{3n}{4n} \times \frac{4n}{5n} \times \frac{1}{r^4}$ + &c, or than the Series $1 + \frac{1}{2r} + \frac{1}{3r^2} + \frac{1}{4r^3} + \frac{1}{5r^4}$ + &c, becauſe the co-efficients of the terms of the former Series are leſs than the co-efficients of the correſponding terms of the latter Series. But the difference between theſe two Serieſes will continually grow leſs and leſs as the number n is ſuppoſed to be greater and greater; becauſe each of the fractions $\frac{n-x}{2n}$, $\frac{2n-x}{3n}$, $\frac{3n-x}{4n}$, $\frac{4n-x}{5n}$, &c, (which are the factors by the continual multiplication of which the co-efficients of the terms of the former Series are produced,) will continually grow greater and greater, while the number n increaſes, and will approach to the ſeveral fractions $\frac{n}{2n}$, $\frac{2n}{3n}$, $\frac{3n}{4n}$, $\frac{4n}{5n}$, &c, or $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, &c, reſpectively, as being the limits of their magnitude. Thus, for example, if n is at firſt only equal to m times x, $\frac{4n-x}{5n}$ will be $= \frac{4mx-x}{5mx} = \frac{4m-1}{5m}$; and, when n is increaſed from m times x to ſome greater multiple of x, as M x, $\frac{4n-x}{5n}$ will be $= \frac{4Mx-x}{5Mx} = \frac{4M-1}{5M}$; which is equal to $\frac{4M}{5M} - \frac{1}{5M} = \frac{4m}{5m} - \frac{1}{5M}$, and therefore is greater than $\frac{4m}{5m} - \frac{1}{5m}$, or than $\frac{4m-1}{5m}$, but is leſs than $\frac{4M}{5M}$, or $\frac{4}{5}$, and never can be quite equal to it, how great ſoever the number M be taken. Therefore, if n is taken of a very great magnitude in compariſon of x, (as, for example, equal to 10,000,000 $\times$ x, or to ſome ſtill greater multiple of x,) the Series $1 + \frac{n-x}{2n} \times \frac{1}{r} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^2} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{3n-x}{4n} \times \frac{1}{r^3} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{3n-x}{4n} \times \frac{4n-x}{5n} \times \frac{1}{r^4}$ + &c will be very nearly equal to the Series $1 + \frac{1}{2r} + \frac{1}{3r^2} + \frac{1}{4r^3} + \frac{1}{5r^4}$ + &c; and it may, by increaſing the mag-

nitude of the number n, be made to approach as near to the ſaid laſt-mentioned Series as we pleaſe.

Let this laſt Series $1 + \frac{1}{2r} + \frac{1}{3r^2} + \frac{1}{4r^3} + \frac{1}{5r^4} +$ &c be denoted by the Letter b. And let the number n be taken of ſo great a magnitude in compariſon of x that the former Series $1 + \frac{n-x}{2n} \times \frac{1}{r} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{1}{r^2} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{3n-x}{4n} \times \frac{1}{r^3} + \frac{n-x}{2n} \times \frac{2n-x}{3n} \times \frac{3n-x}{4n} \times \frac{4n-x}{5n} \times \frac{1}{r^4} +$ &c, which has been denoted above by the letter y, ſhall be very nearly equal to the Series $1 + \frac{1}{2r} + \frac{1}{3r^2} + \frac{1}{4r^3} + \frac{1}{5r^4} +$ &c, or b, which is the limit of it's magnitude. And let this latter Series, or it's value b, be ſubſtituted inſtead of y, or the value of that former Series, in the equation $\frac{S}{r} =$ the Series $1 - \frac{x}{r} \times y + \frac{n-1}{2n} \times \frac{x^2}{r^2} \times y^2 - \sqrt{\frac{n-1}{2n}} \times \frac{n-2}{3n} \times \frac{x^3}{r^3} \times y^3 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{x^4}{r^4} \times y^4 - \sqrt{\frac{n-1}{2n}} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{n-4}{5n} \times \frac{x^5}{r^5} \times y^5 +$ &c. And we ſhall then have $\frac{S}{r} =$ the Series $1 - \frac{x}{r} \times b + \frac{n-1}{2n} \times \frac{x^2}{r^2} \times b^2 - \sqrt{\frac{n-1}{2n}} \times \frac{n-2}{3n} \times \frac{x^3}{r^3} \times b^3 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{x^4}{r^4} \times b^4 - \sqrt{\frac{n-1}{2n}} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{n-4}{5n} \times \frac{x^5}{r^5} \times b^5 +$ &c.

Art. 37. But, when n is a very great number, as 10,000,000, or 10,000,000,000, or ſome ſtill greater number, the ſeveral fractions $\frac{n-1}{2n}$, $\frac{n-2}{3n}$, $\frac{n-3}{4n}$, and $\frac{n-4}{5n}$, &c will become very nearly equal to the fractions $\frac{n}{2n}$, $\frac{n}{3n}$, $\frac{n}{4n}$, and $\frac{n}{5n}$, &c reſpectively, and conſequently to the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, and $\frac{1}{5}$, &c, reſpectively. Therefore theſe latter fractions may be ſubſtituted inſtead of the former in the foregoing equation $\frac{S}{r} =$ the Series $1 - \frac{x}{r} \times b + \frac{n-1}{2n} \times \frac{x^2}{r^2} \times b^2 - \sqrt{\frac{n-1}{2n}} \times \frac{n-2}{3n} \times \frac{x^3}{r^3} \times b^3 + \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{x^4}{r^4}$

$\frac{x^4}{r^4} \times b^4 - \frac{n-1}{2n} \times \frac{n-2}{3n} \times \frac{n-3}{4n} \times \frac{n-4}{5n} \times \frac{x^5}{r^5} \times b^5 +$ &c. And we ſhall then have $\frac{S}{r}$ = the Series $1 - \frac{x}{r} \times b + \frac{1}{2} \times \frac{x^2}{r^2} \times b^2 - \frac{1}{2} \times \frac{1}{3} \times \frac{x^3}{r^3} \times b^3 + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{x^4}{r^4} \times b^4 - \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5} \times \frac{x^5}{r^5} \times b^5 +$ &c = the Series $1 - \frac{x}{r} \times b + \frac{1}{2} \times \frac{x^2}{r^2} \times b^2 - \frac{1}{2.3} \times \frac{x^3}{r^3} \times b^3 + \frac{1}{2.3.4} \times \frac{x^4}{r^4} \times b^4 - \frac{1}{2.3.4.5} \times \frac{x^5}{r^5} \times b^5 +$ &c. Therefore S, or the Sine that correſponds to the given Logarithm x, will be = the Series $r - x \times b + \frac{x^2}{2r} \times b^2 - \frac{x^3}{2.3.r^2} \times b^3 + \frac{x^4}{2.3.4.r^3} \times b^4 - \frac{x^5}{2.3.4.5.r^4} \times b^5 +$ &c.

Art. 38. But b is = the Series $1 + \frac{1}{2r} + \frac{1}{3r^2} + \frac{1}{4r^3} + \frac{1}{5r^4} +$ &c, which may be conſidered on this occaſion as only equal to it's firſt term 1. For, ſince in the foregoing Table of Sines, and their Logarithms, the radius r is = 10,000,000, the Series $1 + \frac{1}{2r} + \frac{1}{3r^2} + \frac{1}{4r^3} + \frac{1}{5r^4}$ will be = $1 + \frac{1}{2 \times 10,000,000} + \frac{1}{3 \times 10,000,000^2} + \frac{1}{4 \times 10,000,000^3} + \frac{1}{5 \times 10,000,000^4} +$ &c $= 1 + \frac{0.000,000,10}{2} +$ &c $= 1 + 0.000,000,05 +$ &c; which exceeds the firſt term 1 by only the ſmall quantity 0.000,000,05 + &c, of which the firſt ſignificant figure enters only in the eighth place of decimal fractions. Therefore, if we ſubſtitute the firſt term, 1, of the ſaid ſeries inſtead of b, or the true value of the whole Series, or of ſeveral of it's firſt terms, in the Series $r - x \times b + \frac{x^2}{2r} \times b^2 - \frac{x^3}{2.3.r^2} \times b^3 + \frac{x^4}{2.3.4.r^3} \times b^4 - \frac{x^5}{2.3.4.5.r^4} \times b^5 +$ &c, which is equal to s, the errors ariſing from ſuch ſubſtitution of 1 inſtead of b will not affect the firſt ſeven decimal figures of the value of the ſaid Series, or the Sine S, and therefore will be of no importance in computing the value of a Sine in Lord Napier's Trigonometrical Canon, which is carried only to ſeven places of figures.

Now, if we ſubſtitute 1 inſtead of b in the foregoing Series, we ſhall have b^2 and b^3 and b^4 and b^5 &c, all equal to 1, as well as b. And conſequently the ſaid Series will become = the Series $r - x + \frac{x^2}{2r} - \frac{x^3}{2.3.r^2} + \frac{x^4}{2.3.4.r^3} - \frac{x^5}{2.3.4.5.r^4}$ + &c, and therefore the Sine S, that correſponds to the given Logarithm x, will be equal to the ſaid Series $r - x + \frac{x^2}{2r} - \frac{x^3}{2.3.r^2} + \frac{x^4}{2.3.4.r^3} - \frac{x^5}{2.3.4.5.r^4}$ + &c the Sine to which it belongs, or of the ratio of which to the radius r, or 10,000,000, it is the Logarithm.

Q. E. I.

An Example of the Computation of the Sine to which a given Logarithm belongs by means of the foregoing Series.

Art. 39. Let the given Logarithm be 10,005. And let it be required to find by means of the foregoing Series $r - x + \frac{x^2}{2r} - \frac{x^3}{2.3.r^2} + \frac{x^4}{2.3.4.r^3} - \frac{x^5}{2.3.4.5.r^4}$ + &c the Sine to which it belongs, or of the ratio of which to the radius r, or 10,000,000, it is the Logarithm.

Then will x be = 10,005, and x^2 will be $(= \overline{10,005}^2) = 100,100,025$, and $\frac{x^2}{r}$ will be $(= \frac{100,100,025}{10,000,000}) = 10.010,002,5$, and $\frac{x^3}{r^2}$ will be $(= \frac{x^2}{r} \times \frac{x}{r} = 10.010,002,5 \times \frac{10,005}{10,000,000} = 10.010,002,5 \times 0.001,000,5) =$ 0.010,010, &c, which is leſs than 1, or the 10,000,000th part of the radius r, or 10,000,000, and therefore muſt be neglected. Therefore, *à fortiori*, the term $\frac{x^3}{2.3.r^2}$, which is only the 6th part of $\frac{x^3}{r^2}$,) and the two following terms $\frac{x^4}{2.3.4.r^3}$, or $\frac{x^4}{24r^3}$, and $\frac{x^5}{2.3.4.5.r^4}$, or $\frac{x^5}{120r^4}$ which are ſtill ſmaller, muſt alſo be neglected, and the whole Series muſt be conſidered as being equal to it's

three

three first terms $r - x + \frac{x^2}{2r}$ or to 10,000,000 — 10,005 + $\frac{10,010,007,5}{2}$ = 10,000,000 — 10,005 + 5,005,001,25, or (neglecting the fraction 005,001,25) = 10,000,000 — 10,005 + 5 = 10,000,005 — 10,005 = 9,990,000. Therefore the Sine to which the given Logarithm 10,005 belongs, will be = 9,990,000.

Q. E. I.

This result agrees with the computation made above in Art. 29, pages 671, 672, where the Sine 9,990,000 was given, and the Logarithm 10,005 was derived from it by the former Series of Mr. Ivory.

Another Example of a like Computation of the Sine to which a given Logarithm belongs, by means of the foregoing Series.

Art. 40. Let the given Logarithm be 100,500. And let it be required to find, by means of the foregoing Series $r - x + \frac{x^2}{2r} - \frac{x^3}{2.3.r^2} + \frac{x^4}{2.3.4.r^3} - \frac{x^5}{2.3.4.5.r^4}$ + &c, the Sine to which it belongs, or of the ratio of which to the radius r, or 10,000,000, it is the Logarithm.

Then will x be = 100,500, and x^2 will be $(= \overline{100,500}^2) = 10,100,250,000$, and $\frac{x^2}{r}$ will be $(= \frac{10,100,250,000}{10,000,000}) = 1010$, and $\frac{x^2}{2r}$ will be $(= \frac{1010}{2}) = 505$, and $\frac{x^3}{2.3.r^2}$ will be $(= \frac{x^2}{2r} \times \frac{x}{3r} = 505 \times \frac{x}{3r} = \frac{505}{3} \times \frac{x}{r} = 168 \times \frac{x}{r} = 168 \times \frac{100,500}{10,000,000} = \frac{16,884,000}{10,000,000}) = 1.688,400 =$, nearly, 2. Therefore $r - x + \frac{x^2}{2r} - \frac{x^3}{2.3.r^2}$ + &c will be (= 10,000,000 — 100,500 + 505 — 2 +

2 + &c = 10,000,505 — 100,502) = 9,900,003, and consequently the Sine to which the given Logarithm 100,500 belongs will be = 9,900,003.

Q. E. I.

This result agrees very nearly with the computation made above in Art. 30, pages 673, 674, where the Sine 9,900,000 was given, and the Logarithm 100,500.0335 was derived from it by the former Series of Mr. Ivory; the two values of the Sine S differing only in the seventh, or last, place of figures, in which we have in the present computation *a* 3 instead of a cypher.

A Third Example of the like Computation of the Sine to which a given Logarithm belongs, by means of the foregoing Series.

Art. 41. Let the given Logarithm be 1,053,605. And let it be required to find, by means of the foregoing Series $r - x + \frac{x^2}{2r} - \frac{x^3}{2.3.r^2} + \frac{x^4}{2.3.4.r^3} - \frac{x^5}{2.3.4.5.r^4}$ + &c, the Sine to which it belongs, or of the ratio of which to the radius r, or 10,000,000, it is the Logarithm.

Here x is = 1,053,605, and consequently x^2 will be $(= \overline{1,053,605}^2) =$ 1,110,083,496,025, and $\frac{x^2}{r}$ will be $(= \frac{1,110,083,496,025}{10,000,000} = 111,008.349,602,5$, and $\frac{x^2}{2r}$ will be $(= \frac{111,008.349,602,5}{2} = 55,504.174,301,25) = 55,504$; and $\frac{x^3}{2.3.r^2}$ will be $(= \frac{x^2}{2r} \times \frac{x}{3r} = 55,504 \times \frac{x}{3r} = \frac{55,504}{3} \times \frac{x}{r} = 18,501.3 \times \frac{1,053,605}{10,000,000} = \frac{19,493,062,186.5}{10,000,000} = 1949.306,218,65) = 1949$; and $\frac{x^4}{2.3.4.r^3}$ will be $(= \frac{x^3}{2.3.r^2} \times \frac{x}{4r} = 1949 \times \frac{x}{4r} = \frac{1949}{4} \times \frac{x}{r} = 487.2 \times \frac{x}{r} =$

487.2

$487.2 \times \frac{1,053,605}{10,000,000} = \frac{513,316,356}{10,000,000}) = 51$; and $\frac{x^5}{2.3.4.5.r^4}$ will be $(= \frac{x^4}{2.3.4.r^3} \times \frac{x}{5r} = 51 \times \frac{x}{5r} = \frac{51}{5} \times \frac{x}{r} = 10.2 \times \frac{x}{r} = 10.2 \times \frac{1,053,605}{10,000,000} = \frac{10,746,771}{10,000,000} = 1.074,677,1) = 1$. Therefore the Series $r - x + \frac{x^2}{2r} - \frac{x^3}{2.3.r^2} + \frac{x^4}{2.3.4.r^3} - \frac{x^5}{2.3.4.5.r^4}$ + &c will be $(= 10,000,000 - 1,053,605 + 55,504 - 1949 + 51 - 1 + \&c = 10,055,555 - 1,055,555) = 9,000,000$, and consequently the Sine to which the given Logarithm 1,053,605 belongs, or of the ratio of which to the radius r, or 10,000,000, it is the Logarithm, will be $= 9,000,000$. Q. E. I.

This result agrees exactly with the computation in Art. 31, pages 676, 677, where the Sine 9,000,000 was given and the Logarithm 1,053,605 was derived from it by the former Series of Mr. Ivory.

A Fourth Example of the like Computation of the Sine to which a given Logarithm belngs, by means of the foregoing Series.

Art. 42. Let the given Logarithm be 6,931, 471. And let it be required to find, by means of the foregoing Series $r - x + \frac{x^2}{2r} - \frac{x^3}{2.3.r^2} + \frac{x^4}{2.3.4.r^3} - \frac{x^5}{2.3.4.5.r^4} + \frac{x^6}{2.3.4.5.6.r^5} - \frac{x^7}{2.3.4.5.6.7.r^6} + \frac{x^8}{2.3.4.5.6.7.8.r^7} + \frac{x^9}{2.3.4.5.6.7.8.9.r^8} + \frac{x^{10}}{2.3.4.5.6.7.8.9.10.r^9} - \frac{x^{11}}{2.3.4.5.6.7.8.9.10.11.r^{10}}$ + &c, the Sine to which it belongs, or of the ratio of which to the radius r, or 10,000,000, it is the Logarithm.

Here x is $= 6,931,471$. Therefore x^2 will be $(= \overline{6,931,471}^2) =$ 48,045,

48,045,290,223,841, and $\frac{x^2}{2}$ will be $(= \frac{48,045,290,223,841}{2}) = 24,022,645,$ 111,920.5, and $\frac{x^2}{2r}$ will be $= \frac{24,022,645,111,920.5}{10,000,000} = 2,402,264.511$, &c; and $\frac{x^3}{2.3.r^2}$ will be $(= \frac{x^2}{2r} \times \frac{x}{3r} = 2,402,264.511$ &c $\times \frac{x}{3r} = \frac{2,402,264.511}{3}$ &c $\times \frac{x}{r} = 800,754.837 \times \frac{6,931,471}{10,000,000} = \frac{5,550,408,930,775}{10,000,000} = 555,040.893$ &c) =, nearly, 555,041; and $\frac{x^4}{2.3.4.r^3}$ will be $(= \frac{x^3}{2.3.r^2} \times \frac{x}{4r} = 555,041 \times \frac{x}{4r} = \frac{555,041}{4} \times \frac{x}{r} = 138,760.2 \times \frac{x}{r} = 138,760.2 \times \frac{6,931,471}{10,000,000} = \frac{961,812,302,254.2}{10,000,000} = 96,181.23$ &c) $= 96,181$;

And $\frac{x^5}{2.3.4.5.r^4}$ will be $(= \frac{x^4}{2.3.4.r^3} \times \frac{x}{5r} = 96,181 \times \frac{x}{5r} = \frac{96,181}{5} \times \frac{x}{r} = 19,236.2 \times \frac{x}{r} = 19,236.2 \times \frac{6,931,471}{10,000,000} = \frac{133,335,162,450.2}{10,000,000}) = 13,333.5$ or (neglecting the fraction .5,) 13,333;

And $\frac{x^6}{2.3.4.5.6.r^5}$ will be $(= \frac{x^5}{2.3.4.5.r^4} \frac{x}{6r} = 13,333 \times \frac{x}{6r} = \frac{13,333}{6} \times \frac{x}{r} = 2,222.1 \times \frac{x}{r}$ 2,222.1 $= \times \frac{6,931,471}{10,000,000} = \frac{15,402,421,709.1}{10,000,000}) = 1540$;

And $\frac{x^7}{2.3.4.5.6.7.r^6}$ will be $(= \frac{x^6}{2.3.4.5.6.r^5} \times \frac{x}{7r} = 1540 \times \frac{x}{7r} = \frac{1540}{7} \times \frac{x}{r} = 220 \times \frac{x}{r} = 220 \times \frac{6,931,471}{10,000,000} = \frac{1,524,923,620}{10,000,000} = 152.492,362,0) =$ 152;

And $\frac{x^8}{2.3.4.5.6.7.8.r^7}$ will be $(= \frac{x^7}{2.3.4.5.6.7.r^6} \times \frac{x}{8r} = 152 \times \frac{x}{8r} = \frac{152}{8} \times \frac{x}{r} = 19 \times \frac{x}{r} = 19 \times \frac{6,931,471}{10,000,000} = \frac{131,697,949}{10,000,000}) = 13$;

And

And $\frac{x^9}{2.3.4.5.6.7.8.9.r^8}$ will be $(= \frac{x^8}{2.3.4.5.6.7.8.r^7} \times \frac{x}{9r} = 13 \times \frac{x}{9r} = \frac{13}{9} \times \frac{x}{r} = 1.444 \times \frac{x}{r} = 1.444 \times \frac{6,931,471}{10,000,000} = \frac{10,009,044.124}{10,000,000} = 1.000,904,$ &c $= 1$. Therefore the Series $r - x + \frac{x^2}{2r} - \frac{x^3}{2.3.r^2} + \frac{x^4}{2.3.4.r^3} - \frac{x^5}{2.3.4.5.r^4} + \frac{x^6}{2.3.4.5.6.r^5} - \frac{x^7}{2.3.4.5.6.7.r^6} + \frac{x^8}{2.3.4.5.6.7.8.r^7} - \frac{x^9}{2.3.4.5.6.7.8.9.r^8} +$ &c will be

$= 10,000,000 - 6,931,471$
$+ 2,402,264 - 555,041$
$+ 96,181 - 13,333$
$+ 1,540 - 152$
$+ 13 - 1$
$+$ &c
$= 12,499,998 - 7,499,998 = 5,000,000.$

Therefore the Sine to which the given Logarithm 6,931,471 belongs, or of the ratio of which to the radius r, or 10,000,000, it is the meaſure, or Logarithm, is $= 5,000,000$, or to half the radius 10,000,000.

Q. E. I.

This reſult agrees with the Computation performed above in Art. 32, pages 678, 679, and 680, where the Sine 5,000,000, or the Sine of 30 degrees, was given, and the Logarithm 6,931,470.86, or 6,931,471, was derived from it by the former Series of Mr. Ivory.

Theſe four Examples are, I preſume, ſufficient to illuſtrate the nature of this ſecond Series communicated to me by Mr. Ivory, and to ſhow it's uſefulneſs in this reverſe operation of deriving from a given Logarithm the value of the Sine to which it belongs; and therefore I ſhall dwell no longer on this ſubject.

Of the Changes made, or intended to be made, afterwards by Lord Napier in the system of Logarithms described in the foregoing Latin Tract, and set-down in the foregoing Table at the end of it.

Art. 43. We have seen that, in the foregoing system of Logarithms, the Logarithm of the ratio of 1,000,000 to 10,000,000, or of 1 to 10, is the whole number 23,025,842. Therefore the Logarithm of the ratio of 1 to 100, or $\overline{10}^2$, will be $= 2 \times 23{,}025{,}842$, or 46,051,684, and the Logarithm of the ratio of 1 to 1000, or $\overline{10}^3$, will be $= 3 \times 23{,}025{,}842$, or 69,077,526, and the Logarithms of the ratios of 1 to 10,000, 100,000, 1,000,000, 10,000,000, &c, or to $\overline{10}^4$, $\overline{10}^5$, 10^6, $\overline{10}^7$, &c, will be equal to the multiples of 23,025,842 by the numbers 4, 5, 6, 7, &c; and consequently, if we were required to add to the ratio of minority of any given Sine to the radius 10,000,000 the ratio of minority of 1 to 10, or of 1 to 100, or $\overline{10}^2$, or of 1 to 1000, or $\overline{10}^3$, or to multiply the said Sine into the fraction $\frac{1}{10}$, or $\frac{1}{100}$, or $\frac{1}{1000}$, by means of Logarithms, we must add to the Logarithm of the said Sine the number 23,025,842, or 46,051,684, or 69,077,526, which numbers differ from each other in almost all their figures; and in like manner, if we were required to add to the ratio of minority of such given Sine to the radius 10,000,000 the ratio of majority of 10 to 1, or of 100 to 1, or of 1000 to 1, or to multiply the said Sine into the whole number 10, or 100, or 1000, by means of Logarithms, we must subtract from the Logarithm of the said Sine one of the said numbers 23,025,842, 46,051,684, and 69,077,526, or add to the Logarithm of the said Sine one of the defective, or negative, numbers $-$ 23,025,842, $-$ 46,051,684 and $-$ 69,077,526, according as the multiplication of the said Sine was to be made into 10, or 100, or 1000. Thus, for example, if the given Sine was 523,360, (which is the Sine of an arch of 3 degrees, and of which the Logarithm in Lord Napier's Table, or the Logarithm of it's ratio to the radius 10,000,000, is 29,500,706,) and we were required to multiply this Sine into the

the fraction $\frac{1}{10}$ by means of the foregoing Table of Logarithms, we must add to the Logarithm, 29,500,706, of the said Sine the number 23,025,842, which is the Logarithm of the fraction $\frac{1}{10}$; and the sum thereby obtained, to wit, 52,526,548, would be the Logarithm of the product, or of 523,360 × $\frac{1}{10}$ or of 52,336: And, if we were required to multiply this Sine 523,360 into the whole number 10 by means of the same Table of Logarithms, we must add to the Logarithm of the said Sine, to wit, 29,500,706, the negative number — 23,025,842, which is in this Table the Logarithm of the whole number 10; and the sum thereby obtained, to wit, the number 6,474,864, would be the Logarithm of the product, or of 523,360 × 10, or of 5,233,600. And thus the three numbers 52,336, and 523,360, and 5,233,600, which have the same significant figures one with the other, and differ only in the decimal places in which the said figures stand, (which makes the second number 523,360 be equal to ten times the first number 52,336, and the third number 5,233,600 be equal to ten times the second number 523,360,) would have for their Logarithms the three numbers 52,526,548, 29,500,706, and 6,474,864, which differ from each other in almost all their figures, and carry with them no marks of belonging to natural numbers that exceed each other in the proportion of 10 to 1, and consequently consist of the same significant figures. This want of resemblance between the Logarithms of natural numbers that exceed each other in the proportion of 10 to 1, and consequently consist of the same significant figures, appeared to Mr. Henry Briggs (who was, at the time of the publication of Lord Napier's foregoing Tract on Logarithms in the year 1614, Professor of Geometry at Gresham College in London, or, in the plainer style of that age, *Geometry-Reader at Gresham House*,) to be an inconvenience, or imperfection, in this excellent Invention of Lord Napier, which rendered it less suited to the decimal notation of numbers that had been for many centuries received into general use throughout Europe, than might have been expected: and therefore, in his Lectures to his Auditors at Gresham College, (in which he undertook to explain to them this new and noble improvement in Arithmetick,) he suggested the following change in Lord Napier's Logarithms as a remedy for this inconvenience; namely, "To diminish them all in the proportion of 23,025,842 (which is the Logarithm of the ratio of 1,000,000 to 10,000,000, or of the ratio of 1 to 10, in Lord Napier's Table,) to 10,000,000;" so that thence-

forwards 10,000,000 ſhould be the Logarithm of the ratio of 1 to 10 inſtead of 23,025,842, and conſequently that 20,000,000 ſhould be the Logarithm of the ratio of 1 to 100, or to $\overline{10}|^2$, inſtead of 46,051,684, and that 30,000,000 ſhould be the Logarithm of the ratio of 1 to 1000, or $\overline{10}|^3$, inſtead of 69,077,526, and, in like manner, that 40,000,000, and 50,000,000, and 60,000,000, and 70,000,000, and 80,000,000, and 90,000,000, &c, (which differ from each other only in the higheſt, or eighth, place of figures,) ſhould be the Logarithms of the ratios of 1 to 10,000, or $\overline{10}|^4$, and of 1 to 100,000, or $\overline{10}|^5$, and of 1 to 1000,000, or $\overline{10}|^6$, and of 1 to 10,000,000, or $\overline{10}|^7$, and of 1 to 100,000,000, or $\overline{10}|^8$, and of 1 to 1000,000,000, or $\overline{10}|^9$, &c, or of 1 to the following powers of 10, reſpectively. And then the three numbers 52,336, 523,360, and 5,233,600, (which have all the ſame ſignificant figures 52,336, but advanced in the two latter numbers to higher decimal places than in the firſt number,) would, if the Logarithm of the firſt of them, to wit, 52,336, be denoted by l, have for their Logarithms the three numbers l, $l - 10{,}000{,}000$, and $l - 20{,}000{,}000$, which would all have the ſame ſignificant figures in the firſt ſeven places of figures, and would differ from each other only in the eighth, or higheſt, place, and in that place would differ only by an unit when the numbers to which they belonged would be greater the one than the other in the proportion of 10 to 1. This reſemblance between the Logarithms of different numbers that conſiſted of the ſame ſignificant figures, and were produced the one from the other by multiplying them either into the fraction $\frac{1}{10}$, or the whole number 10, ſeemed to be very well adapted to the decimal notation of numbers that has been generally adopted in Europe for the laſt ſeven or eight hundred years; and therefore the change of Lord Napier's Logarithms, by which it was obtained, ſeemed to be very obvious, and to have been likely to occur to any man that was converſant with Arithmetical calculations, and much more to ſo great a Mathematician as Mr. Briggs. And Mr. Briggs, when he had thought of making this change in Lord Napier's Logarithms, not only mentioned it to his Auditors at Greſham College, but likewiſe wrote a letter concerning it to Lord Napier himſelf, about the ſpring of the year 1615; and afterwards, in the ſummer of the ſame year, when he took a journey into Scotland, to viſit Lord Napier, and converſe with him upon his late noble invention, he again propoſed it to him. And Lord Napier, (as Mr. Briggs informs us,) expreſſed a compleat approbation of it, and wiſhed that it might be adopted, and even ſaid that he himſelf had had ſome thoughts

of

of adopting it before Mr. Briggs had mentioned it to him; but that, as it had not occurred to him till after he had compleated the computation of his Table of Logarithms upon his firſt plan (by which the number 23,025,842 was the Logarithm of the ratio of 1 to 10,) he had thought it better to publiſh his Table in the preſent form, than to delay the publication of it till he had made this reduction of the ſeveral Logarithms contained in it, in the ſaid proportion of 23,025,842 to 10,000,000, which could not be done without a good deal of time and labour. And he ſaid, further, that, if his invention of Logarithms ſhould meet with the general approbation of Mathematicians, and he ſhould be bleſſed with ſufficient health and leiſure to undertake the computation of his Logarithms upon that improved plan, he hoped that he ſhould one day give a ſecond edition of his work with the Logarithms ſo computed.

This was the firſt improvement made in Napier's original Syſtem of Logarithms, and ſeems to belong almoſt equally to him and Mr. Briggs, though it had been firſt publickly communicated to the world by Mr. Briggs in his Lectures at Greſham College. But it was inſtantly approved-of and adopted by Lord Napier as ſoon as it was propoſed to him by Mr. Briggs; and Lord Napier alſo declared, that the thought of it had already occurred to him before Mr. Briggs had mentioned it, though not till after his Table of Logarithms had been compleated upon the former plan: and the thought is in itſelf ſo natural and obvious that it ſeems highly probable that it ſhould have ſo occurred to him.

But Lord Napier, in his converſations on this ſubject with Mr. Briggs in that ſame ſummer of the year 1615, mentioned to him another change that he propoſed to make in his firſt ſyſtem of Logarithms, and which was of very great importance and utility. This was to chuſe the very ſmall line 1, or the ten-millionth part of the radius 10,000,000, inſtead of the radius itſelf, or the high number 10,000,000, for the quantity of which the Logarithm was to be nothing, or 0, or to which all the Sines, Tangents, and Secants in the quadrant of the circle were to be compared, or which was to become the common conſequent, or ſecond term, of all the ratios of which the ſaid Sines, Tangents, and Secants were the Antecedents, or firſt terms. After this change in the quantity choſen for the common ſtandard of compariſon with all the ſaid Sines, Tangents,

Tangents, and Secants, the Logarithm of a Sine would no longer mean the Logarithm of the ratio of ſuch Sine to the radius, or the high number 10,000,000, which was always a ratio of minority; but it would mean the Logarithm of the ratio of ſuch Sine to the very ſmall line 1, or the 10,000,000th part of the radius 10,000,000, which ratio would be always a ratio of majority: and conſequently the Logarithms of the Sines, and thoſe of the Tangents of arches leſs than 45 degrees, but greater than one minute, would be, all of them, Logarithms of ratios of majority, as well as the Logarithms of the Tangents of arches greater than 45 degrees, and the Logarithms of all the Secants. There would therefore be no longer any occaſion for the diſtribution of theſe Logarithms into two claſſes, called *abundant*, or *affirmative*, or *poſitive*, and *defective*, or *negative*, Logarithms according as they repreſented *ratios of minority*, or *ratios of majority*; but, as all the ratios of the Sines, Tangents, and Secants, of the arches in the Quadrant, that were greater than one minute of a degree, to the aforeſaid ſmall line 1, or the 10,000,000th part of the radius, would be ratios of majority, their Logarithms would be all of the ſame claſs, as being repreſentatives of ratios of the ſame kind, to wit, ratios of majority. And by this circumſtance a great deal of perplexity and difficulty, which had before ariſen from the conſideration of the oppoſite ratios of minority and of majority, and the affirmative and negative Logarithms that repreſented them, would be avoided. And this ſecond change in Lord Napier's firſt ſyſtem of Logarithms, which Lord Napier communicated to Mr. Briggs, was immediately approved-of by Mr. Briggs.

After theſe two improvements on Lord Napier's firſt ſyſtem of Logarithms, that is ſet-forth in the foregoing Latin Tract, and the Table printed at the end of it, the Logarithm of every Sine, Tangent, and Secant in the Quadrant, ſtill continued to be a whole number, as before, but ſignified the Logarithm of the ratio of the Sine, Tangent, or Secant, to which it belonged, to the little line 1, or the 10,000,000th part of the radius 10,000,000, which ratio, when the arch was greater than one minute of a degree, or even than one ſecond minute of a degree, would always be a ratio of majority. And the Logarithm of the ratio of 10 to 1, or (in a ſhorter way of expreſſing it,) the Logarithm of 10, in this ſyſtem, would be the great whole number 10,000,000, and the Logarithms

of

of 100, 1000, 10,000, 100,000, 1000,000, &c, or of $\overline{10}|^2$, $\overline{10}|^3$, $\overline{10}|^4$, $\overline{10}|^5$, $\overline{10}|^6$, &c, would be $2 \times 10{,}000{,}000$, or 20,000,000, and $3 \times 10{,}000{,}000$, or 30,000,000, and $4 \times 10{,}000{,}000$, or 40,000,000, and $5 \times 10{,}000{,}000$, or 50,000,000, and $6 \times 10{,}000{,}000$, or 60,000,000, &c, or the products of the multiplication of the several Indexes of the powers of 10, that is, of $\overline{10}|^1$, $\overline{10}|^2$, $\overline{10}|^3$, $\overline{10}|^4$, $\overline{10}|^5$, $\overline{10}|^6$, &c, in their natural order, into the number 10,000,000. And it does not appear that during the short remainder of Lord Napier's Life, (who died in the year 1618,) any other change besides the two that have been here described, was made, or proposed to be made, in Lord Napier's first system of Logarithms, by either Lord Napier himself, or Mr. Briggs.

A Third Change in Lord Napier's System of Logarithms made afterwards by Mr. Briggs.

Art. 44. But, after the death of Lord Napier, Mr. Briggs (who continued to cultivate this useful invention of Logarithms during the whole remainder of his life, with great industry and success,) reduced the Logarithms of the several powers of 10, to wit, $\overline{10}|^1$, $\overline{10}|^2$, $\overline{10}|^3$, $\overline{10}|^4$, $\overline{10}|^5$, $\overline{10}|^6$, &c from the high numbers 10,000,000, 20,000,000, 30,000,000, 40,000,000, 50,000,000, 60,000,000, &c, to the small numbers 1, 2, 3, 4, 5, 6, &c, or the mere Indexes of those powers; which, (though it causes all the Logarithms of the intermediate numbers between 1 and 10, and between 10 and $\overline{10}|^2$, or 100, and between $\overline{10}|^2$ and $\overline{10}|^3$, or 100 and 1000, and between all the following powers of 10, to be mixt numbers, consisting partly of the small whole numbers, 1, 2, 3, 4, 5, 6, &c in their natural order, and partly of seven figures of decimal fractions, and the Logarithms of the eight whole numbers 2, 3, 4, 5, 6, 7, 8, and 9, which are less than 10, to be mere decimal fractions,) has been found to give us the most convenient system of Logarithms for the purposes of practical Arithmetick that has yet been discovered. And this system of Logarithms is that which is exhibited in Sherwin's Tables of Logarithms, and in most other Tables of Logarithms now in use, not only in Great Britain and Ireland, and the other British dominions, but in the other countries of Europe and the whole civilized and literary world, and which is generally known by the name of *Briggs's System of Logarithms.*

And

And in this ſyſtem that part of every Logarithm which is a whole number is called *the Index* of the Logarithm, or it's *Characteriſtick*. Thus, for example, the Logarithm of the number 43 in this ſyſtem of Briggs (as ſettled at this day,) is 1.633,468,5, and 1 is it's *Index*, or *Characteriſtick*; and the Logarithm of the number 430 (which is equal to 10 times 43) is 2.633,468,5, and 2 is it's *Index*, or *Characteriſtick*; and the Logarithm of the number 4300 (which is equal to 100 times 43) is 3.633,468,5, and 3 is it's *Index, or Characteriſtick*. And theſe integral parts of theſe Logarithms are called their *Indexes*, or *Characteriſticks*, becauſe they *indicate*, or point-out, the power of 10, between which and the next higher power of 10 the numbers correſponding to them lie. Thus, the Index 1, in the aforeſaid Logarithm 1.633, 468,5, ſhows that the number to which it belongs is greater than 10, of which the Logarithm is 1, but leſs than 100, of which the Logarithm is 2; and the Index 2 of the Logarithm 2.633,468,5 ſhows that the number to which it belongs (and which we have ſeen to be 430,) is greater than 100, of which the Logarithm is 2, but leſs than 1000, of which the Logarithm is 3; and the Index 3 of the Logarithm 3.633,468,5 ſhows that the number to which it belongs, (to wit, the number 4300) is greater than 1000, of which the Logarithm is 3, but leſs 10,000, of which the Logarithm is 4.

A Recapitulation of the Three foregoing Changes, or Improvements, made by Lord Napier and Mr. Henry Briggs in Lord Napier's Original Syſtem of Logarithms.

Art. 45. Thus it appears that the original Syſtem of Logarithms invented by Lord Napier (which is ſet-forth in the foregoing Tract of Lord Napier and the Table that accompanies it,) underwent three ſucceſſive changes under the hands of it's Inventor Lord Napier, and his great Co-adjutor, Mr. Henry Briggs. By the firſt of theſe changes (which was ſuggeſted by Mr. Briggs, and mentioned to the Auditors of his Lectures at Greſham College, and communicated to Lord Napier by letter, and afterwards again mentioned to him, in the converſations they had together in the Summer of the year 1615, and which was highly approved-of by Lord Napier,) the Logarithm of the ratio of the Sine 1,000,000,

1,000 000, (which is equal to a tenth part of the radius 10,000,000,) to the radius 10,000,000, or the Logarithm of the ratio of 1 to 10, was diminiſhed from the whole number 23,025,842 to the whole number 10,000,000, or the ſeventh power of the number 10, and all the other Logarithms in Lord Napier's firſt Table were diminiſhed in the ſame proportion; by which means, if any whole number were to be divided by 10, or 100, or 1000, or any higher power of 10, and thereby a ſet of new numbers were to be produced, which would all have the ſame ſignificant figures as the firſt number, but placed in lower decimal places than in the ſaid firſt number, the Logarithms of theſe ſeveral new numbers, or quotients of the diviſion of the firſt number by 10, 100, 1000, &c, might be derived from the Logarithm of the ſaid firſt number, by adding to the ſaid Logarithm the ſeveral whole numbers 10,000,000, 20,000,000, 30,000,000, &c, which would be the Logarithms of the fractions $\frac{1}{10}$, $\frac{1}{100}$, $\frac{1}{1000}$, &c, or of the ſeveral ratios of 1 to 10, of 1 to 100, and of 1 to 1000, &c, inſtead of adding (as was done before,) to the Logarithm of the ſaid firſt number the ſeveral whole numbers 23,025,842, 46,051,684, 69,077,526, &c, which were the Logarithms of the ſame ratios of 1 to 10, of 1 to 100, and of 1 to 1000, &c, in Lord Napier's firſt ſyſtem of Logarithms. But ſtill the radius 10,000,000 continued to have 0 for it's Logarithm, or to be the common ſtandard to which all the Sines, Tangents, and Secants, in the Quadrant were referred, or to be the common conſequent, or ſecond term, of all the ratios of which the ſaid Sines, Tangents, and Secants were the Antecedents, or firſt terms; ſo that the Logarithm of any given Sine, as, for example, of the Sine 5,000,000, or the Sine of an arch of 30 degrees, did not mean (as it now does,) the Logarithm of the ratio of the ſaid Sine, or 5,000,000, to 1, but the Logarithm of the ratio of the ſaid Sine, or 5,000,000, to the radius, or the great whole number 10,000,000, or of the ratio of 1 to 2.

But Lord Napier ſuggeſted to Mr. Briggs, in his converſations with him upon this ſubject in the Summer of the year 1615, a ſecond change in the Syſtem, by which the inconveniences ariſing from the uſe of the word *Logarithm* in that ſenſe might be avoided. And this was "to ceaſe to make 0 be the Logarithm of the radius, or great whole number 10,000,000, and to make it

be the Logarithm of unity, or 1, or of the ten-millionth part of the radius, or 10,000,000; by which change the very ſmall line 1 would become the common ſtandard to which all the Sines, Tangents, and Secants in the Quadrant were compared, or would be the common conſequent, or ſecond term, of all the ratios of which the ſaid Sines, Tangents, and Secants, were the Antecedents, or firſt terms. This propoſal of Lord Napier was greatly approved-of by Mr. Briggs as ſoon as it was mentioned to him: and he adopted it in the Tables which he afterwards publiſhed in the year 1624. This was a very uſeful change. For by it the Logarithms of all the Sines, Tangents, and Secants in the Quadrant became Logarithms of the ratios of the ſaid Sines, Tangents, and Secants to the very ſmall line 1, or the ten-millionth part of the radius; all which ratios in Lord Napier's Trigonometrical Canon (in which the ſmalleſt arch is an arch of one minute of a degree, the Sine of which is much greater than 1, or the ten-millionth part of the radius 10,000,000,) are ratios of majority, as well as the ratios of Secants of all the arches, and of the Tangents of all arches greater than 45 degrees, to the ſaid common conſequent, or little line 1. And thus we are freed from the difficulties ariſing from the contemplation of two oppoſite kinds of ratios, to wit, ratios of minority and ratios of majority; of which, if the former ratios are ſuppoſed to have *affirmative*, or *poſitive*, Logarithms, (which are thoſe which Lord Napier has aſſigned to them,) the latter ratios muſt be repreſented by Logarithms of a contrary kind, or *negative* Logarithms. But, after this uſeful change ſuggeſted by Lord Napier, the magnitudes of the Logarithms of the ſame ratios will continue the ſame as they were before; only the ratios will be changed from ratios of minority into equal and contrary ratios of majority, which will have equal and contrary Lagarithms to thoſe which they had before. Thus, for example, as the Logarithm of the ratio of the Sine 1,000,000 to the radius 10,000,000, or of the ratio of 1 to 10, was, before this ſecond change, equal to 10,000,000, and the Logarithm of the equal and contrary ratio of 10,000,000 to 1,000,000, or of 10 to 1, was equal to 10,000,000, the Logarithm of the ratio of 10 to 1 will, after the ſaid ſecond change, be of the ſame magnitude as before, to wit, equal to 10,000,000, but without any neceſſity of conſidering it as negative, and prefixing the Sign — to it, becauſe the Logarithms of all the Sines, Tangents, and Secants in the Quadrant, (being now become Logarithms of the ratios of the ſaid Sines, Tangents, and Secants, to the little line 1, or the

ten-

ten-millionth part of the radius,) will all be Logarithms of ratios of the same kind, to wit, ratios of majority, and the division of them into two classes of *affirmative*, or *positive*, or *abundant*, and of *negative*, or *defective*, Logarithms, may consequently be laid-aside.

These two changes in Lord Napier's first system of Logarithms were made in Lord Napier's life-time. And, after Lord Napier's death, Mr. Briggs made a further change in the System, by dividing all the Logarithms in it by 10,000,000, or the number which he had hitherto used for the Logarithm of 10, or of the ratio of 10 to 1; by which division the Logarithms of 10, or $\overline{10}^1$, of 100, or $\overline{10}^2$, of 1000, or $\overline{10}^3$, of 10,000, or $\overline{10}^4$, of 100,000, or $\overline{10}^5$, &c, or of the ratios of those numbers to 1, were reduced from 10,000,000, 20,000,000, 30,000,000, 40,000,000, 50,000,000, &c, to the small numbers 1, 2, 3, 4, 5, &c, or to the Indexes of the said powers of 10. And this I take to be the present and most perfect state of Brigg's System of Logarithms.

Art. 46. I have dwelt the longer upon these changes and improvements upon Lord Napier's first system of Logarithms, because they seem to have been but obscurely described in several of the books I have seen upon the subject: and it is only of late, and in consequence of the perusal of the foregoing original Tract of Lord Napier, and of some passages in Mr. Briggs's *Arithmetica Logarithmica*, that I am come to think that I have a just conception of them; in which, after all, I may, perhaps, be mistaken. But, such as it is, I have endeavoured to convey it to my readers in as full and clear a manner as I could. And, in support of this account of these changes (which I take to be a correct one,) I will here cite two short, but curious, passages from the writings of Lord Napier himself and Mr. Briggs, that seem to me to afford just grounds for supposing these changes to be such as I have here represented them.

A Paſſage in Mr. Edward Wright's Tranſlation of Lord Napier's Tract on Logarithms, relating to the firſt of the above-mentioned Changes, or Improvements, of Lord Napier's firſt Syſtem of Logarithms.

Art. 47. The firſt of theſe paſſages occurs in the 19th and 20th pages of Mr. Edward Wright's Tranſlation of the foregoing original Tract of Lord Napier into Engliſh, which was made in the year 1614, or 1615, almoſt immediately after the publication of the original Tract itſelf. This tranſlation was ſhown in manuſcript to Lord Napier, and met with his entire approbation. For Mr. Samuel Wright, the ſon of Mr. Edward Wright, the tranſlator of this Tract, (who died very ſoon after he had compleated his tranſlation, and before he had had time to publiſh it,) publiſhed this Tract in the year 1615, and dedicated it to the Merchants of the Eaſt-India Company, by whom his father had been employed and patronized; and in his Dedication declares, "that his father's care in making this tranſlation was ſo great that he procured "the Author's peruſal of it; who, after great pains taken therein, gave appro-"bation to it both in ſubſtance and form, as he [the Editor] then preſented it "to them." And Lord Napier himſelf alſo added to Mr. Wright's tranſlation of his, Lord Napier's, preface to this work the following words; "But now "ſome of our countrymen in this Iſland, well-affected to theſe ſtudies and the "more publique good, procured a moſt learned Mathematician to tranſlate the "ſame into our vulgar Engliſh tongue; who, after he had finiſhed it, ſent the copy "of it to me to be ſeen and conſidered-on by myſelf. I, having moſt willingly "and gladly done the ſame, finde it to be moſt exact and preciſely conformable "to my mind and the original. Therefore may it pleaſe you, who are inclined "to theſe ſtudies, to receive it from me and the Tranſlator, with as much "good-will as we recommend it unto you. Fare ye well."

Now

Now in this tranſlation, thus approved and recommended by Lord Napier, we find, in pages 19 and 20, the following paſſage, which is not tranſlated from Lord Napier's Latin original, and which, therefore, we may reaſonably ſuppoſe to have been added by Lord Napier to the manuſcript copy of Mr. Wright's tranſlation which had been ſent to him for his peruſal and examination. This paſſage is in theſe words :

"*An Admonition.*"

"But, becauſe the addition and ſubtraction of theſe former numbers "[23,025,842, 46,051,684, and 69,077,526, &c,] may ſeem ſomewhat pain- "ful, I intend (if it ſhall pleaſe God,) in a ſecond edition to ſet-out ſuch "Logarithms as ſhall make thoſe numbers above-written to fall upon decimal "numbers, ſuch as 100,000,000, 200,000,000, 300,000,000, &c, which are "eaſier to be added, or abated, to, or from, any other number."

A Conjecture concerning the meaning of the foregoing Admonition.

Theſe words (which are ſomewhat obſcure,) ſeem to mean "that Lord Napier intended, in a ſecond edition of his Logarithms, to make the whole number 10,000,000 be the Logarithm of the ratio of 1 to 10 inſtead of the whole number 23,025,842, and conſequently to make the whole number 20,000,000 be the Logarithm of the ratio of 1 to 100, and the whole number 30,000,000 be the Logarithm of the ratio of 1 to 1000, inſtead of the whole numbers 46,051,684 and 69,077,526 ; by which change of the numbers 23,025,842, 46,051,684, and 69,077,526 for the ſimple numbers 10,000,000, 20,000,000, and 30,000,000, the diviſions and multiplications of any given numbers by 10, or 100, or 1000, would be performed by adding to, or ſubtracting from

from the Logarithms of such given numbers the simple and easy whole numbers 10,000,000, 20,000,000, and 30,000,000, instead of the less simple and easy whole numbers 23,025,842, 46,051,684, and 69,077,526."

Objections that may be made to the foregoing conjectural Exposition of the said Admonition, together with some Answers to the said Objections.

This seems to me to be the meaning of this passage. Yet this interpretation is liable to two objections. The first is, "that the expression "*to make those numbers above-written, (to wit, the numbers* 23,025,842, 46,051,684, and 69,077,526,) *to fall upon decimal numbers, such as* 100,000,000, 200,000,000, *and* 300,000,000," seems an odd way of speaking, if it means (as I suppose it to do,) to make those numbers 23,025,842, 46,051,684, and 69,077,526 co-incide with, or become equal to, the decimal numbers 10,000,000, 20,000,000, and 30,000,000; which may be done by diminishing them in the proportion of 23,025,842, to 10,000,000: and the second objection is, that Lord Napier mentions the numbers 100,000,000, 200,000,000, and 300,000,000, or a hundred millions, two hundred millions, and three hundred millions, instead of the numbers 10,000,000, 20,000,000, and 30,000,000, or ten millions, twenty millions, and thirty millions, which I suppose him to mean. But this latter objection may be removed either by supposing that a cipher too much is printed in the numbers 100,000,000, 200,000,000, and 300,000,000, by an error of the press, or that Lord Napier (though he had, in his first table of Logarithms, supposed the radius of the circle to be divided into only ten-millions of equal parts, or had taken the ten-millionth part of the radius for the unit of his system, and carried his Sines and their Logarithms only to seven places of figures,) might intend, in his second edition of them, to carry them to one place of figures more, and, for that purpose, to divide the radius of the circle into a hundred millions, instead of ten millions, of equal parts, and to chuse

the

the hundred-millionth part, inſtead of the ten-millionth part, of the radius for the unit of his Syſtem; and for this purpoſe it would be neceſſary, inſtead of diminiſhing the former Logarithm of the ratio of 1 to 10, to wit, the whole number 23,025,842, from 23,025,842 to 10,000,000, to increaſe the ſaid Logarithm from 23,025,842 to 100,000,000, and to increaſe all the other Logarithms of the former ſyſtem in the ſame proportion. In one, or other, of theſe two ways we may, I think, get rid of the ſecond of theſe two objections. But, as to the firſt objection, which is grounded on the impropriety of uſing the expreſſion of "making the numbers 23,025,842, 46,051,684, and 69,077,526, *fall upon* decimal numbers, ſuch as 100,000,000, 200,000,000, and 300,000,000," inſtead of the expreſſion of "making them *become equal to* the ſaid decimal numbers, or diminiſhing, or increaſing, them till they become equal to the ſaid decimal numbers," I know not how to get rid of it, and can only ſay that I can find no meaning in the paſſage but by interpreting this expreſſion in this manner. And therefore, upon the whole matter, I conceive the foregoing explanation of this paſſage to be the true one. But of this my readers muſt judge for themſelves.

Now, if this Admonition of Lord Napier has the meaning which I have here aſcribed to it, the change which Lord Napier declares in it that he had an intention of making in his firſt ſyſtem of Logarithms, is evidently the ſame which Mr. Briggs had ſuggeſted to him, and had before mentioned to his Auditors at Greſham College. And therefore it would have been but a juſt acknowledgment of Mr. Briggs's ſhare in the merit of this change, if Lord Napier had added, at the end of the foregoing *Admonition*, a few words to declare that he had taken the reſolution of making it in purſuance of, or, at leaſt, agreeably to, a ſuggeſtion to that purpoſe that had been made to him by Mr. Briggs; though it had already occurred to himſelf, before he had received this ſuggeſtion from Mr. Briggs, that ſuch a change would be very convenient.

A Passage of Mr. Briggs's Address to his Readers, prefixed to his Arithmetica Logarithmica, *relating to the first and second of the above-mentioned Changes, or Improvements, of Lord Napier's first System of Logarithms.*

Art. 48. The second of these two curious passages is in Mr. Henry Brigg's Address to the reader, prefixed to his great and excellent work, intitled, *Arithmetica Logarithmica*, (which was published at London, in a thin volume *in folio*, in the year 1624,) and is in these words.

Quòd Logarithmi isti diversi sunt ab iis quos clarissimus vir, Baro Merchistonii, *in suo edidit Canone mirifico, non est quod mireris. Ego enim, cum meis Auditoribus,* Londini, *publicè in Collegio* Greshamiensi, *horum doctrinam explicarem, animadverti multo futurum commodius si Logarithmus Sinûs totius servaretur* 0 *(ut in Canone mirifico,) Logarithmus autèm partis decimæ ejusdem Sinûs totius, nempe, Sinûs 5 graduum, 44 minutorum, et 21 secundorum, esset* 10,000,000,000. *Atque eâ de re scripsi statìm ad ipsum auctorem; et quamprimùm per anni tempus et vacationem à publico docendi munere licuit, profectus sum* Edinburgum; *ubi, humanissimè ab eo acceptus, hæsi per integrum mensem. Cum autèm inter nos de horum mutatione sermo haberetur, ille se idem sensisse et cupivisse dicebat: veruntamèn istos, quos jam paraverat, edendos curâsse, donec alios, si per negotia et valetudinem liceret, magìs commodos confecisset. Istam autèm mutationem ita faciendam consebat ut* 0 *esset Logarithmus unitatis, et* 10,000,000,000 *Sinûs totius: Quod ego longè commodissimum esse non potui non agnoscere. Coepi igitùr, ejus hortatu, rejectis illis quos anteà paraveram, de horum calculo seriò cogitare: et, sequenti æstate iterùm profectus* Edinburgum, *horum, quos hîc exhibeo, præcipuos illi ostendi: idem etiàm tertiâ æstate facturus, si DEUS illum nobis tamdiù superstitem esse voluisset.*

In

In this paſſage we find mention made of both the changes in Lord Napier's original Syſtem of Logarithms which were made in the life-time of Lord Napier; to wit, firſt, of that which had been mentioned by Mr. Briggs in his Lectures to his Auditors at Greſham College, which conſiſted in chuſing a whole number in which 1 was the only ſignificant figure, and was followed by ſeveral ciphers (ſuch as the number 10,000,000,) for the Logarithm of the ratio of 1,000,000 to 10,000,000, or of the ratio of the Sine of an arch of 5°, 44′, 21″ to the radius, or of the ratio of 1 to 10, inſtead of the number 23,025,842, which had been the Logarithm of that ratio in Lord Napier's firſt ſyſtem; and, ſecondly, of that which Mr. Briggs declares to have been propoſed by Lord Napier himſelf, and which Mr. Briggs ſays he immediately acceeded-to and acknowledged to be a moſt convenient change, (*quod ego longè commodiſſimum eſſe non potui non agnoſcere*;) and which conſiſted in making 0 be the Logarithm of 1, or the 10,000,000th part of the radius, inſtead of being the Logarithm of the radius itſelf, or 10,000,000, as it had been in Lord Napier's firſt Syſtem.

Art. 49. It may be obſerved, however, that in this paſſage of Mr. Briggs's Addreſs to his Readers, Mr. Briggs ſpeaks of the whole number 10,000,000,000, or ten thouſand millions, inſtead of the whole number 10,000,000, or ten millions, as being the number which he had mentioned both to his auditors at Greſham College and to Lord Napier, and which he had thought to be fitter to be made the Logarithm of the ratio of 1 to 10 than the Logarithm of that ratio in Lord Napier's firſt Syſtem, which was the leſs ſimple whole number 23,025,842. But this ſeems to have been owing to an intention both in Lord Napier and in Mr. Briggs, at the time of theſe converſations on this ſubject at Edinburgh in the ſummer of the year 1615, to compute their Logarithms, and likewiſe the Sines, Tangents, and Secants of all the arches of the Quadrant, to ten decimal places of figures inſtead of ſeven places, and, for that purpoſe, to call the radius of the circle 10,000,000,000, inſtead of 10,000,000, or to chuſe for 1, or the unit of their Syſtem both of Sines, Tangents, and Secants, and of Logarithms, the ten-thouſand-millionth part of the radius inſtead of it's ten-millionth part.

Art. 50. And there is alſo another difficulty in the aforeſaid Latin paſſage of Mr. Briggs, which, I doubt not, will have occurred to the attentive readers

of it, and which therefore I will endeavour to remove. This difficulty relates to the meaning of the words *et* 10,000,000,000 *Sinûs totius*, which, from the words that immediately preceed them, to wit, the words *Istam autèm mutationem ita faciendam censebat ut* 0 *esset Logarithmus unitatis*, must be equivalent to the words *et* [*ut*] 10,000,000,000 [*esset Logarithmus*] *Sinûs totius*. Now these latter words *et ut* 10,000,000,000 *esset Logarithmus Sinûs totius* are incompatible with the former words *ut* 0 *esset Logarithmus unitatis*. For, if *o* is made the Logarithm of 1, or of the 10,000,000,000th part of the radius, and 10,000,000,000, is made the Logarithm of the ratio of 10 to 1, (as Mr. Briggs seems to have supposed in the foregoing Latin passage,) the Logarithm of the ratio of the whole Sine, or radius, 10,000,000,000, or $\overline{10}^{10}$, to 1, must be equal to 10 times the Logarithm of the ratio of 10 to 1, or must be equal to 10 times 10,000,000,000, or to 100,000,000,000, that is, the Logarithm of the radius, or whole Sine, *Logarithmus Sinûs totius*, must be 100,000,000,000, and not 10,000,000,000. And therefore I suppose that in this part of the foregoing passage of Mr. Briggs the number 10,000,000,000 has been set-down instead of 100,000,000,000 by an error of the Press.

FRANCIS MASERES.

Inner Temple,
August 1, 1805.

End of the Observations on the foregoing Tract of Lord Napier, intitled, *Mirifici Logarithmorum Canonis Descriptio.*

OF

JOHN SPEIDELL'S NEW LOGARITHMS,

DERIVED BY SUBTRACTION FROM THE LOGARITHMS OF LORD NAPIER'S FIRST SYSTEM.

IN the foregoing Obſervations on the Tract of Lord Napier, intitled *Mirifici Logarithmorum Canonis Deſcriptio*, I have mentioned every thing that occurred to me as neceſſary to explain the nature and uſe of thoſe Logarithms, and likewiſe to give the reader a clear idea of the three great improvements afterwards made in them by Lord Napier himſelf, and by Mr. Henry Briggs; to wit, firſt, by reducing the Logarithm of the ratio of 1 to 10, or of the equal and contrary ratio of 10 to 1, from the whole number 23,025,842 to the ſimpler whole number 10,000,000, and diminiſhing the Logarithms of all other ratios in the ſame proportion; and, ſecondly, by making 0 become the Logarithm of 1, or of the 10,000,000th, or ten-millionth, part of the radius 10,000,000, inſtead of being the Logarithm of the radius itſelf, or the great whole number 10,000,000, or by chuſing the ſmall line 1, or the 10,000,000th part of the radius, (which is only the 2909th part of the Sine of one minute of a degree,) for the common ſtandard of compariſon with all the Sines, Tangents, and Secants in the Quadrant, inſtead of the radius 10,000,000; and, thirdly, by Mr. Briggs's laſt operation of reducing the Logarithm of the ratio of 10 to 1 from the great whole number 10,000,000 to unity, or 1, and conſequently of diminiſhing the Logarithms of all other ratios in the ſame proportion; by which improvements the modern Tables of Brigg's Logarithms, (which are now in general uſe,) have been brought to their preſent high degree of convenience and perfection. But there was another eminent Mathematician who diſtinguiſhed himſelf at the ſame time by his cultivation of this noble invention of Lord Napier as well as Mr. Briggs, and who made an improvement on it which deſerves to be mentioned in this collection of tracts on the ſubject of Logarithms. This was Mr. John Speidell, a learned and ingenious Engliſh Mathematician, who was a Teacher of Mathematicks in London from about the year 1607 to ſome time after the year 1646, and who, in the year 1619, (which was the year immediately following that in which Lord Napier

died,) publiſhed a Table of *New Logarithms*, different from thoſe of Lord Napier, but derived from them by ſubtracting them all from the great whole number 100,000,000, which is greater than the greateſt Logarithm in Lord Napier's Canon, to wit, 40,317,433, or the Logarithm of the Sine of one minute of a degree. This Table of new Logarithms ſeems to have been well received by the Publick, and probably went through ſeveral Editions: for in an Edition of it publiſhed in the year 1628 (which is the only Edition of it that I have ſeen,) I obſerve there are, at the bottom of the Title-page, the words *the* 10*th Impreſſion*, by which, I ſuppoſe, is meant the 10th Edition. This Edition I have examined. But, as it contains only the Logarithms themſelves, without any explanation of their nature and conſtruction, and without informing his readers *of what ratios* they are the Logarithms, or meaſures, I was not able to underſtand them without the aſſiſtance of my very acute and learned friend, Mr. James Ivory, one of the Inſtructors of the New Military College at Great Marlow in Buckinghamſhire. But he, upon my application for his aſſiſtance, has drawn-up for me the following explanatory account of them, which, with his permiſſion, I ſhall now proceed to inſert in this volume for the information and ſatisfaction of my readers. As for the Logarithms themſelves, though they were, at the time of their firſt publication, a uſeful improvement on Lord Napier's firſt ſyſtem of Logarithms, yet I conceive them to be ſo compleatly ſuperſeded by the modern Tables of Mr. Briggs's Logarithms that are now in general uſe, that it would be quite uſeleſs to reprint them. Mr. Ivory's account of them is as follows.

AN ACCOUNT

OF A

TABLE OF LOGARITHMS,

Publiſhed by JOHN SPEIDELL, *an eminent Engliſh Mathematician, in the year* 1619, *and afterwards again in the year* 1628, *under the title of* New Logarithms, extracted from and out of thoſe of Lord Napier.

BY JAMES IVORY, M. A.

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Article 1. In Lord Napier's Canon the Logarithms are meaſures of the ratios of minority of the Sines of the ſeveral arches of the Quadrant to the radius. And therefore the Logarithms of the Sines in his Table conſtantly decreaſe from the Logarithm of the Sine of one minute of a degree (which is the ſmalleſt arch contained in Lord Napier's Canon) to 0, (which is the Logarithm of the radius 10,000,000,) becauſe the ratio of minority which the Sine of an arch bears to the radius, (of which ratio Lord Napier's Logarithm of the Sine is the meaſure,) decreaſes continually from a great ratio of minority to a very ſmall ratio of minority, while the Sine increaſes from the ſmall Sine of one minute and approaches very nearly to an equality with the radius. The ratios of the Sines to the radius are, all of them, ratios of minority throughout the whole extent of the Quadrant, and their Logarithms are therefore all of one ſort, that is, either all affirmative or all negative. But the caſe is different with the ratios which the Tangents of the ſeveral arches of the Quadrant bear to the radius: for theſe ratios are ratios of minority only ſo long as the Tangents belong to arches leſs than 45 degrees, and are afterwards, all, ratios of majority; and conſequently

sequently the Logarithms of the Tangents of the arches that are less than 45 degrees will be of a contrary kind to the Logarithms of the Tangents of arches that are greater than the said arch: and therefore, if the Logarithms of one of these kinds of ratios be considered as *positive*, or *affirmative*, the Logarithms of the ratios of the other kind must be considered as *negative*. And, if Lord Napier's Table had included the Secants of the arches in the Quadrant as well as their Sines and Tangents, the Logarithms of the ratios of the Secants to the radius (which are, all of them, ratios of majority,) would have been of the same kind as the Logarithms of the ratios of the Tangents of arches that were greater than 45° to the radius, but of a different kind from the Logarithms of the ratios of the Sines to the radius, and from the Logarithms of the ratios of the Tangents to the radius in the first half of the Quadrant, in which the arches were less than 45 degrees.

Art. 2. It was soon discovered to be troublesome and perplexing in practice to have two sorts of Logarithms in the Canon, and it became desirable to remedy this inconvenience. Now to do this was the main object of John Speidell's *New Logarithms*. And for this purpose he chuses a whole number which is greater than the greatest Logarithm contained in Lord Napier's Table, and subtracts from the said whole number each of the Logarithms contained in the said Table. Now, as the least Sine that is set-down in Lord Napier's Table is the Sine of one minute of a degree, which bears a greater ratio of minority to the radius than that of any greater Sine to the same quantity, it is evident that the Logarithm of the Sine of one minute, or the Logarithm of the ratio of the said Sine to the radius, must be greater than the Logarithm of any other Sine in the Quadrant. But the Sine of one minute in Lord Napier's Canon is equal to the whole number 177,433, and it's Logarithm is the whole number 40,317,483. Therefore the whole number 40,317,433 is the greatest Logarithm set-down in Lord Napier's Canon. Now this number is less than the whole number 100,000,000, or 10 times the radius 10,000,000, and consequently may be subtracted from it; and therefore, *à fortiori*, all the other Logarithms in Lord Napier's Canon may likewise be subtracted from it. And Mr. Speidell, in framing his new Logarithms, made choice of this number 100,000,000 for the whole number from which he resolved to subtract all the Logarithms of Lord Napier's Canon; and the remainders of these several

subtractions

ſubtractions are the Logarithms of his new Table. We will now inquire what theſe new numbers, that are ſet-down in Speidell's Table, and called *New Logarithms*, will ſignify, when clearly and ſcientifically interpreted.

Art. 3. Although Lord Napier's Table of Sines and their Logarithms is carried only ſo far as to include the Logarithm of the ratio of the Sine of one minute to the Radius, yet we may ſuppoſe it to be carried as much farther as we pleaſe. We will therefore ſuppoſe it to be carried ſo much farther as to include the Sine that has to the radius 10,000,000 the ſtill greater ratio of minority of which the whole number 100,000,000, (or 10 times the radius 10,000,000,) is the meaſure, or would be the Logarithm in Lord Napier's Table, if that Table had been carried to ſmaller Sines than the Sine of one minute. Let the letter *a* denote this unknown, but determinate, Sine, which is leſs than the Sine of one minute, and of which the Logarithm is 100,000,000; and let S denote any of the Sines that are ſet-down in Lord Napier's Table; and let *r* be put for 10,000,000, or the radius of the circle. Then it is evident that the ratio of the ſmall Sine *a* to the radius *r* will be compounded of the ratio of the ſaid Sine *a* to the Sine S, and of the ratio of the Sine S to the radius *r*. Therefore the ratio of the Sine *a* to the Sine S will be equal to the exceſs of the ratio of the Sine *a* to the radius *r* above the ratio of the Sine S to the radius *r*; and conſequently the meaſure, or Logarithm, of the ratio of the Sine *a* to the Sine S will be equal to the exceſs of the meaſure, or Logarithm, of the ratio of the Sine *a* to the radius *r*, above the meaſure, or Logarithm, of the ratio of the Sine S to the radius *r*, that is, to the exceſs of the Logarithm 100,000,000 above Lord Napier's Logarithm of the Sine S. Now this exceſs is the number ſet-down in Speidell's Table of New Logarithms as the Logarithm correſponding to the Sine S. And therefore the numbers ſet-down in Speidell's Table under the Title of *New Logarithms* are the meaſures, or Logarithms, of the ſeveral ratios of minority of the Sine *a* (which is leſs than the Sine of one minute,) to the Sines of the ſeveral arches of the Quadrant to which the ſaid Logarithms are annexed, or correſpond.

Art. 4. If it be required to aſſign the numerical value of the ſmall Sine *a*, of which the Logarithm is 100,000,000, it may be done as follows. If, in Lord Napier's Table of Sines and Logarithms, the letter *r* be put for the radius of

of the circle, and the letter S for any Sine in it, and the letter x for the Logarithm of the ratio of the Sine S to the radius r, we shall have $S = \frac{\overline{r-1}^{x}}{r^{x-1}}$, as has been shown above in page 665. Therefore $\frac{S}{r}$ will be $(= \frac{\overline{r-1}^{x}}{r^{x-1}} \times \frac{1}{r} = \frac{\overline{r-1}^{x}}{r \times r^{x-1}} = \frac{\overline{r-1}^{x}}{r^{x}} = \overline{\frac{r-1}{r}}^{x}) = \overline{1 - \frac{1}{r}}^{x}$. But, when the Logarithm x is $= 100{,}000{,}000$, or $10 \times 10{,}000{,}000$, or $10 \times r$, the Sine S will be equal to a; and consequently $\frac{S}{r}$ will be $= \frac{a}{r}$, and $\overline{1 - \frac{1}{r}}^{x}$ will be $= \overline{1 - \frac{1}{r}}^{10 \times r}$, and $\frac{a}{r}$ (which is $= \frac{S}{r}$,) will be $= \overline{1 - \frac{1}{r}}^{10 \times r}$. Therefore $\overline{\frac{a}{r}}^{\frac{1}{10}}$ will be $(= \overline{1 - \frac{1}{r}}^{10r \times \frac{1}{10}}) = \overline{1 - \frac{1}{r}}^{r} =$ (by the binomial Theorem,) to the Series $1 - r \times \frac{1}{r} + r \times \frac{r-1}{2} \times \frac{1}{r^2} - r \times \frac{r-1}{2} \times \frac{r-2}{3} \times \frac{1}{r^3} + r \times \frac{r-1}{2} \times \frac{r-2}{3} \times \frac{r-3}{4} \times \frac{1}{r^4} - r \times \frac{r-1}{2} \times \frac{r-2}{3} \times \frac{r-3}{4} \times \frac{r-4}{5} \times \frac{1}{r^5} +$ &c $=$ the Series $1 - 1 + \frac{r-1}{2r} - \overline{\frac{r-1}{2r}} \times \frac{r-2}{3r} + \frac{r-1}{2r} \times \frac{r-2}{3r} \times \frac{r-3}{4r} - \overline{\frac{r-1}{2r}} \times \frac{r-2}{3r} \times \frac{r-3}{4r} \times \frac{r-4}{5r} +$ &c $=$ the Series $\frac{r-1}{2r} - \overline{\frac{r-1}{2r}} \times \frac{r-2}{3r} + \frac{r-1}{2r} \times \frac{r-2}{3r} \times \frac{r-3}{4r} - \overline{\frac{r-1}{2r}} \times \frac{r-2}{3r} \times \frac{r-3}{4r} \times \frac{r-4}{5r} +$ &c.

But, because r is equal to the great number 10,000,000, the binomial quantities $r-1$, $r-2$, $r-3$, $r-4$, &c will be very nearly equal to, though a little less than, r, or 10,000,000, and consequently the Series $\frac{r-1}{2r} - \overline{\frac{r-1}{2r}} \times \frac{r-2}{3r} + \frac{r-1}{2r}$

$+ \frac{r-1}{2r} \times \frac{r-2}{3r} \times \frac{r-3}{4r} - \overline{\frac{r-1}{2r} \times \frac{r-2}{3r} \times \frac{r-3}{4r} \times \frac{r-4}{5r}}$ + &c will be very nearly equal to the Series $\frac{r}{2r} - \frac{r}{2r} \times \frac{r}{3r} + \frac{r}{2r} \times \frac{r}{3r} \times \frac{r}{4r} - \frac{r}{2r} \times \frac{r}{3r} \times \frac{r}{4r} \times \frac{r}{5r}$ + &c, or to the Series $\frac{1}{2} - \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} - \frac{1}{2} \times \frac{1}{3} \times \frac{1}{4} \times \frac{1}{5}$ + &c, or to the Series $\frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}$ + &c.

Therefore $\overline{\frac{a}{r}}^{\frac{1}{10}}$ will be very nearly = the Series $\frac{1}{2} - \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} - \frac{1}{2 \cdot 3 \cdot 4 \cdot 5}$ + &c = (by the computation of this Series made in the first volume of the *Scriptores Logarithmici*, page 373,) the decimal fraction 0.367,879,4. And consequently $\frac{a}{r}$ will be very nearly = $\overline{0.367,879,4}^{10}$, and a will be very nearly = $r \times \overline{0.367,879,4}^{10}$, or 10,000,000 × $\overline{0.367,879,4}^{10}$. Now $\overline{0.367,879,4}^{10}$ is nearly equal to $\frac{454,000}{10,000,000,000} = \frac{454}{10,000,000}$, and consequently 10,000,000 × $\overline{0.367,879,4}^{10}$ will be nearly = 10,000,000 × $\frac{454}{10,000,000}$ = 454. Therefore the Sine a will be nearly = 454.

Q. E. I.

It follows therefore that the numbers set-down in Speidell's Table under the title of *New Logarithms* are the measures, or Logarithms, of the several ratios of minority of the small Sine denoted by the whole number 454 to the Sines of the several arches of the Quadrant to which the said Logarithms are annexed, or correspond, the radius of the circle being r, or 10,000,000.

Examples of the Connection between Speidell's New Logarithms of given Sines, and Napier's Logarithms of the same Sines.

Art. 5. The Sine of one minute is 2909. And I find $\frac{454}{2909}$ to be $=$ $\frac{1,560,722}{10,000,000}$. And it will be found that the Logarithm of the Sine of one minute in Speidell's Table, which is the measure of the ratio of 454 to 2909, is equal to the Logarithm corresponding to the Sine 1,560,722 in Napier's Canon, which is the measure of the ratio of 1,560,722, to the radius 10,000,000, which is equal to the former ratio of 454 to 2909. For the Logarithm of the Sine of one minute in Speidell's Table is 185,743, and the Logarithm of the Sine 1,560,722, or 1,561,472, (which is the nearest Sine to 1,560,722, that is set-down in Napier's Table,) is 18,569,557, which is nearly equal to 18,574,300, which would be the Logarithm of the Sine of one minute in Speidell's Table, if the Logarithms had been carried to the same number of figures in his Table as in Lord Napier's.

As another example of the truth of this explanation of Speidell's New Logarithms, let us consider the Sine of one degree. Now in Lord Napier's Table, (in which the radius of the circle is called 10,000,000,) the Sine of one degree is $=$ 174,524. And I find $\frac{454}{174,524}$ to be $=$ $\frac{26,013}{10,000,000}$. And the Logarithm of the Sine of one degree in Speidell's Table is 595,172, or (adding two ciphers to it, in order to make it fit to be compared with the Logarithms of Lord Napier's Table,) 59,517,200. And the Sine in Napier's Canon that comes nearest to the Sine 26,013 is 26,180, the Logarithm of which is 59,453,453; which is nearly equal to the Logarithm 59,517,200, as it ought to be, they being measures of the two equal ratios of 454 to 174,524, and of 26,013 to 10,000,000.

Art.

Art. 6. Speidell's contrivance for baniſhing from the Canon the troubleſome diſtinction of Logarithms that were to be added, and Logarithms that were to be ſubtracted, and for ſubſtituting in it's place a new Table containing only one ſort of Logarithms, amounts therefore to this: "That he has pitched-upon a "convenient line that is leſs than any of the Sines and Tangents that are in-"tended to be comprehended in the Table of Sines and Tangents and their "Logarithms, and has filled-up the Canon with numbers, derived from the "Logarithms contained in Lord Napier's Table by Subtraction, which new "numbers are meaſures, or Logarithms, of the ratios of minority of the ſaid "ſmall aſſumed Line to the ſeveral Sines and Tangents, &c contained in Lord "Napier's Canon, or Table, of Sines and Logarithms, inſtead of ſetting-down "the meaſures, or Logarithms, of the ratios of the ſaid Sines, and Tangents, "&c to the radius of the circle, as Lord Napier had done." In this way of proceeding it is manifeſt that this Canon, or Table, will contain only one ſort of Logarithms, to wit, the Logarithms, or meaſures, of ratios of minority. And, becauſe the Logarithm, or meaſure, of the ratio of minority of the ſmall aſſumed line *a* to any Sine S that is ſet-down in the Table, is preciſely equal in magnitude to the Logarithm, or meaſure, of the contrary ratio of majority of S to *a*, we may affirm of the Logarithms of ſuch a table as Speidell's, compoſed of one ſort of Logarithms only, "that they are the Logarithms, or meaſures, of the "ſeveral ratios of majority of the Sines and Tangents, &c, to which they re-"ſpectively belong, to the ſaid ſmall, aſſumed, line *a*," with the ſame degree of propriety as we can ſay of them, "that they are the Logarithms, or meaſures, of "the ratios of minority, of the ſaid ſmall, aſſumed, line *a* to the ſeveral Sines, "and Tangents, &c, to which they reſpectively belong."

Art. 7. We may therefore obſerve, that the principle upon which Speidell's Table of New Logarithms is conſtructed (as far as the object aimed-at was to introduce into it only one ſort of Logarithms inſtead of two,) when viewed in a ſcientific manner, is the very ſame with that on which the Tables now in uſe for the purpoſes of Trigonometry are formed. For the Logarithms contained in theſe Trigonometrical Tables are the meaſures of the ratios of majority of the ſeveral Sines, Tangents, &c contained in them, to the thouſand-millionth part

of the radius, which is leſs than any Sine, or Tangent, in the Table; or (which comes to the ſame thing,) they are the meaſures of the ratios of minority of the thouſand-millionth part of the radius to the ſeveral Sines, Tangents, &c, contained in the ſaid Tables.

Art. 8. The Logarithm of the radius in Speidell's Table, being the meaſure of the ratio of the ſmall line *a*, or 454, to the radius, would have been 100,000,000, or 10 × 10,000,000, or 10 × *r*, if all the places of figures in the numbers contained in Lord Napier's Table had been retained in Speidell's Table. But, as Speidell has choſen to reject the two laſt figures of the numbers in Lord Napier's Canon, the Logarithm of the radius in his table is only $\left(\frac{100,000,000}{100}\right.$, or) 1,000,000.

Art. 9. What has been ſaid above concerning Speidell's Table of New Logarithms relates only to the Logarithms of the Sines throughout the Quadrant, and of the Tangents in the firſt half of the Quadrant. For, as to the numbers ſet-down in that Table as the Logarithms of the Tangents in the latter half of the Quadrant, they are not the Logarithms of thoſe Tangents, though they are called ſo, but are the remainders that ariſe by ſubtracting the Logarithms of the Tangents contained in the firſt half of the Quadrant (in which the arches are leſs than 45 degrees,) from 1,000,000, or the Logarithm of the radius. And, in like manner, the numbers called *Secants* in Speidell's Table, (as if they were the Logarithms of the Secants of the arches ſet-down in the Quadrant,) are not (as they might be taken to be fromthe title given them,) the Logarithms of the Secants of the arches ſet-down in the Table, but are the remainders which ariſe by ſubtracting the correſponding Co-ſines from 1,000,000, or the Logarithm of the radius. And Speidell's intention, in ſetting-down theſe remainders in his Table inſtead of the proper Logarithms of theſe Tangents and Secants, was to baniſh the operation of *Subtraction* from the Arithmetical operations that would be neceſſary to be performed in applying his Table of Logarithms to Practice, and to work by *Addition* only; as is often done in the uſe of our modern Tables of Logarithms by the means of *Arithmetical Complements*. This he ſeems to think

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an improvement of ſome conſequence, and plumes himſelf upon it, as having made a happy diſcovery. See the 17th and 18th pages of the Tract intitled *A Brief Treatiſe of Sphærical Triangles*, which is prefixed to his Table of New Logarithms.

Art. 10. Speidell's Table of New Logarithms, in the Edition of the year 1628, takes-up ninety pages of a ſmall quarto ſize, every page containing the Logarithms belonging to the Sines and Co-ſines, Tangents and Co tangents, Secants and Co-ſecants, of 30 different arches of the Quadrant, that exceed each other by one minute of a Degree. Each of theſe pages is divided by parallel lines into eight different columns, the firſt of which, or that fartheſt on the left-hand part of the page, contains the number of minutes in 30 different ſucceſſive arches, over and above the number of whole Degrees contained in the ſaid arches, the number of which Degrees is ſet-down at the top of the page, as it's title. And this firſt column, containing theſe 30 minutes, has the capital letter M. for *Minutes*, placed over it. The ſecond column of the page contains the Logarithms of the Sines of the ſeveral arches, which conſiſt of the Degrees ſet-down at the top of the page, and the Minutes of a Degree ſet-down in the firſt column of the page; which (as has been obſerved above in Art. 6,) will be the Logarithms of the ratios of the Sines of the ſaid arches, reſpectively, to the ſmall line *a*, or 454, if the Sines be the ſame, or contain the ſame number of decimal figures as in Lord Napier's Canon, or the radius be 10,000,000, or be ſuppoſed to be divided into ten million equal parts; but, if the radius be only $\frac{10,000,000}{100}$, or 100,000, and conſequently the numbers expreſſing the lengths of the Sines contain two figures fewer than the numbers expreſſing their lengths in Lord Napier's Table, (which is the caſe in Speidell's Table of Logarithms publiſhed in the year 1628,) the ſaid Logarithms of the Sines will be the Logarithms of the ratios of the Sines to the ſmall line $\frac{454}{100}$, or 4.54. And this ſecond column has the word *Sine* placed over it. The third column of the page contains the Logarithms of the Co-ſines of the ſame arches, or of the Sines of the complements of the ſaid arches to an arch of 90 Degrees, and has the abridged word *Comp.* for *Complement*, placed over it. The fourth column contains

the

the Logarithms of the Tangents of the ſame arches, and has the word *Tangent* placed over it. The fifth column contains the Logarithms of the Co-tangents of the ſame arches, or of the Tangents of the complements of the ſaid arches to an arch of 90 Degrees, and has the abridged word *Comp.* for *Complement* placed over it. The ſixth column contains the Logarithms of the Secants of the ſame arches, and has the word *Secant* placed over it. The ſeventh column contains the Logarithms of the Co-ſecants, or of the Secants of the complements of the ſaid arches to an arch of 90 degrees, and has the abridged word *Comp.* for *Complement*, placed over it. And the eighth and laſt column, or that which is neareſt to the right ſide of the page, contains a liſt of 30 ſucceſſive Minutes of a Degree, that decreaſe gradually from the top of the page to the bottom of it, and are the complements of the minutes that are ſet-down even with them in the firſt column to 60 minutes, or the number of minutes contained in a whole degree. And this eighth column, conſiſting of Minutes of a Degree, has no title placed above it, but has the capital letter *M.* for *Minutes*, placed under it at the bottom of the page. And at the bottom of the page, on the right-hand ſide, is placed the number of Degrees which is the complement of the number of Degrees ſet-down at the top of the page to 89 Degrees. And under the ſecond column of the page, which has the word *Sine* placed over it, is placed the abridged word *Comp.* for *Complement*, ſignifying that the Logarithms of the Sines of the arches reckoned from the top of the page, and of which the additional minutes are ſet-down in the firſt column, are the Logarithms of the Co-ſines of the arches, reckoned upwards from the bottom of the page, and of which the additional minutes are ſet-down in the eighth column, becauſe the Sines of the arches reckoned from the top of the page are the Co-ſines of the arches reckoned from the bottom of the page, thoſe two arches being the complements of each other to a Quadrant, or arch of 90 Degrees. And under the third column, (over which is placed, at the top of the page, the word *Comp.*) is placed, at the bottom of the page, the word *Sine.* And under the fourth column (which has the word *Tangent* placed over it,) is placed the word *Comp.* for *Complement.* And under the fifth column (which has the word *Comp.* placed over it,) is placed the word *Tangent.* And under the ſixth column (which has the word *Secant* placed over it,) is placed the word *Comp.* for *Complement.* And

under

under the ſeventh column (which has the word *Comp.* placed over it,) is placed the word *Secant.* But this deſcription of this Table will be better underſtood by ſetting down the two firſt pages of it; which are as follows :

Deg. 0. *Numbers for the*

M.	*Sine.*	*Comp.*	*Tangent.*	*Comp.*	*Secant.*	*Comp.*	
0	000,000	100,000	000,000	000,000	0	000,000	60
1	185,743	100,000	185,743	814,257	0	814,257	59
2	255,058	100,000	255,058	744,942	0	744,942	58
3	295,604	100,000	295,604	704,396	0	704,396	57
4	324,373	100,000	324,373	675,627	0	675,627	56
5	346,687	100,000	346,687	653,313	0	653,313	55
6	364,919	100,000	364,919	635,081	0	635,081	54
7	380,334	100,000	380,334	619,666	0	619,666	53
8	393,687	100,000	393,687	606,313	0	606,313	52
9	405,465	100,000	405,466	594,534	0	594,535	51
10	416,001	100,000	416,002	583,998	0	583,999	50
11	425,532	999,999	425,533	574,467	1	574,468	49
12	434,234	999,999	434,234	565,766	1	565,766	48
13	442,238	999,999	442,239	557,761	1	557,762	47
14	449,649	999,999	449,649	550,351	1	550,351	46
15	456,548	999,999	456,549	543,451	1	543,452	45
16	463,002	999,999	463,003	536,997	1	536,998	44
17	469,064	999,999	469,064	530,936	1	530,936	43
18	474,780	999,999	474,781	525,219	1	525,220	42
19	480,186	999,998	480,188	519,812	2	519,814	41
20	485,316	999,998	485,316	514,684	2	514,684	40
21	490,195	999,998	490,195	509,805	2	509,805	39
22	494,847	999,998	494,849	505,151	2	505,153	38
23	499,292	999,998	499,294	500,706	2	500,708	37
24	503,548	999,998	503,550	496,450	2	496,452	36
25	507,630	999,997	507,632	492,368	3	492,370	35
26	511,552	999,997	511,555	488,445	3	488,448	34
27	515,326	999,997	515,329	484,671	3	484,674	33
28	518,962	999,997	518,966	481,034	3	481,038	32
29	522,471	999,996	522,475	477,525	4	477,529	31
30	525,861	999,996	525,865	474,135	4	474,139	30
	Comp.	*Sine.*	*Comp.*	*Tangent.*	*Comp.*	*Secant.*	*M.*

Numbers for the Deg. 89.

Deg. o. *Numbers for the*

M.	*Sine.*	*Comp.*	*Tangent.*	*Comp.*	*Secant.*	*Comp.*	
30	525,861	999,996	525,865	474,135	4	474,139	30
31	529,140	999,996	529,144	470,856	4	470,860	29
32	532,315	999,996	532,320	467,680	4	467,685	28
33	535,392	999,995	535,397	464,603	5	464,608	27
34	538,377	999,995	538,382	461,618	5	461,623	26
35	541,276	999,995	541,281	488,719	5	458,724	25
36	544,093	999,995	544,099	455,901	5	455,907	24
37	546,833	999,994	546,839	453,161	6	453,167	23
38	549,500	999,994	549,506	450,494	6	450,500	22
39	552,097	999,994	552,103	447,897	6	447,903	21
40	554,629	999,993	554,635	445,365	7	445,371	20
41	557,098	999,993	557,105	442,895	7	442,902	19
42	559,507	999,993	559,515	440,485	7	440,493	18
43	561,860	999,992	561,868	438,132	8	438,140	17
44	564,159	999,992	564,167	435,833	8	435,841	16
45	566,406	999,991	566,415	433,585	9	433,594	15
46	568,604	999,991	568,613	431,387	9	431,396	14
47	570,755	999,991	570,764	429,236	9	429,245	13
48	572,860	999,990	572,870	427,130	10	427,140	12
49	574,922	999,990	574,932	425,068	10	425,078	11
50	576,942	999,989	576,952	423,048	11	423,058	10
51	578,922	999,989	578,933	421,067	11	421,078	9
52	580,864	999,989	580,875	419,125	11	419,136	8
53	582,768	999,988	582,780	417,220	12	417,232	7
54	584,637	999,988	586,650	415,350	12	415,363	6
55	586,472	999,987	588 485	413,515	13	413,528	5
56	588,274	999,987	588,287	411,713	13	411,726	4
57	589,934	999,986	589,947	410,053	14	410,066	3
58	591,783	999,986	591,797	408,203	14	408,217	2
59	593,492	999,985	593,507	406,493	15	406,508	1
60	595,172	999,985	595,188	404,812	15	404,828	0
	Comp.	*Sine.*	*Comp.*	*Tangent.*	*Comp.*	*Secant.*	*M.*

Numbers for the 89. Deg.

Art. 11. But the reader would be enabled to form a juſter notion of this Table of Speidell's Logarithms if the titles of ſome of the columns were changed for others that more correctly expreſſed the nature of the numbers contained in them. Thus, inſtead of the title *Secant* at the top of the ſixth column, I would write *Arithmetical Complement of the Logarithm of Co-ſine*; and inſtead of the word *Comp.* for *Complement*, at the top of the ſeventh column, (which ſeems to denote the Logarithm of the Co-ſecant of the arch, or of the Secant of it's complement to an arch of 90 degrees,) I would write *Arithmetical Complement of the Logarithm of Sine.* And the numbers ſet-down in the fifth column, (which has the word *Comp.* for *Complement*, placed over it, and ſeems to denote the Logarithm of the Co-tangent of an arch leſs than 45 degrees, or of the Tangent of an arch, that is the complement of ſuch an arch, to 90 degrees, that is, the Logarithm of the Tangent of an arch that is greater than 45 degrees,) ought to be intitled *Arithmetical Complement of the Logarithm of Co-tangent.*

The uſe made of the Arithmetical Complements of Logarithms, to avoid the operation of ſubtraction in working the proportions that occur in Spherical Trigonometry, is founded on the two following propoſitions, to wit, firſt, "That the Co-ſecant of a circular arch is to the radius of the circle in the ſame proportion as the radius is to the Sine of the ſame arch," and, ſecondly, "That the tangent of any circular arch is to the radius in the ſame proportion as the radius is to the Co-tangent of the ſame arch;" which propoſitions co-incide with the 4th axiom mentioned in page 2nd of the tract of John Speidell, intitled *A Brief Treatiſe of Sphærical Triangles*, prefixed to his Table of New Logarithms, and taken from Petiſcus. In two Lemmas, one in page 16, and the other in page 20, of the ſame treatiſe of Speidell, it is demonſtrated that the addition of the Arithmetical Complements of the Logarithms, as it is commonly practiſed, is the ſame in arithmetical effect with the ſubtraction of the Logarithms themſelves, of which the Arithmetical Complements have been ſo added.

A Deſcription of another Table of Logarithms publiſhed by the ſaid John Speidell, after his New Logarithms, in the ſaid Edition of the year 1628.

Art. 12. After the Table of New Logarithms of Sines, Tangents, and Secants, publiſhed

published in the year 1628, Speidell has given us another Table of Logarithms relating to numbers in general, without any reference to Trigonometry, in which are contained the Logarithms of all the natural numbers 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c. as far as the number 1000, by which he underſtands, (as mathematicians in general do at this day,) the Logarithms of the ratios of thoſe numbers to unity, or 1. Theſe Logarithms are the ſame as Lord Napier's Logarithms of the ſame numbers, the Logarithm of the ratio of 10 to 1 being 2,302,584, which is Napier's Logarithm of that ratio. The column of the numbers 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, &c, to 1000, is placed in the middle of the page; and the Logarithms of the numbers are placed in the firſt column on the left-hand ſide of the page. The next, or ſecond, column of the page contains the differences of the Logarithms ſet down in the preceeding, or firſt, column; and the next column to that, or the third column, contains the Arithmetical Complements of the Logarithms to the number 10,000,000. The numbers in the column of Logarithms are to be uſed when the given number is a multiplier; the numbers in the column of Arithmetical Complements are to be uſed when the given number is a diviſor. The three columns on the right of the middle column contain numbers that are the halves of thoſe which are contained in the three columns on the left-hand ſide of the middle column. The column immediately on the right hand of the middle column contains the halves of the Logarithms in the firſt column, which numbers are therefore the Logarithms of the ſquare-roots of the numbers in the middle column: and this column is to be uſed when the ſquare-root of the given number is a multiplier. And the numbers in the eighth column, or the laſt column on the right-hand ſide of the page, are the Arithmetical Complements of the half-Logarithms to 5,000,000, and not to 10,000,000; and this column is to be uſed when the ſquare-root of the given number is a diviſor.

End of Mr. Ivory's Account of Mr. John Speidell's New Logarithms.

This Second Table of Logarithms, publiſhed by J. Speidell, may, it is ſuppoſed, be of conſiderable uſe to Mathematicians in cultivating ſome Branches of the Doctrine of Fluxions: and therefore I have here re-printed it, as follows:

THE

THE SECOND TABLE OF LOGARITHMS,

Published by Mr. John Speidell in the Edition of the year 1628, *being the Logarithms of the Ratios of the several natural Numbers* 2, 3, 4, 5, 6, *&c, continued to* 1000, *to* 1, *according to Lord Napier's System.*

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
000,000		0,000,000	1	000,000		5,000,000	1
	693,147				346,573		
693,147		9,306,853	2	346,573		4,653,427	2
	405,465				202,733		
1,098,612		8,901,388	3	549,306		4,450,694	3
	287,682				143,841		
1,386,294		8,613,706	4	693,147		4,306,853	1.
	223,143				111,752		
1,609,437		8,390,563	5	804,719		4,195,281	1
	182,321				91,160		
1,791,759		8,208,241	6	895,879.		4,104,121	2
	154,150				77,075		
1.945,909		8,054,091	7	972,955		4,027,045	3
	133,531				66,765		
2,079,441		7,920,559	8	1,039,720		3,960,280	2.
	117,783				58,892		
2,197,224		7,802,776	9	1,098,612		3,901,388	1
	105,360				52,680		
2,302,584		7,697,416	10	1,151,292		3,848,708	2
	95,310				47,655		
2,397,894		7,602,106	1	1,198,947		3,801,053	3
	87,011				43,506		
2,484,906		7,515,094	2	1,242,453		3,757,547	3.
	80,043				40,021		
2,564,948		7,435,052	3	1,282,474		3,717,526	1
	74,108				37,054		
2,639,056		7,360,944	4	1,319,528		3,680,472	2
	68,993				34,497		
2,708,049		7,291,951	15	1,354,025		3,645,975	3
	64,538				32,269		
2,772,588		7,227,412	6	1,386,294		3,613,706	4.
	60,624				30,312		
2,833,212		7,166,788	7	1,416,606		3,583,394	1
	57,158				28,579		
2,890,371		7,109,629	8	1,445,185		3,554,815	2
	54,067				27,033		
2,944,438		7,055,562	9	1,472,219		3,527,781	3
	51,293				25,647		
2,995,731		7,004,269	20	1,497,866		3,502,134	5.
	48,790				24,395		
3,044,521		6,955,479	1	1,522,261		3,477,739	1
	46,520				23,260		
3,091,041		6,908,959	2	1,545,521		3,454,479	2
	44,452				22,226		
3,135,493		6,864,507	3	1,567,747		3,432,253	3
	42,560				21,280		
3,178,053		6,821,947	4	1,589,026		3,410,974	6.
	40,822				20,411		
3,218,875		6,781,125	25	1,609,437		3,390,563	1
	39,220				19,610		
3,258,095		6,741,905	6	1,629,048		3,370,952	2
	37,740				18,870		
3,295,836		6,704,164	7	1,647,918		3,352,082	3
	36,367				18,184		
3,332,203		6,667,797	8	1,666,102		3,333,898	7.
	35,091				17,545		
3,367,295		6,632,705	9	1,683,647		3,316,353	1
	33,902				16,951		
3,401,196		6,598,804	30	1,700,598		3,299,402	2
	32,790				16,394		
3,433,986		6,566,014	1	1,716,993		3,283,007	3
	31,749				15,874		
3,465,735		6,534,265	2	1,732,867		3,267,133	8.
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
8	3,465,735		6,534,265	2	1,732,867		3,267,133
		30,771				15,386	
1	3,496,506		6,503,494	3	1,748,253		3,251,747
		29,853				14,927	
2	3,526,359		6,473,641	4	1,763,180		3,236,820
		28,988				14,494	
3	3,555,347		6,444,653	35	1,777,673		3,222,327
		28,171				14,085	
9	3,583,518		6,416,482	6	1,791,759		3,208,241
		27,399				13,699	
1	3,610,917		6,389,083	7	1,805,458		3,194,542
		26,668				13,334	
2	3,637,585		6,362,415	8	1,818,792		3,181,208
		25,975				12,988	
3	3,663,560		6,336,440	9	1,831,780		3,168,220
		25,318				12,659	
10	3,688,878		6,311,122	40	1,844,439		3,155,561
		24,693				12,346	
1	3,713,571		6,286,429	1	1,856,785		3,143,215
		24,098				12,049	
2	3,737,668		6,262,332	2	1,868,834		3,131,166
		23,530				11,765	
3	3,761,199		6,238,801	3	1,880,599		3,119,401
		22,990				11,495	
11	3,784,188		6,215,812	4	1,892,094		3,107,906
		22,473				11,236	
1	3,806,661		6,193,339	45	1,903,331		3,096,669
		21,979				10,989	
2	3,828,640		6,171,360	6	1,914,320		3,085,680
		21,506				10,753	
3	3,850,146		6,149,854	7	1,925,073		3,074,927
		21,053				10,527	
10	3,871,200		6,128,800	8	1,935,600		3,064,400
		20,619				10,310	
1	3,891,819		6,108,181	9	1,945,909		3,054,091
		20,202				10,101	
2	3,912,022		6,087,978	50	1,956,011		3,043,989
		19,803				9,901	
3	3,931,824		6,068,176	1	1,965,912		3,034,088
		19,418				9,709	
1.1	3,951,242		6,048,758	2	1,975,621		3,024,379
		19,048				9,524	
1	3,970,290		6,029,710	3	1,985,145		3,014,855
		18,692				9,346	
2	3,988,983		6,011,017	4	1,994,491		3,005,509
		18,349				9,175	
3	4,007,332		5,992,668	55	2,003,666		2,996,334
		18,018				9,009	
1.2	4,025,350		5,974,650	6	2,012,675		2,987,325
		17,699				8,849	
1	4,043,050		5,956,950	7	2,021,525		2,978,475
		17,392				8,696	
2	4,060,441		5,939,559	8	2,030,221		2,969,779
		17,095				8,547	
3	4,077,536		5,922,464	9	2,038,768		2,961,232
		16,807				8,404	
1.3	4,094,343		5,905,657	60	2,047,172		2,952,828
		16 529				8,265	
1	4,110,872		5,889,128	1	2,055,436		2,944,564
		16,261				8,130	
2	4,127,133		5,872,867	2	2,063,566		2,936,434
		16,000				8,000	
3	4,143,133		5,856,867	3	2,071,567		2,928,433
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
4,143,133		5,856,867	3	2,071,567		2,928,433	3
	15,748				7,874		
4,158,882		5,841,118	4	2,079,441		2,920,559	1.4
	15,504				7,752		
4,174,386		5,825,614	65	2,087,193		2,912,807	1
	15,267				7,634		
4,189,653		5,810,347	6	2,094,827		2,905,173	2
	15,038				7,519		
4,204,691		5,795,309	7	2,102,346		2,897,654	3
	14,815				7,407		
4,219,506		5,780,494	8	2,109,753		2,890,247	1.5
	14,599				7,299		
4,234,105		5,765,895	9	2,117,052		2,882,948	1
	14,389				7,194		
4,248,494		5,751,506	70	2,124,247		2,875,753	2
	14,185				7,092		
4,262,678		5,737,322	1	2,131,339		2,868,661	3
	13,986				6,993		
4,276,665		5,723,335	2	2,138,332		2,861,668	1.6
	13,793				6,897		
4,290,458		5,709,542	3	2,145,229		2,854,771	1
	13,606				6,803		
4,304,063		5,695,937	4	2,152,032		2,847,968	2
	13,423				6,712		
							3
4,317,487		5,682,513	75	2,158,743		2,841,257	
	13,245				6,623		
4,330,732		5,669,268	6	2,165,366		2,834,634	1.7
	13,072				6,536		
4,343,804		5,656,196	7	2,171,902		2,828,098	1
	12,903				6,452		
							2
4,356,707		5,643,293	8	2,178,354		2,821,646	
	12,739				6,369		
4,369,446		5,630,554	9	2,184,723		2,815,277	3
	12,579				6,289		
4,382,025		5,617,975	80	2,191,013		2,808,987	1.8
	12,423				6,211		
							1
4,394,448		5,605,552	1	2,197,224		2,802,776	
	12,270				6,135		
4,406,718		5,593,282	2	2,203,359		2,796,641	2
	12,121				6,061		
4,418,839		5,581,161	3	2,209,419		2,790,581	3
	11,976				5,988		
							1.9
4,430,815		5,569,185	4	2,215,408		2,784,592	
	11,834				5,917		
4,442,650		5,557,350	85	2,221,325		2,778,675	1
	11,696				5,848		
4,454,346		5,545,654	6	2,227,173		2,772,827	2
	11,561				5,780		
							3
4,465,906		5,534,094	7	2,232,953		2,767,047	
	11,429				5,714		
4,477,335		5,522,665	8	2,238,668		2,761,332	1.10
	11,300				5,650		
4,488,635		5,511,365	9	2,244,317		2,755,683	1
	11,173				5,587		
							2
4,499,808		5,500,192	90	2,249,904		2,750,096	
	11,050				5,525		
4,510,858		5,489,142	1	2,255,429		2,744,571	3
	10,929				5,464		
4,521,787		5,478,213	2	2,260,893		2,739,107	1.11
	10,811				5,406		
							1
4,532,598		5,467,402	3	2,266,299		2,733,701	
	10,695				5,348		
4,543,293		5,456,707	4	2,271,647		2,728,353	2
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences Logarithms.*	*Halves of Arithmetical Complements.*
2	4,543,293	10,582	5,456,707	4	2,271,647	5,291	2,728,353
3	4,553,875	10,471	5,446,125	95	2,276,938	5,236	2,723,062
2.0	4,564,346	10,363	5,435,654	6	2,282,173	5,181	2,717,827
1	4,574,709	10,256	5,425,291	7	2,287,355	5,128	2,712,645
2	4,584,966	10,152	5,415,034	8	2,292,483	5,076	2,707,517
3	4,595,118	10,050	5,404,882	9	2,297,559	5,025	2,702,441
2.1	4,605,168	9,950	5,394,832	100	2,302,584	4,975	2,697,416
1	4,615,119	9,852	5,384,881	1	2,307,559	4,926	2,692,441
2	4,624,971	9,756	5,375,029	2	2,312,486	4,878	2,687,514
3	4,634,727	9,662	5,365,273	3	2,317,364	4,831	2,682,636
2.2	4,644,389	9,570	5,355,611	4	2,322,195	4,785	2,677,805
1	4,653,959	9,478	5,346,041	105	2,326,979	4,739	2,673,021
2	4,663,437	9,390	5,336,563	6	2,331,719	4,695	2,668,281
3	4,672,827	9,302	5,327,173	7	2,336,414	4,651	2,663,586
2.3	4,682,129	9,217	5,317,871	8	2,341,065	4,608	2,658,935
1	4,691,346	9,132	5,308,654	9	2,345,673	4,566	2,654,327
2	4,700,479	9,050	5,299,521	110	2,350,239	4,525	2,649,761
3	4,709,528	8,969	5,290,472	1	2,354,764	4,484	2,645,236
2.4	4,718,497	8,889	5,281,503	2	2,359,249	4,444	2,640,751
1	4,727,386	8,811	5,272,614	3	2,363,693	4,405	2,636,307
2	4,736,197	8,734	5,263,803	4	2,368,098	4,367	2,631,902
3	4,744,930	8,658	5,255,070	115	2,372,465	4,329	2,627,535
2.5	4,753,588	8,584	5,246,412	6	2,376,794	4,292	2,623,206
1	4,762,172	8,511	5,237,828	7	2,381,086	4,255	2,618,914
2	4,770,683	8,439	5,229,317	8	2,385,341	4,219	2,614,659
3	4,779,122	8,368	5,220,878	9	2,389,561	4,184	2,610,439
2.6	4,787,490	8,299	5,212,510	120	2,393,745	4,149	2,606,255
1	4,795,789	8,230	5,204,211	1	2,397,894	4,115	2,602,106
2	4,804,019	8,163	5,195 981	2	2,402,010	4,082	2,597,990
3	4,812,183	8,097	5,187,817	3	2,406,091	4,049	2,593,909
2.7	4,820,280	8,032	5,179,720	4	2,410,140	4,016	2,589,860
1	4,828,312		5,171,688	125	2,414,156		2,585,844
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
4,828,312		5,171,688	125	2,414,156		2,585,844	1
	7,968				3,984		
4,836,280		5,163,720	6	2,418,140		2,581,860	2
	7,905				3,953		
4,844,185		5,155,815	7	2,422,093		2,577,907	3
	7,843				3,922		
4,852,028		5,147,972	8	2,426,014		2,573,986	2.8
	7,782				3,891		
4,859,811		5,140,189	9	2,429,905		2,570,095	1
	7,722				3,861		
4,867,533		5,132,467	130	2,433,766		2,566,234	2
	7,663				3,831		
4,875,196		5,124,804	1	2,437,598		2,562,402	3
	7,605				3,802		
4,882,800		5,117,200	2	2,441,400		2,558,600	2.9
	7,547				3,774		
4,890,347		5,109,653	3	2,445,174		2,554,826	1
	7,490				3,745		
4,897,838		5,102,162	4	2,448,919		2,551,081	2
	7,435				3,717		
4,905,273		5,094,727	135	2,452,636		2,547,364	3
	7,380				3,690		
4,912,653		5,087,347	6	2,456,327		2,543,673	2.10
	7,326				3,663		
4,919,979		5,080,021	7	2,459,990		2,540,010	1
	7,273				3,636		
4,927,252		5,072,748	8	2,463,626		2,536,374	2
	7,220				3,610		
4,934,472		5,065,528	9	2,467,236		2,532,764	3
	7,168				3,584		
4,941,640		5,058,350	140	2,470,820		2,529,180	2.11
	7,118				3,559		
4,948,758		5,051,242	1	2,474,379		2,525,621	1
	7,067				3,534		
4,955,825		5,044,175	2	2,477,913		2,522,087	2
	7,018				3,509		
4,962,843		5,037,157	3	2,481,421		2,518,579	3
	6,969				3,484		
4,969,811		5,030,189	4	2,484,906		2,515,094	3.0
	6,920				3,460		
4,976,732		5,023,268	145	2,488,366		2,511,634	1
	6,873				3,436		
4,983,605		5,016,395	6	2,491,802		2,508,198	2
	6,826				3,413		
4,990,431		5,009,569	7	2,495,215		2,504,785	3
	6,780				3,390		
4,997,210		5,002,790	8	2,498,605		2,501,395	3.1
	6,734				3,367		
5,003,944		4,996,056	9	2,501,972		2,498,028	1
	6,689				3,345		
5,010,633		4,989,367	150	2,505,317		2,494,683	2
	6,645				3,322		
5,017,278		4,982,722	1	2,508,639		2,491,361	3
	6,601				3,300		
5,023,879		4,976,121	2	2,511,939		2,488,061	3.2
	6,557				3,279		
5,030,436		4,969,564	3	2,515,218		2,484,782	1
	6,515				3,257		
5,036,951		4,963,049	4	2,518,475		2,481,525	2
	6,472				3,236		
5,043,423		4,956,577	155	2,521,712		2,478,288	3
	6,431				3,215		
5,049,854		4,950,146	6	2,524,927		2,475,073	3.3
6,907,753		3,092,247		4,605,128		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
3·3	5,049,854	6,390	4,950,146	6	2,524,927	3,195	2,475,073
1	5,056,244	6,349	4,943,756	7	2,528,122	3,175	2,471,878
2	5,062,593	6,309	4,937,407	8	2,531,297	3,155	2,468,703
3	5,068,902	6,270	4,931,098	9	2,534,451	3,135	2,465,549
3·4	5,075,172	6,230	4,924,828	160	2,537,586	3,115	2,462,414
1	5,081,402	6,192	4,918,598	1	2,540,701	3,096	2,459,299
2	5,087,594	6,154	4,912,406	2	2,543,797	3,077	2,456,203
3	5,093,748	6,116	4,906,252	3	2,546,874	3,058	2,453,126
3·5	5,099,865	6,079	4,900,135	4	2,549,932	3,039	2,450,068
1	5,105,944	6,042	4,894,056	165	2,552,972	3,021	2,447,028
2	5,111,986	6,006	4,888,014	6	2,555,993	3,003	2,444,007
3	5,117,992	5,970	4,882,008	7	2,558,996	2,985	2,441,004
3·6	5,123,962	5,935	4,876,038	8	2,561,981	2,967	2,438,019
1	5,129,897	5,900	4,870,103	9	2,564,948	2,945	2,435,052
2	5,135,797	5,865	4,864,203	170	2,567,898	2,933	2,432,102
3	5,141,662	5,831	4,858,338	1	2,570,831	2,915	2,429,169
3·7	5,147,493	5,787	4,852,507	2	2,573,746	2,899	2,426,254
1	5,153,290	5,864	4,846,710	3	2,576,645	2,882	2,423,355
2	5,159,053	5,731	4,840,947	4	2,579,527	2,867	2,420,473
3	5,164,784	5,698	4,835,216	175	2,582,392	2,849	2,417,608
3·8	5,170,482	5,666	4,829,518	6	2,585,241	2,833	2,414,759
1	5,176,148	5,634	4,823,852	7	2,588,074	2,817	2,411,926
2	5,181,782	5,602	4,818,218	8	2,590,891	2,801	2,409,109
3	5,187,384	5,571	4,812,616	9	2,593,692	2,786	2,406,308
3·9	5,192,955	5,549	4,807,045	180	2,596,477	2,770	2,403,523
1	5,198,495	5,510	4,801,505	1	2,599,248	2,755	2,400,752
2	5,204,005	5,480	4,795,995	2	2,602,002	2,740	2,397,998
3	5,209,484	5,450	4,790,516	3	2,604,742	2,725	2,395,258
3·10	5,214,934	5,420	4,785,066	4	2,607,467	2,710	2,392,533
1	5,220,354	5,391	4,779,646	185	2,610,177	2,695	2,389,823
2	5,225,745	5,362	4,774,255	6	2,612,872	2,681	2,387,128
3	5,231,107		4,768,893	7	2,615,553		2,384,447
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
5,231,107		4,768,893	7	2,615,553		2,384,447	3
	5,333				2,667		
5,236,440		4,763,560	8	2,618,220		2,381,780	3.11
	5,305				2,653		
5,241,745		4,758,255	9	2,620,873		2,379,127	1
	5,277				2,639		
5,247,022		4,752,978	190	2,623,511		2,376,489	2
	5,249				2,625		
5,252,271		4,747,729	1	2,626,136		2,373,864	3
	5,222				2,611		
5,257,493		4,742,507	2	2,628,747		2,371,253	4.0
	5,195				2,597		1
5,262,688		4,737,312	3	2,631,344		2,368,656	
	5,168				2,584		
5,267,856		4,732,144	4	2,633,928		2,366,072	2
	5,141				2,571		
5,272,998		4,727,002	195	2,636,499		2,363,501	3
	5,115				2,558		4.1
5,278,113		4,721,887	6	2,639,056		2,360,944	
	5,089				2,545		
5,283,202		4,716,798	7	2,641,601		2,358,399	1
	5,063				2,532		
5,288,265		4,711,735	8	2,644,133		2,355,867	2
	5,038				2,519		3
5,293,303		4,706,697	9	2,646,651		2,353,349	
	5,012				2,506		
5,298,315		4,701,685	200	2,649,158		2,350,842	4.2
	4,988				2,494		
5,303,303		4,696,697	1	2,651,651		2,348,349	1
	4,963				2,481		2
5,308,266		4,691,734	2	2,654,133		2,345,86,	
	4,938				2,469		
5,313,204		4,686,796	3	2,656,602		2,343,398	3
	4,915				2,457		
5,318,118		4,681,882	4	2,659,059		2,340,941	4 3
	4,890				2,445		1
5,323,008		4,676,992	205	2,661,504		2,338,496	
	4,866				2,433		
5,327,874		4,672,126	6	2,663,937		2,336,063	2
	4,843				2,421		
5,332,717		4,667,283	7	2,666,358		2,333,642	3
	4,819				2,410		4.4
5,337,536		4,662,464	8	2,668,768		2,331,232	
	4,796				2,398		
5,342,332		4,657,668	9	2,671,166		2,328,834	1
	4,773				2,387		
5,347,106		4,652,894	210	2,673,553		2,326,447	2
	4,751				2,375		3
5,351,856		4,648,144	1	2,675,928		2,324,072	
	4,728				2,364		
5,356,584		4,643,416	2	2,678,292		2,321,708	4.5
	4,706				2,353		
5,361,290		4,638,710	3	2,680,645		2,319,355	1
	4,684				2,342		2
5,365,974		4,634,026	4	2,682,987		2,317,013	
	4,662				2,331		
5,370,636		4,629,364	215	2,685,318		2,314,682	3
	4,640				2,320		
5,375,276		4,624,724	6	2,687,639		2,312,362	4 6
	4,619				2,309		1
5,379,895		4,620,105	7	2,689,948		2,310,052	
	4,598				2,299		
5,384,493		4,615,507	8	2,692,247		2,307,753	2
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
		4,577				2,288	
3	5,389,070	4,556	4,610,930	9	2,694,535	2,278	2,305,465
4.7	5,393,626	4,535	4,606,374	220	2,696,813	2,268	2,303,187
1	5,398,161	4,515	4,601,839	1	2,699,080	2,257	2,300,920
		4,494				2,247	
2	5,402,675	4,475	4,597,325	2	2,701,338	2,237	2,298,662
3	5,407,170	4,454	4,592,830	3	2,703,585	2,227	2,296,415
4.8	5,411,644		4,588,356	4	2,705,822		2,294,178
		4,435				2,217	
1	5,416,098	4,415	4,583,902	225	2,708,049	2,208	2,291,951
2	5,420,533	4,396	4,579,467	6	2,710,266	2,198	2,289,734
3	5,424,948		4,575,052	7	2,712,474		2,287,526
		4,376				2,188	
4.9	5,429,344	4,357	4,570,656	8	2,714,672	2,179	2,285,328
1	5,433,720	4,338	4,566,280	9	2,716,860	2,169	2,283,140
2	5,438,077		4,561,923	230	2,719,039		2,280,961
		4,319				2,160	
3	5,442,416	4,301	4,557,584	1	2,721,208	2,150	2,278,792
4.10	5,446,735	4,283	4,553,265	2	2,723,368	2,141	2,276,632
1	5,451,036		4,548,964	3	2,725,518		2,274,482
		4,264				2,132	
2	5,455,319	4,246	4,544,681	4	2,727,660	2,123	2,272,340
3	5,459,583	4,228	4,540,417	235	2,729,792	2,114	2,270,208
4.11	5,463,830		4,536,170	6	2,731,915		2,268,085
		4,211				2,105	
1	5,468,058	4,193	4,531,942	7	2,734,029	2,097	2,265,971
2	5,472,269	4,175	4,527,731	8	2,736,134	2,087	2,263,866
3	5,476,462		4,523,538	9	2,738,231		2,261,769
		4,158				2,079	
5.0	5,480,637	4,141	4,519,363	240	2,740,318	2,071	2,259,682
1	5,484,795	4,124	4,515,205	1	2,742,397	2,062	2,257,603
2	5,488,936		4,511,064	2	2,744,468		2,255,532
		4,107				2,053	
3	5,493,059	4,090	4,506,941	3	2,746,530	2,045	2,253,470
5.1	5,497,166	4,073	4,502,834	4	2,748,583	2,037	2,251,417
1	5,501,256		4,498,744	245	2,750,628		2,249,372
		4,057				2,028	
2	5,505,329	4,040	4,494,671	6	2,752,665	2,020	2,247,335
3	5,509,386	4,024	4,490,614	7	2,754,693	2,012	2,245,307
5.2	5,513,427		4,486,573	8	2,756,713		2,243,287
		4,008				2,004	
1	5,517,451	4,004	4,482,549	9	2,758,725	2,002	2,241,275
2	5,521,459		4,478,541	250	2,760,729		2,239,271
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
5,525,451	3,992	4,474,549	1	2,762,725	1,996	2,237,275	3
5,529,427	3,976	4,470,573	2	2,764,714	1,989	2,235,286	5·3
5,533,387	3,960	4,466,613	3	2,766,694	1,980	2,233,306	1
	3,945				1,972		
5,537,332	3,929	4,462,668	4	2,768,666	1,965	2,231,334	2
5,541,261	3,914	4,458,739	255	2,770,631	1,957	2,229,369	3
5,545,175	3,899	4,454,825	6	2,772,588	1,949	2,227,412	5·4
5,549,074	3,883	4,450,926	7	2,774,537	1,942	2,225,463	1
5,552,958	3,868	4,447,042	8	2,776,479	1,934	2,223,521	2
5,556,826	3,853	4,443,174	9	2,778,413	1,927	2,221,587	3
5,560,680	3.839	4,439,320	260	2,780,340	1,919	2,219,660	5·5
5,564,518	3,824	4,435,482	1	2,782,259	1,912	2,217,741	1
5,568,342	3,809	4,431,658	2	2,784,171	1,905	2,215,829	2
5,572,152	3,795	4.427,848	3	2,786,076	1,898	2,213,924	3
5,575,947	3,785	4,424,053	4	2,787,974	1,890	2,212,026	5.6
5,579,728	3,766	4 420,272	265	2,789,864	1,883	2,210,136	1
5,583,494	3,752	4,416,506	6	2,791,747	1,876	2,208,253	2
5,587,247	3,738	4,412,753	7	2,793,623	1,869	2,206,377	3
5,590,985	3,724	4,409,015	8	2,795,492	1,863	2,204,508	5·7
5,594,709	3,710	4,405,291	9	2,797,355	1,855	2,202,645	1
5,598,420	3,697	4,401,580	270	2,799,210	1,848	2,200,790	2
5,602,117	3,683	4,397,883	1	2,801,058	1,842	2,198,940	3
5,605,800	3,670	4,394,200	2	2,802,900	1,835	2,197,100	5.8
5,609 470	3,656	4,390,530	3	2,804,735	1,828	2,195,265	1
5,613,126	3,643	4,386,874	4	2,806,563	1,822	2,193,437	2
5,616,769	3,630	4,383,231	275	2,808,385	1,815	2,191,615	3
5,6 0,399	3,617	4,379,601	6	2,810,199	1,808	2,189,801	5·9
5,624,015	3,604	4,375,985	7	2,812,008	1,802	2,187,992	1
5,627,619	3,592	4,372,381	8	2,813,810	1,795	2,186,190	2
5,631,210	3,578	4,368,790	9	2,815,605	1,789	2,184,395	3
5,634,788	3,565	4,365,212	280	2,817,394	1,782	2,182,606	5.10
5,638,353	3,552	4,361,647	1	2,819,176	1,776	2,180,824	1
5,641,905		4,358,095	2	2,820,952		2,179,048	2
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
2	5,641,905		4,358,095	2	2,820,952		2,179,048
		3,540				1,770	
3	5,645,445		4,354,555	3	2,822,722		2,177,278
		3,527				1,764	
5.11	5,648,972		4,351,028	4	2,824,486		2,175,514
		3,515				1,758	
1	5,652,487		4,347,513	285	2,826,244		2,173,756
		3,503				1,751	
2	5,655,990		4,344,010	6	2,827,995		2,172,005
		3,590				1,745	
3	5,659,480		4,340,520	7	2,829,740		2,170,260
		3,478				1,739	
6.0	5,662,958		4,337,042	8	2,831,479		2,168,521
		3,466				1,733	
1	5,666,425		4,333,575	9	2,833,212		2,166,788
		3,454				1,727	
2	5,669,879		4,330,121	290	2,834,939		2,165,061
		3,442				1,721	
3	5,673,321		4,326,679	1	2,836,661		2,163,339
		3,430				1,715	
6.1	5,676,752		4,323,238	2	2,838,376		2,161,624
		3,418				1,709	
1	5,680,170		4,323,830	3	2,840,085		2,159,915
		3,407				1,704	
2	5,683,578		4,314,122	4	2,841,789		2,158,211
		3,395				1,698	
3	5,686,973		4,313,027	295	2,843,487		2,156,513
		3,384				1,692	
6.2	5,690,357		4,309,643	6	2,845,179		2,154,821
		3,373				1,686	
1	5,693,730		4,306,270	7	2,846,865		2,153,135
		3,361				1,681	
2	5,697,091		4,302,909	8	2,848,546		2,151,454
		3,350				1,675	
3	5,700,441		4,299,559	9	2,850,221		2,149,779
		3,339				1,669	
6.3	5,703,780		4,296,220	300	2,851,890		2,148,110
		3,328				1,664	
1	5,707,108		4,292,892	1	2,853,554		2,146,446
		3,317				1,658	
2	5,710,425		4,289,575	2	2,855,212		2,144,788
		3,305				1,653	
3	5,713,731		4,286,269	3	2,856,865		2,143,135
		3,295				1,648	
6.4	5,717,026		4,282,974	4	2,858,513		2,141,487
		3,284				1,642	
1	5,720,310		4,279,690	305	2,860,155		2,139,845
		3,273				1,636	
2	5,723,583		4,276,417	6	2,861,791		2,138,209
		3,263				1,632	
3	5,726,846		4,273,154	7	2,863,423		2,136,577
		3,252				1,626	
6.5	5,730,098		4,269,902	8	2,865,049		2,133,951
		3,242				1,621	
1	5,733,339		4,266,661	9	2,866,670		2,133,330
		3,231				1,615	
2	5,736,570		4,263,420	310	2,868,285		2,131,715
		3,221				1,610	
3	5,739,791		4,260,209	1	2,869,895		2,130,105
		3,210				1,605	
6.6	5,743,001		4,256,999	2	2,871,501		2,128,499
		3,200				1,600	
1	5,746,201		4,253,799	3	2,873,011		2,126,899
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
5,746,201		4,253,799	3	2,873,101		2,126,899	1
	3,190				1,595		
5,749,391		4,250,609	4	2,874,695		2,125,305	2
	3,180				1,590		
5,752,570		4,247,430	315	2,876,285		2,123,715	3
	3,170				1,585		
5,755,740		4,244,260	6	2,877,870		2,122,130	6.7
	3,160				1,580		
5,758,900		4,241,100	7	2,879,450		2,120,550	1
	3,150				1,575		
5,762,049		4,237,951	8	2,881,025		2,118,975	2
	3,140				1,570		
5,765,189		4,234,811	9	2,882,594		2,117,406	3
	3,130				1,565		
5,768,319		4,231,681	320	2,884,159		2,115,841	6.8
	3,120				1,560		
5,771,439		4,228,561	1	2,885,719		2,114,281	1
	3,110				1,555		
5,774,549		4,225,451	2	2,887,275		2,112,725	2
	3,101				1,550		
5,777,650		4,222,350	3	2,888,825		2,111,175	3
	3,091				1,545		
5,780,741		4,219,259	4	2,890,371		2,109,629	6.9
	3,082				1,541		
5,783,823		4,216,177	325	2,891,912		2,108,088	1
	3,072				1,536		
5,786,895		4,213,105	6	2,893,448		2,106,552	2
	3,063				1,531		
5,789,958		4,210,042	7	2,894,979		2,105,021	3
	3,053				1,527		
5,793,011		4,206,989	8	2,896,506		2,103,494	6.10
	3,044				1,522		
5,796,056		4,203,944	9	2,898,028		2,101,972	1
	3,035				1,517		
5,799,090		4,200,910	330	2,899,545		2,100,455	2
	3,026				1,513		
5,802,116		4,197,884	1	2,901,058		2,098,942	3
	3,017				1,508		
5,805,133		4,194,867	2	2,902,566		2,097,434	6.11
	3,008				1,504		
5,808,140		4,191,860	3	2,904,070		2,095,930	1
	2,999				1,499		
5,811,139		4,188,861	4	2,905,569		2,094,431	2
	2,990				1,495		
5,814,128		4,185,872	335	2,907,064		2,092,936	3
	2,980				1,490		
5,817,109		4,182,891	6	2,908,554		2,091,446	7.0
	2,972				1,486		
5,820,081		4,179,919	7	2,910,040		2,089,960	1
	2,963				1,482		
5,823,044		4,176,956	8	2,911,522		2,088,478	2
	2,954				1,477		
5,825,998		4,174,002	9	2,912,999		2,087,001	3
	2,945				1,473		
5,828,943		4,171,057	340	2,914,472		2,085,528	7.1
	2,937				1,468		
5,831,880		4,168,120	1	2,915,940		2,084,060	1
	2,928				1,464		
5,834,809		4,165,191	2	2,917,404		2,082,596	2
	2,920				1,460		
5,837,728		4,162,272	3	2,918,864		2,081,136	3
	2,911				1,456		
5,840,639		4,159,361	4	2,920,320		2,079,680	7.2
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
7.2	5,840,639		4,159,361	4	2,920,320		2,079,680
		2,903				1,451	
1	5,843,542		4,156,458	345	2,921,771		2,078,229
		2,894				1,447	
2	5,846,437		4,153,563	6	2,923,218		2,076,782
		2,886				1,443	
3	5,849,323		4,150,677	7	2,924,661		2,075,339
		2,878				1,439	
7.3	5,852,200		4,147,800	8	2,926,100		2,073,900
		2,869				1,435	
1	5,855,070		4,144,930	9	2,927,535		2,072,465
		2,861				1,430	
2	5,857,931		4,142,069	350	2,928,965		2,071,035
		2,853				1,427	
3	5,860,784		4,139,216	1	2,930,392		2,069,608
		2,845				1,422	
7.4	5,863,629		4,136,371	2	2,931,814		2,068,186
		2,837				1,419	
1	5,866,466		4,133,534	3	2,933,233		2,066,767
		2,829				1,414	
2	5,869,295		4,130,705	4	2,934,647		2,065,353
		2,821				1,410	
3	5,872,116		4,127,884	355	2,936,058		2,063,942
		2,813				1,406	
7.5	5,874,929		4,125,071	6	2,937,464		2,062,536
		2,805				1,403	
1	5,877,734		4,122,266	7	2,938,887		2,061,133
		2,797				1,398	
2	5,880,531		4,119,469	8	2,940,265		2,059,735
		2,789				1,395	
3	5,883,320		4,116,680	9	2,941,660		2,058,340
		2,782				1,391	
7.6	5,886,102		4,113,898	360	2,943,051		2,056,949
		2,774				1,387	
1	5,888,876		4,111,124	1	2,944,438		2,055,562
		2,766				1,383	
2	5,891,642		4,108,358	2	2,945,821		2,054,179
		2,759				1,379	
3	5,894,401		4,105,599	3	2,947,200		2,052,800
		2,751				1,375	
7.7	5,897,152		4,102,848	4	2,948,576		2,051,424
		2,744				1,372	
1	5,899,895		4,100,105	365	2,949,948		2,050,052
		2,736				1,368	
2	5,902,631		4,097,369	6	2,951,316		2,048,684
		2,729				1,364	
3	5,905,360		4,094,640	7	2,952,680		2,047,320
		2,721				1,360	
7.8	5,908,081		4,091,919	8	2,954,040		2,045,960
		2,714				1,357	
1	5,910,794		4,089,206	9	2,955,397		2,044,603
		2,706				1,353	
2	5,913,501		4,086,499	370	2,956,750		2,043,250
		2,699				1,350	
3	5,916,200		4,083,800	1	2,958,100		2,041,900
		2,692				1,346	
7.9	5,918,892		4,081,108	2	2,959,446		2,040,554
		2,685				1,342	
1	5,921,576		4,078,424	3	2,960,788		2,039,212
		2,677				1,339	
2	5,924,254		4,075,746	4	2,962,127		2,037,873
		2,670				1,335	
3	5,926,924		4,073,076	375	2,963,462		2,036,538
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
5,926,924		4,073,076	375	2,963,462		2,036,538	3
	2,663				1,331		
5,929,587		4,070,413	6	2,964,793		2,035,207	7.10
	2,656				1,328		
5,932,243		4,067,757	7	2,966,121		2,033,879	1
	2,649				1,325		
5,934,892		4,065,108	8	2,967,446		2,032,554	2
	2,642				1,321		
5,937,534		4,062,466	9	2,968,767		2,031,233	3
	2,635				1,318		
5,940,169		4,059,831	380	2,970,085		2,029,915	7.11
	2,628				1,314		
5,942,797		4,057,203	1	2,971,399		2,028,601	1
	2,621				1,310		
5,945,418		4,054,582	2	2,972,709		2,027,291	2
	2,614				1,307		
5,948,033		4,051,967	3	2,974,016		2,025,984	3
	2,608				1,304		
5,950,640		4,049,360	4	2,975,320		2,024,680	8.0
	2,601				1,300		
5,953,241		4,046,759	385	2,976,621		2,023,379	1
	2,594				1,297		
5,955,835		4,044,165	6	2,977,918		2,022,082	2
	2,587				1,294		
5,958,422		4,041,578	7	2,979,211		2,020,789	3
	2,581				1,290		
5,961,003		4,038,997	8	2,980,502		2,019,498	8.1
	2,574				1,287		
5,963,577		4,036,423	9	2,981,789		2,018,211	1
	2,567				1,283		
5,966,145		4,033,855	390	2,983,072		2,016,928	2
	2,561				1,281		
5,968,705		4,031,295	1	2,984,353		2,015,647	3
	2,554				1,277		
5,971,260		4,028,740	2	2,985,630		2,014,370	8.2
	2,548				1,274		
5,973,807		4,026,193	3	2,986,904		2,013,096	1
	2,541				1,271		
5,976,349		4,023,651	4	2,988,174		2,011,826	2
	2,535				1,267		
5,978,884		4,021,116	395	2,989,442		2,010,558	3
	2,528				1,264		
5,981,412		4,018,588	6	2,990,706		2,009,294	8.3
	2,522				1,261		
5,983,934		4,016,066	7	2,991,967		2,008,033	1
	2,516				1,258		
5,986,450		4,013,550	8	2,993,225		2,006,775	2
	2,509				1,255		
5,988,959		4,011,041	9	2,994,480		2,005,520	3
	2,503				1,251		
5,991,462		4,008,538	400	2,995,731		2,004,269	8.4
	2,497				1,248		
5,993,959		4,006,041	1	2,996,980		2,003,020	1
	2,491				1,245		
5,996,450		4,003,550	2	2,998,225		2,001,775	2
	2,484				1,242		
5,998,934		4,001,066	3	2,999,467		2,000,533	3
	2,478				1,239		
6,001,413		3,998,587	4	3,000,706		1,999,294	8.5
	2,472				1,236		
6,003,885		3,996,115	405	3,001,942		1,998,058	1
	2,466				1,233		
6,006,351		3,993,649	6	3,003,175		1,996,825	2
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
2	6,006,351		3,993,649	6	3,003,175		1,996,825
		2,460				1,230	
3	6,008,811		3,991,189	7	3,004,405		1,995,595
		2,454				1,227	
8.6	6,011,265		3,988,735	8	3,005,632		1,994,368
		2,448				1,224	
1	6,013,713		3,986,287	9	3,006,856		1,993,144
		2,442				1,221	
2	6,016,155		3,983,845	410	3,008,077		1,991,923
		2,436				1,218	
3	6,018,591		3,981,409	1	3,009,295		1,990,705
		2,430				1,215	
8.7	6,021,021		3,978,979	2	3,010,511		1,989,489
		2,424				1,212	
1	6,023,445		3,976,555	3	3,011,723		1,988,277
		2,418				1,209	
2	6,025,864		3,974,136	4	3,012,932		1,987,068
		2,412				1,206	
3	6,028,276		3,971,724	415	3,014,138		1,985,862
		2,407				1,203	
8.8	6,030,683		3,969,317	6	3,015,342		1,984,658
		2,401				1,200	
1	6,033,084		3,966,916	7	3,016,542		1,983,458
		2,395				1,198	
2	6,035,479		3,964,521	8	3,017,740		1,982,260
		2,389				1,195	
3	6,037,869		3,962,131	9	3,018,934		1,981,066
		2,384				1,192	
8.9	6,040,252		3,959,748	420	3,020,126		1,979,874
		2,378				1,189	
1	6,042,631		3,957,369	1	3,021,315		1,978,685
		2,373				1,186	
2	6,045,003		3,954,997	2	3,022,502		1,977,498
		2,367				1,183	
3	6,047,370		3,952,630	3	3,023,685		1,976,315
		2,361				1,181	
8.10	6,049,731		3,950,269	4	3,024,866		1,975,134
		2,356				1,178	
1	6,052,087		3,947,913	425	3,026,043		1,973,957
		2,350				1,175	
2	6,054,437		3,945,563	6	3,027,219		1,972,781
		2,345				1,172	
3	6,056,782		3,943,218	7	3,028,391		1,971,609
		2,339				1,169	
8.11	6,059,121		3,940,879	8	3,029,560		1,970,440
		2,334				1,167	
1	6,061,455		3,938,545	9	3,003,727		1,969,273
		2,328				1,164	
2	6,063,783		3,936,217	430	3,031,891		1,968,109
		2,323				1,161	
3	6,066,106		3,933,894	1	3,033,053		1,966,947
		2,318				1,159	
9.0	6,068,423		3,931,577	2	3,034,212		1,965,788
		2,312				1,156	
1	6,070,735		3,929,265	3	3,035,368		1,964,632
		2,307				1,153	
2	6,073,042		3,926,958	4	3,036,521		1,963,479
		2,301				1,150	
3	6,075,344		3,924,656	435	3,037,672		1,962,328
		2,296				1,148	
9.1	6,077,640		3,922,360	6	3,038,820		1,961,180
		2,291				1,145	
1	6,079,931		3,920,069	7	3,039,965		1,960,035
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
6,079,931	2,286	3,920,069	7	3,039,965	1,143	1,960,035	1
6,082,217	2,280	3,917,783	8	3,041,108	1,140	1,958,892	2
6,084,497	2,275	3,915,503	9	3,042,249	1,138	1,957,751	3
6,086,772	2,270	3,913,228	440	3,043,386	1,135	1,956,614	9.2
6,089,043	2,265	3,910,957	1	3,044,521	1,132	1,955,479	1
6,091,308	2,260	3,908,692	2	3,045,654	1,130	1,954,346	2
6,093,567	2,255	3,906,433	3	3,046,784	1,127	1,953,216	3
6,095,822	2,250	3,904,178	4	3,047,911	1,125	1,952,089	9.3
6,098,072	2,245	3,901,928	445	3,049,036	1,122	1,950,964	1
6,100,317	2,240	3,899,683	6	3,050,158	1,120	1,949,842	2
6,102,556	2,235	3,897,444	7	3,051,278	1,117	1,948,722	3
6,104,791	2,230	3,895,209	8	3,052,395	1,115	1,947,605	9.4
6,107,021	2,225	3,892,979	9	3,053,510	1,112	1,946,490	1
6,109,245	2,220	3,890,755	450	3,054,623	1,110	1,945,377	2
6,111,465	2,215	3,888,535	1	3,055,733	1,107	1,944,267	3
6,113,680	2,210	3,886,320	2	3,056,840	1,105	1,943,160	9.5
6,115,890	2,205	3,884,110	3	3,057,945	1,103	1,942,055	1
6,118,095	2,200	3,881,905	4	3,059,047	1,100	1,940,953	2
6,120,295	2,195	3,879,705	455	3,060,148	1,098	1,939,852	3
6,122,491	2,191	3,877,509	6	3,061,245	1,095	1,938,755	9.6
6,124,681	2,186	3,875,319	7	3,062,341	1,093	1,937,659	1
6,126,867	2,181	3,873,133	8	3,063,433	1,091	1,936,567	2
6,129,048	2,176	3,870,952	9	3,064,524	1,088	1,935,476	3
6,131,224	2,172	3,868,776	460	3,065,612	1,086	1,934,388	9.7
6,133,396	2,167	3,866,604	1	3,066,698	1,083	1,933,302	1
6,135,563	2,162	3,864,437	2	3,067,781	1,081	1,932,219	2
6,137,725	2,157	3,862,275	3	3,068,862	1,079	1,931,138	3
6,139,882	2,153	3,860,118	4	3,069,941	1,076	1,930,059	9.8
6,142,035	2,148	3,857,965	465	3,071,018	1,074	1,928,982	1
6,144,183	2,144	3,855,817	6	3,072,092	1,072	1,927,908	2
6,146,327	2,139	3,853,673	7	3,073,163	1,070	1,926,837	3
6,148,466		3,851,534	8	3,074,233		1,925,767	9.9
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
1	6,150,600		3,849,400	9	3,075,300		1,924,700
		2,130				1,065	
2	6,152,730		3,847,270	470	3,076,365		1,923,635
		2,125				1,063	
3	6,154,856		3,845,144	1	3,077,428		1,922,572
		2,121				1,060	
9.10	6,156,977		3,843,023	2	3,078,488		1,921,512
		2,116				1,058	
1	6,159,093		3,840,907	3	3,079,547		1,920,453
		2,112				1,056	
2	6,161,205		3,838,795	4	3,080,603		1,919,397
		2,108				1,054	
3	6,163,313		3,836,687	475	3,081,656		1,918,344
		2,103				1,052	
9.11	6,165,416		3,834,584	6	3,082,708		1,917,292
		2,099				1,049	
1	6,167,514		3,832,486	7	3,083,757		1,916,243
		2,084				1,047	
2	6,169,608		3,830,392	8	3,084,804		1,915,196
		2,090				1,045	
3	6,171,698		3,828,302	9	3,085,849		1,914,151
		2,086				1,043	
10.0	6,173,784		3,826,216	480	3,086,892		1,913,108
		2,081				1,040	
1	6,175,865		3,824,135	1	3,087,932		1,912,068
		2,077				1,038	
2	6,177,942		3,822,058	2	3,088,971		1,911,029
		2,072				1,036	
3	6,180,014		3,819,986	3	3,090,007		1,909,993
		2,068				1,034	
10.1	6,182,083		3,817,917	4	3,091,041		1,908,959
		2,064				1,032	
1	6,184,147		3,815,853	485	3,092,073		1,907,927
		2,060				1,030	
2	6,186,206		3,813,794	6	3,093,103		1,906,897
		2,056				1,028	
3	6,188,262		3,811,738	7	3,094,131		1,905,869
		2,051				1,026	
10.2	6,190,313		3,809,687	8	3,095,157		1,904,843
		2,047				1,023	
1	6,192,360		3,807,640	9	3,096,180		1,903,820
		2,043				1,021	
2	6,194,403		3,805,597	490	3,097,202		1,902,798
		2,039				1,019	
3	6,196,442		3,803,558	1	3,098,221		1,901,779
		2,035				1,017	
10.3	6,198,476		3,801,524	2	3,099,238		1,900,762
		2,030				1,015	
1	6,200,507		3,799,493	3	3,100,253		1,899,747
		2,026				1,013	
2	6,202,533		3,797,467	4	3,101,267		1,898,733
		2,022				1,011	
3	6,204,555		3,795,445	495	3,102,278		1,897,722
		2,018				1,009	
10.4	6,206,574		3,793,426	6	3,103,287		1,896,713
		2,014				1,007	
1	6,208,588		3,791,412	7	3,104,294		1,895,706
		2,010				1,005	
2	6,210,598		3,789,402	8	3,105,299		1,894,701
		2,006				1,003	
3	6,212,604		3,787,396	9	3,106,302		1,893,698
		2,002				1,001	
10.5	6,214,606		3,785,394	500	3,107,303		1,892,697
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
6,216,604		3,783,396	1	3,108,302		1,891,698	1
	1,994				997		
6,218,598		3,781,402	2	3,109,299		1,890,701	2
	1,990				995		
6,220,588		3,779,412	3	3,110,294		1,889,706	3
	1,986				993		
6,222,574		3,777,426	4	3,111,287		1,888,713	10.6
	1,982				991		
6,224,556		3,775,444	505	3,112,278		1,887,722	1
	1,978				989		
6,226,534		3,773,466	6	3,113,267		1,886,733	2
	1,974				987		
6,228,509		3,771,491	7	3,114,254		1,885,746	3
	1,970				985		
6,230,479		3,769,521	8	3,115,240		1,884,760	10.7
	1,967				983		
6,232,446		3,767,554	9	3,116,223		1,883,777	1
	1,963				981		
6,234,408		3,765,592	510	3,117,204		1,882,796	2
	1,959				979		
6,236,367		3,763,633	1	3,118,184		1,881,816	3
	1,955				977		
6,238,322		3,761,678	2	3,119,161		1,880,839	10.8
	1,951				975		
6,240,274		3,759,726	3	3,120,137		1,879,863	1
	1,947				973		
6,242,221		3,757,779	4	3,121,110		1,878,890	2
	1,944				972		
6,244,165		3,755,835	515	3,122,082		1,877,918	3
	1,940				970		
6,246,104		3,753,896	6	3,123,052		1,876,948	10.9
	1,936				968		
6,248,041		3,751,959	7	3,124,020		1,875,980	1
	1,932				966		
6,249,973		3,750,027	8	3,124,986		1,875,014	2
	1,929				965		
6,251,902		3,748,098	9	3,125,951		1,874,049	3
	1,925				962		
6,253,826		3,746,174	520	3,126,913		1,873,087	10.10
	1,921				961		
6,255,748		3,744,252	1	3,127,874		1,872,126	1
	1,917				959		
6,257,665		3,742,335	2	3,128,833		1,871,167	2
	1,914				957		
6,259,579		3,740,421	3	3,129,790		1,870,210	3
	1,910				955		
6,261,489		3,738,511	4	3,130,745		1,869,255	10.11
	1,906				953		
6,263,396		3,736,604	525	3,131,698		1,868,302	1
	1,903				951		
6,265,299		3,734,701	6	3,132,649		1,867,351	2
	1,900				950		
6,267,198		3,732,802	7	3,133,599		1,866,401	3
	1,896				948		
6,269,094		3,730,906	8	3,134,547		1,865,453	11.0
	1,892				946		
6,270,986		3,729,014	9	3,135,493		1,864,507	1
	1,889				944		
6,272,875		3,727,125	530	3,136,437		1,863,563	2
	1,885				943		
6,274,760		3,725,240	1	3,137,380		1,862,620	3
	1,881				941		
6,276,641		3,723,359	2	3,138,321		1,861,679	11.1
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
11.1	6,276,641		3,723,359	2	3,138,321		1,861,679
		1,878				939	
1	6,278,519		3,721,481	3	3,139,260		1,860,740
		1,874				937	
2	6,280,393		3,719,607	4	3,140,197		1,859,803
		1,871				935	
3	6,282,264		3,717,736	535	3,141,132		1,858,868
		1,867				934	
11.2	6,284,132		3,715,868	6	3,142,066		1,857,934
		1,864				932	
1	6,285,996		3,714,004	7	3,142,998		1,857,002
		1,861				930	
2	6,287,856		3,712,144	8	3,143,928		1,856,072
		1,857				928	
3	6,289,713		3,710,287	9	3,144,857		1,855,143
		1,854				926	
11.3	6,291,567		3,708,433	540	3,145,783		1,854,217
		1,850				925	
1	6,293,417		3,706,583	1	3,146,708		1,853,292
		1,847				923	
2	6,295,264		3,704,736	2	3,147,632		1,852,368
		1,843				921	
3	6,297,107		3,702,893	3	3,148,553		1,851,447
		1,840				920	
11.4	6,298,947		3,701,053	4	3,149,473		1,850,527
		1,837				918	
1	6,300,783		3,699,217	545	3,150,392		1,849,608
		1,833				916	
2	6,302,617		3,697,383	6	3,151,308		1,848,692
		1,830				915	
3	6,304,446		3,695,554	7	3,152,223		1,847,777
		1,826				913	
11.5	6,306,273		3,693,727	8	3,153,136		1,846,864
		1,823				912	
1	6,308,096		3,691,904	9	3,154,048		1,845,952
		1,820				910	
2	6,309,916		3,690,084	550	3,154,958		1,845,042
		1,816				908	
3	6,311,732		3,688,268	1	3,155,866		1,844,134
		1,813				907	
11.6	6,313,546		3,686,454	2	3,156,773		1,843,227
		1,810				905	
1	6,315,356		3,684,644	3	3,157,678		1,842,322
		1,806				903	
2	6,317,162		3,682,838	4	3,158,581		1,841,419
		1,803				902	
3	6,318,966		3,681,034	555	3,159,483		1,840,517
		1,800				900	
11.7	6,320,766		3,679,234	6	3,160,383		1,839,617
		1,797				898	
1	6,322,563		3,677,437	7	3,161,281		1,838,719
		1,794				897	
2	6,324,357		3,675,643	8	3,162,178		1,837,822
		1,790				895	
3	6,326,147		3,673,853	9	3,163,074		1,836,926
		1,787				893	
11.8	6,327,934		3,672,066	560	3,163,967		1,836,033
		1,784				892	
1	6,329,719		3,670,281	1	3,164,859		1,835,141
		1,781				890	
2	6,331,499		3,668,501	2	3,165,750		1,834,250
		1,778				889	
3	6,333,277		3,666,723	3	3,166,639		1,833,361
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
6,333,277		3,666,723	3	3,166,639		1,833,361	3
	1,775				887		
6,335,052		3,664,948	4	3,167,526		1,832,474	11.9
	1,771				886		
6,336,823		3,663,177	565	3,168,412		1,831,588	1
	1,768				884		
6,338,592		3,661,408	6	3,169,296		1,830,704	2
	1,765				882		
6,340,357		3,659,643	7	3,170,178		1,829,822	3
	1,762				881		
6,342,119		3,657,881	8	3,171,060		1,828,940	11.10
	1,759				879		
6,343,878		3,656,122	9	3,171,939		1,828,061	1
	1,756				878		
6,345,634		3,654,366	570	3,172,817		1,827,183	2
	1,753				876		
6,347,387		3,652,613	1	3,173,693		1,826,307	3
	1,750				875		
6,349,137		3,650,863	2	3,174,568		1,825,432	11.11
	1,747				874		
6,350,883		3,649,117	3	3,175,442		1,824,558	1
	1,744				872		
6,352,627		3,647,373	4	3,176,314		1,823,686	2
	1,741				870		
6,354,368		3,645,632	575	3,177,184		1,822,816	3
	1,738				869		
6,356,105		3,643,895	6	3,178,053		1,821,947	12.0
	1,735				867		
6,357,840		3,642,160	7	3,178,920		1,821,080	1
	1,732				866		
6,359,571		3,640,429	8	3,179,786		1,820,214	2
	1,729				864		
6,361,300		3,638,700	9	3,180,650		1,819,350	3
	1,726				863		
6,363,026		3,636,974	580	3,181,513		1,818,487	12.1
	1,723				861		
6,364,748		3,635,252	1	3,182,374		1,817,626	1
	1,720				860		
6,366,468		3,633,532	2	3,183,234		1,816,766	2
	1,717				858		
6,368,185		3,631,815	3	3,184,092		1,815,908	3
	1,714				857		
6,369,899		3,630,101	4	3,184,949		1,815,051	12.2
	1,711				855		
6,371,609		3,628,391	585	3,185,805		1,814,195	1
	1,708				854		
6,373,317		3,626,683	6	3,186,659		1,813,341	2
	1,705				852		
6,375,022		3,624,978	7	3,187,511		1,812,489	3
	1,702				851		
6,376,725		3,623,275	8	3,188,362		1,811,638	12.3
	1,699				850		
6,378,424		3,621,576	9	3,189,212		1,810,788	1
	1,696				848		
6,380,120		3,619,880	590	3,190,060		1,809,940	2
	1,694				847		
6,381,814		3,618,186	1	3,190,907		1,809,093	3
	1,691				845		
6,383,504		3,616,496	2	3,191,752		1,808,248	12.4
	1,688				844		
6,385,192		3,614,808	3	3,192,596		1,807,404	1
	1,686				842		
6,386,877		3,613,123	4	3,193,438		1,806,562	2
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
2	6,386,877		3,613,123	4	3,193,438		1,806,562
		1,682				841	
3	6,388,559		3,611,441	595	3,194,280		1,805,720
		1,679				839	
12.5	6,390,238		3,609,762	6	3,195,119		1,804,881
		1,676				838	
1	6,391,915		3,608,085	7	3,195,957		1,804,043
		1,673				837	
2	6,393,588		3,606,412	8	3,196,794		1,803,206
		1,671				836	
3	6,395,259		3,604,741	9	3,197,630		1,802,370
		1,668				834	
12.6	6,396,927		3,603,073	600	3,198,464		1,801,536
		1,665				832	
1	6,398,593		3,601,407	1	3,199,296		1,800,704
		1,663				831	
2	6,400,255		3,599,745	2	3,200,128		1,799,872
		1,660				830	
3	6,401,915		3,598,085	3	3,200,957		1,799,043
		1,657				829	
12.7	6,403,572		3,596,428	4	3,201,786		1,798,214
		1,654				827	
1	6,405,226		3,594,774	605	3,202,613		1,797,387
		1,652				826	
2	6,406,878		3,593,122	6	3,203,439		1,796,561
		1,649				824	
3	6,408,526		3,591,474	7	3,204,263		1,795,737
		1,646				823	
12.8	6,410,172		3,589,828	8	3,205,086		1,794,914
		1,643				822	
1	6,411,816		3,588,184	9	3,205,908		1,794,092
		1,641				820	
2	6,413,457		3,586,543	610	3,206,728		1,793,072
		1,638				819	
3	6,415,095		3,584,905	1	3,207,547		1,792,453
		1,635				818	
12.9	6,416,730		3,583,270	2	3,208,365		1,791,635
		1,633				816	
1	6,418,363		3,581,637	3	3,209,181		1,790,819
		1,630				815	
2	6,419,993		3,580,007	4	3,209,996		1,790,004
		1,627				814	
3	6,421,620		3,578,380	615	3,210,810		1,789,190
		1,625				812	
12.10	6,423,245		3,576,755	6	3,211,622		1,788,378
		1,622				811	
1	6,424,867		3,575,133	7	3,212,433		1,787,567
		1,619				810	
2	6,426,486		3,573,514	8	3,213,243		1,786,757
		1,617				808	
3	6,428,103		3,571,897	9	3,214,051		1,785,949
		1,614				807	
12.11	6,429,717		3,570,283	620	3,214,859		1,785,141
		1,612				806	
1	6,431,329		3,568,671	1	3,215,664		1,784,336
		1,609				805	
2	6,432,938		3,567,062	2	3,216,469		1,783,531
		1,606				803	
3	6,434,544		3,565,456	3	3,217,272		1,782,728
		1,604				802	
13.0	6,436,148		3,563,852	4	3,218,074		1,781,926
		1,601				801	
1	6,437,749		3,562,251	625	3,218,875		1,781,125
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
6,437,749		3,562,251	625	3,218,875		1,781,125	1
	1,599				799		
6,439,348		3,560,652	6	3,219,674		1,780,326	2
	1,596				798		
6,440,944		3,559,056	7	3,220,472		1,779,528	3
	1,594				797		
6,442,538		3,557,462	8	3,221,269		1,778,731	13.1
	1,591				795		
6,444,129		3,555,871	9	3,222,064		1,777,936	1
	1,589				794		
6,445,717		3,554,283	630	3,222,859		1,777,141	2
	1,586				793		
6,447,303		3,552,697	1	3,223,652		1,776,348	3
	1,584				792		
6,448,887		3,551,113	2	3,224,443		1,775,557	13.2
	1,581				791		
6,450,468		3,549,532	3	3,225,234		1,774,766	1
	1,579				790		
6,452,047		3,547,953	4	3,226,023		1,773,977	2
	1,576				788		
6,453,623		3,546,377	635	3,226,811		1,773,189	3
	1,573				787		
6,455,196		3,544,804	6	3,227,598		1,772,402	13.3
	1,571				786		
6,456,767		3,543,233	7	3,228,384		1,771,616	1
	1,568				784		
6,458,336		3,541,664	8	3,229,168		1,770,832	2
	1,566				783		
6,459,902		3,540,098	9	3,229,951		1,770,049	3
	1,564				782		
6,461,466		3,538,534	640	3,230,733		1,769,267	13.4
	1,561				781		
6,463,027		3,536,973	1	3,231,514		1,768,486	1
	1,559				779		
6,464,586		3,535,414	2	3,232,293		1,767,707	2
	1,556				778		
6,466,142		3,533,858	3	3,233,071		1,766,929	3
	1,554				777		
6,467,696		3,532,304	4	3,233,848		1,766,152	13.5
	1,552				776		
6,469,248		3,530,752	645	3,234,624		1,765,376	1
	1,549				775		
6,470,797		3,529,203	6	3,235,399		1,764,601	2
	1,547				774		
6,472,344		3,527,656	7	3,236,172		1,763,828	3
	1,544				772		
6,473,888		3,526,112	8	3,236,944		1,763,056	13.6
	1,542				771		
6,475,430		3,524,570	9	3,237,715		1,762,285	1
	1,540				770		
6,476,970		3,523,030	650	3,238,485		1,761,515	2
	1,537				769		
6,478,507		3,521,493	1	3,239,254		1,760,746	3
	1,535				767		
6,480,042		3,519,958	2	3,240,021		1,759,979	13.7
	1,533				766		
6,481,575		3,518,425	3	3,240,787		1,759,213	1
	1,530				765		
6,483,105		3,516,895	4	3,241,552		1,758,448	2
	1,528				764		
6,484,633		3,515,367	655	3,242,316		1,757,684	3
	1,526				763		
6,486,158		3,513,842	6	3,243,079		1,756,921	13.8
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
13.8	6,486,158		3,513,842	6	3,243,079		1,756,921
		1,523				762	
1	6,487,682		3,512,318	7	3,243,841		1,756,159
		1,521				760	
2	6,489,203		3,510,797	8	3,244,601		1,755,399
		1,519				759	
3	6,490,721		3,509,279	9	3,245,361		1,754,639
		1,516				758	
13.9	6,492,237		3,507,763	660	3,246,119		1,753,881
		1,514				757	
1	6,493,751		3,506,249	1	3,246,876		1,753,124
		1,512				756	
2	6,495,263		3,504,737	2	3,247,632		1,752,368
		1,509				754	
3	6,496,773		3,503,227	3	3,248,386		1,751,614
		1,507				753	
13.10	6,498,280		3,501,720	4	3,249,140		1,750,860
		1,505				752	
1	6,499,785		3,500,215	665	3,249,892		1,750,108
		1,503				751	
2	6,501,287		3,498,713	6	3,250,644		1,749,356
		1,500				750	
3	6,502,788		3,497,212	7	3,251,394		1,748,606
		1,498				749	
13.11	6,504,286		3,495,714	8	3,252,143		1,747,857
		1,496				748	
1	6,505,782		3,494,218	9	3,252,891		1,741,109
		1,494				747	
2	6,507,275		3,492,725	670	3,253,638		1,746,362
		1,491				745	
3	6,508,767		3,491,233	1	3,254,383		1,745,617
		1,489				744	
14.0	6,510,256		3,489,744	2	3,255,128		1,744,872
		1,487				743	
1	6,511,743		3,488,257	3	3,255,871		1,744,129
		1,485				742	
2	6,513,228		3,486,772	4	3,256,614		1,743,386
		1,482				741	
3	6,514,710		3,485,290	675	3,257,355		1,742,645
		1,480				740	
14.1	6,516,191		3,483,809	6	3,258,095		1,741,905
		1,478				739	
1	6,517,669		3,482,331	7	3,258,834		1,741,166
		1,476				738	
2	6,519,145		3,480,855	8	3,259,572		1,740,428
		1,474				737	
3	6,520,619		3,479,381	9	3,260,309		1,739,691
		1,472				736	
14.2	6,522,090		3,477,910	680	3,261,045		1,738,955
		1,470				735	
1	6,523,560		3,476,440	1	3,261,780		1,738,220
		1,467				734	
2	6,525,027		3,474,973	2	3,262,514		1,737,486
		1,465				733	
3	6,526,492		3,473,508	3	3,263,246		1,736,754
		1,463				732	
14.3	6,527,955		3,472,045	4	3,263,978		1,736,022
		1,461				731	
1	6,529,416		3,470,584	685	3,264,708		1,735,292
		1,459				730	
2	6,530,875		3,469,125	6	3,265,438		1,734,562
		1,457				729	
3	6,532,332		3,467,668	7	3,266,166		1,733,834
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
6,532,332		3,467,668	7	3,266,166		1,733,834	3
	1,454				727		
6,533,786		3,466,214	8	3,266,893		1,733,107	14.4
	1,452				726		
6,535,239		3,464,761	9	3,267,619		1,732,381	1
	1,450				725		
6,536,689		3,463,311	690	3,268,345		1,731,655	2
	1,448				724		
6,538,137		3,461,863	1	3,269,069		1,730,931	3
	1,446				723		
6,539,584		3,460,416	2	3,269,792		1,730,208	14.5
	1,444				722		
6,541,028		3,458,972	3	3,270,514		1,729,486	1
	1,442				721		
6,542,470		3,457,530	4	3,271,235		1,728,765	2
	1,440				720		
6,543,909		3,456,091	695	3,271,955		1,728,045	3
	1,438				719		
6,545,347		3,454,653	6	3,272,674		1,727,326	14.6
	1,436				718		
6,546,783		3,453,217	7	3,273,391		1,726,609	1
	1,434				717		
6,548,217		3,451,783	8	3,274,108		1,725,892	2
	1,432				716		
6,549,648		3,450,352	9	3,274,824		1,725,176	3
	1,430				715		
6,551,078		3,448,922	700	3,275,539		1,724,461	14.7
	1,427				713		
6,552,505		3,447,495	1	3,276,253		1,723,747	1
	1,425				713		
6,553,931		3,446,069	2	3,276,965		1,723,035	2
	1,423				712		
6,555,354		3,444,646	3	3,277,677		1,722,323	3
	1,421				711		
6,556,776		3,443,224	4	3,278,388		1,721,612	14.8
	1,419				710		
6,558,195		3,441,805	705	3,279,098		1,720,902	1
	1,417				709		
6,559,613		3,440,387	6	3,279,806		1,720,194	2
	1,415				708		
6,561,028		3,438,972	7	3,280,514		1,719,486	3
	1,413				707		
6,562,442		3,437,558	8	3,281,221		1,718,779	14.9
	1,411				706		
6,563,853		3,436,147	9	3,281,927		1,718,073	1
	1,409				705		
6,565,263		3,434,737	710	3,282,631		1,717,369	2
	1,407				704		
6,566,670		3,433,330	1	3,283,335		1,716,665	3
	1,405				703		
6,568,075		3,431,925	2	3,284,038		1,715,962	14.10
	1,403				702		
6,569,479		3,430,521	3	3,284,739		1,715,261	1
	1,402				701		
6,570,881		3,429,119	4	3,285,440		1,714,560	2
	1,400				700		
6,572,280		3,427,720	715	3,286,140		1,713,860	3
	1,398				699		
6,573,678		3,426,322	6	3,286,839		1,713,161	14.11
	1,396				698		
6,575,073		3,424,927	7	3,287,537		1,712,463	1
	1,394				697		
6,576,467		3,423,533	8	3,288,234		1,711,766	2
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
		1,392				696	
3	6,577,859	1,390	3,422,141	9	3,288,929	695	1,711,071
15.0	6,579,249	1,388	3,420,751	720	3,289,624	694	1,710,376
1	6,580,637	1,386	3,419,363	1	3,290,318	693	1,709,682
2	6,582,023	1,384	3,417,977	2	3,291,011	692	1,708,989
3	6,583,407	1,382	3,416,593	3	3,291,703	691	1,708,297
15.1	6,584,789	1,380	3,415,211	4	3,292,394	690	1,707,606
1	6,586,169	1,378	3,413,831	725	3,293,085	689	1,706,915
2	6,587,548	1,376	3,412,452	6	3,293,774	688	1,706,226
3	6,588,924	1,374	3,411,076	7	3,294,462	687	1,705,538
15.2	6,590,299	1,373	3,409,701	8	3,295,149	686	1,704,851
1	6,591,671	1,371	3,408,329	9	3,295,836	685	1,704,164
2	6,593,042	1,369	3,406,958	730	3,296,521	684	1,703,479
3	6,594,411	1,367	3,405,589	1	3,297,205	684	1,702,795
15.3	6,595,778	1,365	3,404,222	2	3,297,889	683	1,702,111
1	6,597,143	1,363	3,402,857	3	3,298,572	682	1,701,428
2	6,598,507	1,361	3,401,493	4	3,299,253	681	1,700,747
3	6,599,868	1,360	3,400,132	735	3,299,934	680	1,700,066
15.4	6,601,228	1,358	3,398,772	6	3,300,614	679	1,699,386
1	6,602,585	1,356	3,397,415	7	3,301,293	678	1,698,707
2	6,603,941	1,354	3,396,059	8	3,301,971	677	1,698,029
3	6,605,295	1,352	3,394,705	9	3,302,648	676	1,697,352
15.5	6,606,648	1,350	3,393,352	740	3,303,324	675	1,696,676
1	6,607,998	1,349	3,392,002	1	3,303,999	674	1,696,001
2	6,609,347	1,347	3,390,653	2	3,304,673	673	1,695,327
3	6,610,694	1,345	3,389,306	3	3,305,347	672	1,694,653
15.6	6,612,029	1,343	3,387,961	4	3,306,019	671	1,693,981
1	6,613,382	1,341	3,386,618	745	3,306,691	671	1,693,309
2	6,614,723	1,340	3,385,277	6	3,307,362	670	1,692,638
3	6,616,063	1,338	3,383,937	7	3,308,031	669	1,691,969
15.7	6,617,401	1,336	3,382,599	8	3,308,700	668	1,691,300
1	6,618,737	1,334	3,381,263	9	3,309,368	667	1,690,632
2	6,620,071		3,379,929	750	3,310,035		1,689,965
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	Differences of Logarithms.	Arithmetical Complements of Logarithms to 10,000,000	Numbers.	Halves of Logarithms.	Halves of Differences of Logarithms.	Halves of Arithmetical Complements.	
					666		
6,621,403	1,332	3,378,597	1	3,310,702		1,689,298	3
6,622,734	1,331	3,377,266	2	3,311,367	665	1,688,633	15.8
6,624,063	1,329	3,375,937	3	3,312,031	664	1,687,969	1
	1,327				664		
6,625,390	1,325	3,374,610	4	3,312,695		1,687,305	2
6,626,715	1,324	3,373,285	755	3,313,358	663	1,686,642	3
6,628,039		3,371,961	6	3,314,019	662	1,685,981	15.9
	1,322				661		
6,629,361	1,320	3,370,639	7	3,314,680		1,685,320	1
6,630,681	1,318	3,369,319	8	3,315,340	660	1,684,660	2
6,631,999		3,368,001	9	3,316,000	659	1,684,000	3
	1,317				658		
6,633,316	1,315	3,366,684	760	3,316,658		1,683,342	15.10
6,634,631	1,313	3,365,369	1	3,317,315	657	1,682,685	1
6,635,944		3,364,056	2	3,317,972	657	1,682,028	2
	1,312				656		
6,637,256	1,310	3,362,744	3	3,318,628		1,681,372	3
6,638,565	1,308	3,361,435	4	3,319,283	655	1,680,717	15.11
6,639,873		3,360,125	765	3,319,937	654	1,680,063	1
	1,306				653		
6,641,180	1,305	3,358,820	6	3,320,590		1,679,410	2
6,642,484	1,303	3,357,516	7	3,321,242	652	1,678,758	3
6,643,787		3,356,213	8	3,321,894	651	1,678,106	16.0
	1,301				651		
6,645,088	1,300	3,354,912	9	3,322,544		1,677,456	1
6,646,388	1,298	3,353,612	770	3,323,194	650	1,676,806	2
6,647,686		3,352,314	1	3,323,843	649	1,676,157	3
	1,296				648		
6,648,982	1,294	3,351,018	2	3,324,491		1,675,509	16.1
6,650,277	1,293	3,349,723	3	3,325,138	647	1,674,862	1
6,651,569		3,348,431	4	3,325,785	646	1,674,215	2
	1,291				646		
6,652,861	1,289	3,347,139	775	3,326,430		1,673,570	3
6,654,150	1,288	3,345,850	6	3,327,075	645	1,672,925	16.2
6,655,438		3,344,562	7	3,327,719	644	1,672,281	1
	1,286				643		
6,656,724	1,284	3,343,276	8	3,328,362		1,671,638	2
6,658,009	1,282	3,341,991	9	3,329,004	642	1,670,996	3
6,659,291		3,340,709	780	3,329,646	641	1,670,354	16.3
	1,281				641		
6,660,573	1,280	3,339,427	1	3,330,286		1,669,714	1
6,661,852		3,338,148	2	3,330,926	640	1,669,074	2
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms,	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
2	6,661,852		3,338,148	2	3,330,926		1,669,074
		1,278				639	
3	6,663,130		3,336,870	3	3,331,565		1,668,435
		1,276				638	
16.4	6,664,407		3,335.593	4	3,332,203		1,667,797
		1,275				637	
1	6,665,681		3,334,319	785	3,332,841		1,667,159
		1,273				637	
2	6,666,954		3,333,046	6	3,333,477		1,666,523
		1,271				636	
3	6,668,226		3,331,774	7	3,334,113		1,665,887
		1,270				635	
16.5	6,669,496		3,330,504	8	3,334,748		1,665,252
		1,268				634	
1	6,670,764		3,329,236	9	3,335,382		1,664,618
		1,267				633	
2	6,672,030		3,327,970	790	3,336,015		1,663,985
		1,265				633	
3	6,673,295		3,326,705	1	3,336,648		1,663,352
		1,263				632	
16.6	6,674,559		3,325,441	2	3,337,279		1,662,721
		1,262				631	
1	6,675,821		3,324,179	3	3,337,910		1,662,090
		1,260				630	
2	6,677,081		3,322,919	4	3,338,540		1,661,460
		1,259				629	
3	6,678,340		3,321,660	795	3,339,170		1,660,830
		1,257				628	
16.7	6,679,597		3,320,403	6	3,339,798		1,660,202
		1,255				628	
1	6,680,852		3,319,148	7	3,340,426		1,659,574
		1,254				627	
2	6,682,106		3,317,894	8	3,341,053		1,658,947
		1,252				626	
3	6,683,358		3,316,642	9	3,341,679		1,658,321
		1,251				625	
16.8	6,684,609		3,315,391	800	3,342,305		1,657,695
		1,249				624	
1	6,685,858		3,314,142	1	3,342,929		1,657,071
		1,248				624	
2	6,687,106		3,312,894	2	3,343,553		1,656,447
		1,246				623	
3	6,688,352		3,311,648	3	3,344,176		1,655,824
		1,244				622	
16.9	6,689,597		3,310,403	4	3,344,798		1,655,202
		1,243				621	
1	6,690,840		3,309,160	805	3,345,420		1,654,580
		1,241				621	
2	6,692,081		3,307,919	6	3,346,041		1,653,958
		1,240				620	
3	6,693,321		3,306,679	7	3,346,661		1,653,339
		1,238				619	
16.10	6,694,560		3,305,440	8	3,347,280		1,652,720
		1,237				618	
1	6,695,796		3,304,204	9	3,347,898		1,652,102
		1,235				618	
2	6,697,032		3,302,968	810	3,348,516		1,651,484
		1,234				617	
3	6,698,266		3,301,734	1	3,349,133		1,650,867
		1,232				616	
16.11	6,699,498		3,300,502	2	3,349,749	615	1,650,251
		1,231					
1	6,700,729		3,299,271	3	3,350,364		1,649,636
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
6,700,729		3,299,271	3	3,350,364		1,649,636	1
	1,229				615		
6,701,958		3,298,042	4	3,350,979		1,649,021	2
	1,228				614		
6,703,186		3,296,814	815	3,351,593		1,648,407	3
	1,226				613		
6,704,412		3,295,588	6	3,352,206		1,647,794	17.0
	1,225				612		
6,705,637		3,294,363	7	3,352,818		1,647,182	1
	1,223				612		
6,706,860		3,293,140	8	3,353,430		1,646,570	2
	1,222				611		
6,708,082		3,291,918	9	3,354,041		1,645,959	3
	1,220				610		
6,709,302		3,290,698	820	3,354,651		1,645,349	17.1
	1,219				609		
6,710,521		3,289,479	1	3,355,260		1,644,740	1
	1,217				609		
6,711,738		3,288,262	2	3,355,869		1,644,131	2
	1,216				608		
6,712,954		3,287,046	3	3,356,477		1,643,523	3
	1,214				607		
6,714,168		3,285,832	4	3,357,084		1,642,916	17.2
	1,213				606		
6,715,381		3,284,619	825	3,357,690		1,642,310	1
	1,211				606		
6,716,592		3,283,408	6	3,358,296		1,641,704	2
	1,210				605		
6,717,802		3,282,198	7	3,358,901		1,641,099	3
	1,208				604		
6,719,011		3,280,989	8	3,359,505		1,640,495	17.3
	1,207				603		
6,720,218		3,279,782	9	3,360,109		1,639,891	1
	1,206				603		
6,721,423		3,278,577	830	3,360,712		1,639,288	2
	1,204				602		
6,722,627		3,277,373	1	3,361,314		1,638,686	3
	1,203				601		
6,723,830		3,276,170	2	3,361,915		1,638,085	17.4
	1,201				601		
6,725,031		3,274,969	3	3,362,516		1,637,484	1
	1,200				600		
6,726,231		3,273,769	4	3,363,115		1,636,885	2
	1,198				599		
6,727,429		3,272,571	835	3,363,715		1,636,285	3
	1,197				598		
6,728,626		3,271,374	6	3,364,313		1,635,687	17.5
	1,195				598		
6,729,822		3,270,178	7	3,364,911		1,635,089	1
	1,194				597		
6,731,016		3,268,984	8	3,365,508		1,634,492	2
	1,193				596		
6,732,208		3,267,792	9	3,366,104		1,633,896	3
	1,191				596		
6,733,399		3,266,601	840	3,366,700		1,633,300	17.6
	1,190				595		
6,734,589		3,265,411	1	3,367,295		1,632,705	1
	1,188				594		
6,735,778		3,264,222	2	3,367,889		1,632,111	2
	1,187				593		
6,736,964		3,263,036	3	3,368,482		1,631,518	3
	1,186				593		
6,738,150		3,261,850	4	3,369,075		1,630,925	17.7
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
17.7	6,738,150		3,261,850	4	3,369,075		1,630,925
		1,184				592	
1	6,739,334		3,260,666	845	3,369,667		1,630,333
		1,183				591	
2	6,740,517		3,259,483	6	3,370,258		1,629,742
		1,182				591	
3	6,741,698		3,258,302	7	3,370,849		1,629,151
		1,180				590	
17.8	6,742,878		3,257,122	8	3,371,439		1,628,561
		1,179				589	
1	6,744,057		3,255,943	9	3,372,028		1,627,972
		1,177				589	
2	6,745,234		3,254,766	850	3,372,617		1,627,383
		1,176				588	
3	6,746,410		3,253,590	1	3,373,205		1,626,795
		1,174				587	
17.9	6,747,584		3,252,416	2	3,373,792		1,626,208
		1,173				587	
1	6,748,757		3,251,243	3	3,374,379		1,625,621
		1,172				586	
2	6,749,929		3,250,071	4	3,374,964		1,625,036
		1,170				585	
3	6,751,099		3,248,901	855	3,375,549		1,624,451
		1,169				584	
17.10	6,752,268		3,247,732	6	3,376,134		1,623,866
		1,167				584	
1	6,753,435		3,246,565	7	3,376,718		1,623,282
		1,166				583	
2	6,754,602		3,245,398	8	3,377,301		1,622,699
		1,165				582	
3	6,755,766		3,244,234	9	3,377,883		1,622,117
		1,163				582	
17.11	6,756,930		3,243,070	860	3,378,465		1,621,535
		1,162				581	
1	6,758,092		3,241,908	1	3,379,046		1,620,954
		1,161				580	
2	6,759,253		3,240,747	2	3,379,626		1,620,374
		1,159				580	
3	6,760,412		3,239,588	3	3,380,206		1,619,794
		1,158				579	
18.0	6,761,570		3,238,430	4	3,380,785		1,619,215
		1,156				578	
1	6,762,727		3,237,273	865	3,381,363		1,618,637
		1,155				578	
2	6,763,882		3,236,118	6	3,381,941		1,618,059
		1,154				577	
3	6,765,036		3,234,964	7	3,382,518		1,617,482
		1,153				576	
18.1	6,766,189		3,233,811	8	3,383,095		1,616,905
		1,151				576	
1	6,767,341		3,232,659	9	3,383,670		1,616,330
		1,150				575	
2	6,768,491		3,231,509	870	3,384,245		1,615,755
		1,149				574	
3	6,769,639		3,230,361	1	3,384,820		1,615,180
		1,147				574	
18.2	6,770,787		3,229,213	2	3,385,393		1,614,607
		1,146				573	
1	6,771,933		3,228,067	3	3,385,967		1,614,033
		1,145				572	
2	6,773,078		3,226,922	4	3,386,539		1,613,461
		1,143				572	
3	6,774,221		3,225,779	875	3,387,111		1,612,889
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,686,931		2,302,584		3,133,069

Logarithms.	Differences of Logarithms.	Arithmetical Complements of Logarithms to 10,000,000.	Numbers.	Halves of Logarithms.	Halves of Differences of Logarithms.	Halves of Arithmetical Complements.	
6,774,221		3,225,779	875	3,387,111		1,612,889	3
	1,142				571		
6,775,364		3,224,636	6	3,387,682		1,612,318	18.3
	1,141				570		
6,776,504		3,223,496	7	3,388,252		1,611,748	1
	1,140				570		
6,777,644		3,222,356	8	3,388,822		1,611,178	2
	1,138				569		
6,778,782		3,221,218	9	3,389,391		1,610,609	3
	1,137				569		
6,779,919		3,220,081	880	3,389,960		1,610,040	18.4
	1,136				568		
6,781,055		3,218,945	1	3,390,528		1,609,472	1
	1,134				567		
6,782,190		3,217,810	2	3,391,095		1,608,905	2
	1,133				567		
6,783,323		3,216,677	3	3,391,661		1,608,339	3
	1,132				566		
6,784,455		3,215,545	4	3,392,227		1,607,773	18.5
	1,131				565		
6,785,585		3,214,415	885	3,392,793		1,607,207	1
	1,129				565		
6,786,714		3,213,286	6	3,393,357		1,606,643	2
	1,128				564		
6,787,842		3,212,158	7	3,393,921		1,606,079	3
	1,127				563		
6,788,969		3,211,031	8	3,394,485		1,605,515	18.6
	1,126				563		
6,790,095		3,209,905	9	3,395,047		1,604,953	1
	1,124				562		
6,791,219		3,208,781	890	3,395,609		1,604,391	2
	1,123				561		
6,792,342		3,207,658	1	3,396,171		1,603,829	3
	1,122				561		
6,793,464		3,206,536	2	3,396,732		1,603,268	18.7
	1,120				560		
6,794,584		3,205,416	3	3,397,292		1,602,708	1
	1,119				560		
6,795,703		3,204,297	4	3,397,852		1,602,148	2
	1,118				559		
6,796,821		3,203,179	895	3,398,411		1,601,589	3
	1,117				558		
6,797,938		3,202,062	6	3,398,969		1,601,031	18.8
	1,115				558		
6,799,053		3,200,947	7	3,399,527		1,600,473	1
	1,114				557		
6,800,168		3,199,832	8	3,400,084		1,599,916	2
	1,113				556		
6,801,280		3,198,720	9	3,400,640		1,599,360	3
	1,112				556		
6,802,392		3,197,608	900	3,401,196		1,598,804	18.9
	1,111				555		
6,803,503		3,196,497	1	3,401,751		1,598,249	1
	1,109				555		
6,804,612		3,195,388	2	3,402,306		1,597,694	2
	1,108				554		
6,805,720		3,194,280	3	3,402,860		1,597,140	3
	1,107				553		
6,806,827		3,193,173	4	3,403,413		1,596,587	18.10
	1,106				553		
6,807,932		3,192,068	905	3,403,966		1,596,034	1
	1,104				552		
6,809,037		3,190,963	6	3,404,518		1,595,482	2
6,907,753		3,092,247		4,605,168		5,394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

	Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
2	6,809,037	1,103	3,190,963	6	3,404,518	552	1,595,482
3	6,810,140	1,102	3,189,860	7	3,405,070	551	1,594,930
18.11	6,811,242	1,101	3,188,758	8	3,405,621	550	1,594,379
1	6,812,343	1,099	3,187,657	9	3,406,171	550	1,593,829
2	6,813,442	1,098	3,186,558	910	3,406,721	549	1,593,279
3	6,814,540	1,097	3,185,460	1	3,407,270	549	1,592,730
19.0	6,815,637	1,096	3,184,363	2	3,407,819	548	1,592,181
1	6,816,733	1,095	3,183,267	3	3,408,367	547	1,591,633
2	6,817,828	1,094	3,182,172	4	3,408,914	547	1,591,086
3	6,818,922	1,092	3,181,078	915	3,409,461	546	1,590,539
19.1	6,820,014	1,091	3,179,986	6	3,410,007	546	1,589,993
1	6,821,105	1,090	3,178,895	7	3,410,552	545	1,589,448
2	6,822,195	1,089	3,177,805	8	3,411,097	544	1,588,903
3	6,823,284	1,088	3,176,716	9	3,411,642	544	1,588,358
19.2	6,824,371	1,086	3,175,629	920	3,412,186	543	1,587,814
1	6,825,457	1,085	3,174,543	1	3,412,729	543	1,587,271
2	6,826,543	1,084	3,173,457	2	3,413,271	542	1,586,729
3	6,827,627	1,083	3,172,373	3	3,413,813	541	1,586,187
19.3	6,828,710	1,082	3,171,290	4	3,414,355	541	1,585,645
1	6,829,791	1,081	3,170,209	925	3,414,896	540	1,585,104
2	6,830,872	1,079	3,169,128	6	3,415,436	540	1,584,564
3	6,831,951	1,078	3,168,049	7	3,415,976	539	1,584,024
19.4	6,833,029	1,077	3,166,971	8	3,416,515	539	1,583,485
1	6,834,106	1,076	3,165,894	9	3,417,053	538	1,582,947
2	6,835,182	1,075	3,164,818	930	3,417,591	537	1,582,409
3	6,836,257	1,074	3,163,743	1	3,418,128	537	1,581,872
19.5	6,837,330	1,072	3,162,670	2	3,418,665	536	1,581,335
1	6,838,403	1,071	3,161,597	3	3,419,201	536	1,580,799
2	6,839,474	1,070	3,160,526	4	3,419,737	535	1,580,263
3	6,840,544	1,069	3,159,456	935	3,420,272	534	1,579,728
19.6	6,841,613	1,068	3,158,387	6	3,420,806	534	1,579,194
1	6,842,681		3,157,319	7	3,421,340		1,578,660
	6,907,753		3,092,247		4,605,168		5,394,832
	3,453,876		6,866,931		2,302,584		3,133,069

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to* 10,000,000.	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*	
6,842,681		3,157,319	7	3,421,340		1,578,660	1
	1,067				533		
6,843,747		3,156,253	8	3,421,874		1,578,126	2
	1,066				533		
6,844,813		3,155,187	9	3,422,406		1,577,594	3
	1,064				532		
6,845,877		3,154,123	940	3,422,939		1,577,061	19.7
	1,063				532		
6,846,941		3,153,059	1	3,423,470		1,576,530	1
	1,062				531		
6,848,003		3,151,997	2	3,424,001		1,575,999	2
	1,061				531		
6,849,064		3,150,936	3	3,424,532		1,575,468	3
	1,060				530		
6,850,124		3,149,876	4	3,425,062		1,574,938	19.8
	1,059				529		
6,851,182		3,148,818	945	3,425,591		1,574,409	1
	1,058				529		
6,852,240		3,147,760	6	3,426,120		1,573,880	2
	1,057				528		
6,853,297		3,146,703	7	3,426,648		1,573,352	3
	1,055				528		
6,854,352		3,145,648	8	3,427,176		1,572,824	19.9
	1,054				527		
6,855,406		3,144,594	9	3,427,703		1,572,297	1
	1,053				527		
6,856,459		3,143,541	950	3,428,230		1,571,770	2
	1,052				526		
6,857,512		3,142,488	1	3,428,756		1,571,244	3
	1,051				526		
6,858,562		3,141,438	2	3,429,281		1,570,719	19.10
	1,050				525		
6,859,612		3,140,388	3	3,429,806		1,570,194	1
	1,049				524		
6,860,661		3,139,339	4	3,430,331		1,569,669	2
	1,048				524		
6,861,709		3,138,291	955	3,430,854		1,569,146	3
	1,047				523		
6,862,755		3,137,245	6	3,431,378		1,568,622	19.11
	1,045				523		
6,863,801		3,136,199	7	3,431,900		1,568,100	1
	1,044				522		
6,864,845		3,135,155	8	3,432,423		1,567,577	2
	1,043				522		
6,865,889		3,134,111	9	3,432,944		1,567,056	3
	1,042				521		
6,866,931		3,133,069	960	3,433,465		1,566,535	20.0
	1,041				521		
6,867,972		3,132,028	1	3,433,986		1,566,014	
	1,040				520		
6,869,012		3,130,988	2	3,434,506		1,565,494	
	1,039				520		
6,870,051		3,129,949	3	3,435,025		1,564,975	
	1,038				519		
6,871,089		3,128,911	4	3,435,544		1,564,456	
	1,037				518		
6,872,126		3,127,874	965	3,436,063		1,563,937	
	1,036				518		
6,873,161		3,126,839	6	3,436,581		1,563,419	
	1,035				517		
6,874,196		3,125,804	7	3,437,098		1,562,902	
	1,034				517		
6,875,230		3,124,770	8	3,437,615		1,562,385	
6,907,753		3,092,247		4,605,168		5 394,832	
3,453,876		6,866,931		2,302,584		3,133,069	

Logarithms.	*Differences of Logarithms.*	*Arithmetical Complements of Logarithms to 10,000,000.*	*Numbers.*	*Halves of Logarithms.*	*Halves of Differences of Logarithms.*	*Halves of Arithmetical Complements.*
6,876,262		3,123,738	9	3,438,131		1,561,869
	1,031				516	
6,877,294		3,122,706	970	3,438,647		1,561,353
	1,030				515	
6,878,324		3,121,676	1	3,439,162		1,560,838
	1,029				515	
6,879,353		3,120,647	2	3,439,677		1,560,323
	1,028				514	
6,880,382		3,119,618	3	3,440,191		1,559,809
	1,027				514	
6,881,409		3,118,591	4	3,440,704		1,559,296
	1,026				513	
6,882,435		3,117,565	975	3,441,217		1,558,783
	1,025				513	
6,883,460		3,116,540	6	3,441,730		1,558,270
	1,024				512	
6,884,484		3,115,516	7	3,442,242		1,557,758
	1,023				511	
6,885,507		3,114,493	8	3,442,754		1,557,246
	1,022				511	
6,886,529		3,113,471	9	3,443,265		1,556,735
	1,021				510	
6,887,550		3,112,450	980	3,443,775		1,556,225
	1,020				510	
6,888,570		3,111,430	1	3,444,285		1,555,715
	1,019				509	
6,889,589		3,110,411	2	3,444,794		1,555,206
	1,018				509	
6,890,607		3,109,393	3	3,445,303		1,554,697
	1,017				508	
6,891,623		3,108,377	4	3,445,812		1,554,188
	1,016				508	
6,892,639		3,107,361	985	3,446,320		1,553,680
	1,015				507	
6,893,654		3,106,346	6	3,446,827		1,553,173
	1,014				507	
6,894,667		3,105,333	7	3,447,334		1,552,666
	1,013				506	
6,895,680		3,104,320	8	3,447,840		1,552,160
	1,012				506	
6,896,692		3,103,308	9	3,448,346		1,551,654
	1,011				505	
6,897,702		3,102,298	990	3,448,851		1,551,149
	1,010				505	
6,898,712		3,101,288	1	3,449,356		1,550,644
	1,009				504	
6,899,721		3,100,279	2	3,449,860		1,550,140
	1,008				504	
6,900,728		3,099,272	3	3,450,364		1,549,636
	1,007				503	
6,901,735		3,098,265	4	3,450,867		1,549,133
	1,006				503	
6,902,740		3,097,260	995	3,451,370		1,548,630
	1,005				502	
6,903,745		3,096,255	6	3,451,872		1,548,128
	1,004				502	
6,904,748		3,095,252	7	3,452,374		1,547,626
	1,003				501	
6,905,751		3,094,249	8	3,452,875		1,547,125
	1,002				501	
6,906,752		3,093,248	9	3,453,376		1,546,624
	1,001				500	
6,907,753		3,092,247	1000	3,453,876		1,546,124
6,907,753		3,092,247		4,605,168		5,394,832
3,453,876		6,866,931		2,302,584		3,133,069

A

COMPUTATION

OF THE

LENGTH OF THE TANGENT OF A CIRCULAR ARCH OF ONE MINUTE OF A DEGREE,

BY *MR. RAPHSON'S* METHOD OF RESOLVING HIGH EQUATIONS BY APPROXIMATION;

Derived from the Tangent of an Arch of 45 *Degrees (which is equal to the Radius of the Circle,) by Means of one Quinquisection, two Trisections, a second Quinquisection, a third Trisection, and two Bisections;*

TOGETHER WITH

A SUBSEQUENT COMPUTATION OF THE LENGTH OF THE SINE OF THE SAID ARCH OF ONE MINUTE,

DERIVED FROM THE LENGTH OF ITS TANGENT BY THE RULES OF PLANE TRIGONOMETRY.

PERFORMED IN THE YEAR 1805,

BY DR. ANDREW MACKAY, L. L. D.

LATE OF ABERDEEN IN SCOTLAND, A FELLOW OF THE ROYAL SOCIETY OF EDINBURGH.

A

COMPUTATION

OF THE

Length of the Tangent of one Minute of a Degree,

DERIVED FROM

THE TANGENT OF AN ARCH OF 45 DEGREES.

BY DR. ANDREW MACKAY, L. L. D.

Article 1. IN one of the preceding Tracts of the preſent volume, in pages 451, 452, 453, &c, - - - 474, I have exhibited a curious and valuable computation of the length of the Sine of a circular Arch of one Minute of a Degree, performed by the learned Dr. Charles Hutton (Profeſſor of Mathematicks in the Royal Military Academy at Woolwich in Kent,) by divers ſections of a circular arch of 60 Degrees, the chord of which is known to be equal to the radius of the Circle. And that method of finding the length of an arch of one Minute is, as I conceive, the beſt that can be taken for that purpoſe without having recourſe to the infinite Series which Sir Iſaac Newton has given us for expreſſing the length of the Sine of a circular arch in powers of the arch and radius, to wit, the Series $a - \frac{a^3}{2.3.r^2} + \frac{a^5}{2.3.4.5.r^4} - \frac{a^7}{2.3.4.5.6.7.r^6} +$ &c, or $a - \frac{a^3}{6r^2} + \frac{a^5}{120\,r^4} - \frac{a^7}{5040\,r^6} +$ &c, in which r denotes the radius

of the circle, and *a* any arch of it that does not exceed 90 degrees, or the fourth part of the whole circumference. For by this Series the value of the Sine of an arch of one Minute may be computed to a great degree of exactneſs by computing only two, or, at moſt, three, terms of the ſaid Series; the labour of which would be trifling in compariſon of that employed in the aforeſaid computation of Dr. Hutton. But the inveſtigation of this infinite Series cannot be underſtood without a knowledge of Sir Iſaac Newton's Method of Fluxions, and likewiſe of ſome one of the Methods of reverting an infinite Series; which are matters of conſiderable ſubtlety and difficulty. And hence it ſeems to be expedient, in explaining the doctrine of Plane Trigonometry and the conſtruction of a Table of Sines and Tangents, to exhibit ſome other mode of finding the length of a Sine of an arch of one Minute, (which is the grand foundation of ſuch a Table,) that may be underſtood by the help of only Euclid's Elements of Geometry, and the doctrine of the reſolution of Algebräick Equations by Raphſon's ſimple and perſpicuous Method of Approximation. And of theſe more elementary modes of finding the length of this Sine, I take that given by Dr. Hutton in the above-mentioned computation, and derived from the properties of the Chords of multiple arcs, to be the beſt. But the ſaid Sine may alſo be obtained, though with rather more labour, by deriving the Tangent of the ſaid arch of one Minute from the Tangent of an arch of 45 Degrees (which is known to be equal to the radius of the circle,) by performing divers ſections of the ſaid arch of 45 degrees by means of the Tangents of the ſeveral arches inſtead of their Chords, or by reſolving the ſeveral equations that expreſs the relations between the Tangent of any given arch that is leſs than 90 Degrees, and the Tangents of it's half, and it's third part, and it's fourth part, and it's fifth part, and other aliquot parts denoted by any whole number whatſoever, and, laſtly, (when we ſhall, by theſe means, have obtained the length of the ſaid Tangent of an arch of one Minute to the propoſed Degree of exactneſs,) by deriving the Sine of the ſame arch from it's ſaid Tangent by the known rules of Trigonometry. And, if the value of the Sine of an arch of one Minute thus derived from it's Tangent, after having derived the ſaid Tangent from the Tangent of an arch of 45 Degrees, ſhall be found to agree, to a great number of figures, with the value of it before-derived from the Chord of an arch of 60 Degrees by the above-mentioned computation of Dr. Hutton, it will be a ſtrong confirmation of the truth of both the computations, and of

the

the exactneſs of the two numbers obtained by them (ſo far as the figures of the ſaid two numbers ſhall agree,) for the value of the Sine of one Minute, which value is the grand foundation of the Trigonometrical Canon, or the Table of the numeral values of Sines and Tangents. And from this conſideration, I have thought it might be worth while to enter upon this ſecond method of obtaining the length of a Sine of one Minute, and to exhibit to my Readers a careful and compleat computation of it, with which the learned Dr. Mackay has, with great ſkill and uncommon efforts of induſtry, been ſo kind as to furniſh me.

Of the Equations that expreſs the relations between the Tangent of any given circular Arch that is leſs than 90 Degrees, or the fourth part of the whole Circumference, and the Tangent of the half, and the third part, and the fourth part, and the fifth part, and any other aliquot part, of the ſaid Arch.

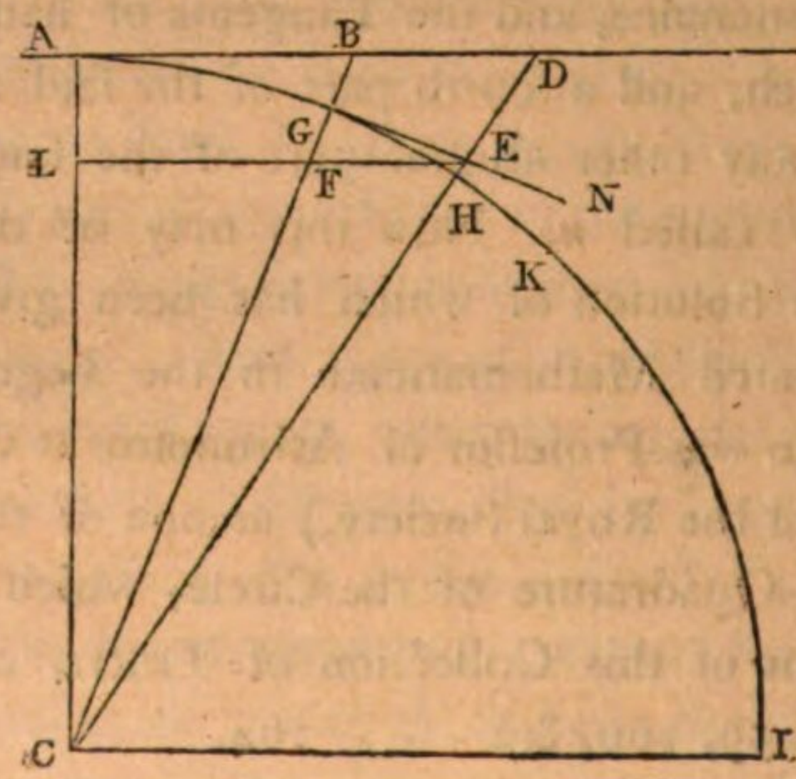
Art. 2. Now, in order to explain this Method of computing the length of the Sine of an arch of one minute, it will in the firſt place, be neceſſary to ſhow how we may find the equations that expreſs the relations between the Tangent of any given arch of a circle that is leſs than 90 Degrees, or the fourth part of the whole circumference, and the Tangents of half the ſaid arch, and a third part of the ſaid arch, and a fourth part of the ſaid arch, and a fifth part of the ſaid arch, and any other aliquot part of the ſaid arch that is denoted by any whole number called *n*. Now this may be done by means of the following Problem, the Solution of which has been given us by Mr. John Machin (a very celebrated Mathematician in the beginning of the laſt, or eighteenth, century, who was Profeſſor of Aſtronomy at Greſham College, and one of the Secretaries of the Royal Society,) as one of the preliminary propoſitions to his excellent Quadrature of the Circle, which has been ſet-forth at large in the third volume of this Collection of Tracts, called *Scriptores Logarithmici*, in pages 158, 159, 160, &c - - - 164.

A Problem.

A Problem.

Art. 3. The radius of a circle, and the tangent of two arches in it which together are leſs than an arch of 90 Degrees, or a quarter of the whole circumference, being given; it is required to find the tangent of their ſum.

Let CAI (in the figure here under following) be a Quadrant of a circle, of which the point C is the center, and the right lines CA and CI are the two radiuses that bound the ſaid Quadrant. And in this quadrantal arch AI take the two contiguous arches AG and GH. From the point A draw the right line AM touching the circle in the point A; and from the point G draw the right line GN, touching the circle in the point G. And from the center C, through the points G and H, draw the right lines CG and CH, and produce them till the line CG meets the Tangent AM in the point B and the line CH meets the Tangent GN in the point E and the Tangent AM in the point D. Then will AB and GE be the Tangents of the arches AG and GH, and AD will be the Tangent of the arch AH, which is the ſum of the two arches AG and GH. Therefore the two Tangents AB and GE are ſuppoſed to be known, and likewiſe the radius CA; and the Tangent AD is ſuppoſed to be unknown, and is required to be found by means of the radius CA and the two Tangents AB and GE, which are known.



Mr.

Mr. Machin's Solution.

Let r be put for the radius CA, and a for the Tangent AB of the arch AG, and b for the Tangent GE of the arch GH; and let x be put for the Tangent AD, which is ſought.

From the extreme point E of the ſecond Tangent GE draw the right line EL parallel to the Tangent AD, and meeting the radius CA in L; and let the point in which it cuts the line CG be called F.

Then will the triangles ABC and EFG be both of them ſimilar to the triangle CFL, and conſequently ſimilar to each other. Therefore CA will be to CB as EG is to EF; that is, r will be to $\sqrt{rr+aa}$ as b is to EF. And conſequently EF will be $= \frac{b \times \sqrt{rr+aa}}{r}$.

Further, CA will be to AB as EG is to GF, or r will be to a as b is to GF. Therefore GF will be $= \frac{ab}{r}$. And conſequently CF (which is $=$ CG $-$ GF,) will be $= r - \frac{ab}{r} = \frac{rr-ab}{r}$.

But from the ſimilarity of the triangles CBD, CFE, we ſhall have CF to EF as CB to BD, that is, $\frac{rr-ab}{r}$ to $\frac{b \times \sqrt{rr+aa}}{r}$ as $\sqrt{rr+aa}$ to BD, or (becauſe $\frac{rr-ab}{r}$ is to $\frac{b \times \sqrt{rr+aa}}{r}$ as $rr-ab$ is to $b \times \sqrt{rr+aa}$,) we ſhall have $rr-ab$ to $b \times \sqrt{rr+aa}$ as $\sqrt{rr+aa}$ is to BD. Therefore BD will be $= \frac{b \times \sqrt{rr+aa} \times \sqrt{rr+aa}}{rr-ab} = \frac{b \times rr+aa}{rr-ab}$. And conſequently x, or AD, (which is $=$ AB $+$ BD,) will be $= a + \frac{b \times \overline{rr+aa}}{rr-ab} = \frac{rra-aab+rrb+aab}{rr-ab} = \frac{rra+rrb}{rr-ab} = \frac{rr \times \overline{a+b}}{rr-ab}$; that is, AD, the tangent of AH, the ſum of the two arches AG

and

and GH, will be to $a+b$, the ſum of the two tangents AB and GE of the ſaid two arches, as the ſquare of the radius CA is to the exceſs of the ſaid ſquare above the rectangle under the two tangents AB and GE of the ſaid two arches.

Q. E. I.

Note. This proportion of the Tangent of the Sum of the two arches to the ſum of their Tangents has been alſo demonſtrated in the form of a Theorem by the learned Dr. Smith, (formerly Maſter of Trinity College, Cambridge,) in his compleat Treatiſe of Opticks, and from thence was transferred into my Elements of Plane Trigonometry, Prop. 25, pages 72, 73, 74, 75, and 76.

Art. 4. Coroll. 1. Now let the ſecond arch GH be ſuppoſed to be equal to the firſt arch AG. Then will the Tangent GE be equal to the Tangent AB; that is, b will be $= a$. Therefore in this caſe $a+b$ will be $(= a+a) = 2a$, and ab will be $= aa$, and x, or $\frac{rr \times \overline{a+b}}{rr-ab}$, will be $= \frac{rr \times 2a}{rr-aa}$; that is, AD, or the Tangent of the arch AH, (which in this caſe is double of the arch AG,) will be $= \frac{rr \times 2a}{rr-aa}$. And conſequently $\overline{rr-aa} \times$ AD, or $\overline{rr-aa} \times x$, will be $= rr \times 2a$, or $rrx-aax$ will be $= 2rra$, and $2rra + aax$ will be $= rrx$; that is, the relation between a, the Tangent of the ſingle arch AG, and x, the Tangent of the double arch AH, will be expreſſed by the equation $2rra + aax = rrx$, or $xaa + 2rra = rrx$. Therefore, if we vary the notation of the Tangents a and x, and denote the former Tangent a by the ſmall letter t, and the latter Tangent x by the capital letter T, the relation between the Tangent of the ſingle arch AG and the Tangent of the double arch AH will be expreſſed by the equation $\mathrm{T}\,tt + 2rrt = rr\,\mathrm{T}$. And conſequently, if, inſtead of ſuppoſing the Tangent t of the ſingle arch AG to be known, and the Tangent T of the double arch AH to be ſought, we ſuppoſe the Tangent T of the double arch AH to be known, and the Tangent t of the ſingle arch AG to be ſought, the ſaid leſſer Tangent t may be derived from the ſaid greater Tangent T by reſolving the equation $\mathrm{T}\,tt + 2rrt = rr\,\mathrm{T}$, or (if we divide all the terms by T,) the equation $tt + \frac{2rr}{\mathrm{T}} \times t = rr$; which is a quadratick equation properly prepared for reſolution. This is called *biſecting* a circular arch by means of it's Tangent.

Art.

Art. 5. Coroll. 2. Now let us ſuppoſe that a third arch HK is taken in the quadrantal arch AI from the point H towards the point I, that is equal to each of the two former arches AG and GH, the ſaid two arches being, as in the foregoing Corollary, ſuppoſed to be equal to each other. Then, it is evident, the Tangent of this third arch HK muſt be equal to *a*, or *t*, the tangent of the firſt arch AG.

It has been ſhewn in the foregoing Corollary that the Tangent AD of the double arch AH is $= \frac{rr \times 2a}{rr-aa}$. For the ſake of brevity let this quantity be denoted by the capital letter A.

Now, ſince it is ſhewn in the foregoing Problem that, when any two arches of which the two Tangents are *a* and *b*, and the ſum of which is leſs than the arch of a quadrant, are contiguous to each other, the Tangent of the ſum of thoſe arches will be $= \frac{rr \times a+b}{rr-ab}$, it will follow that, when the arch HK, of which the Tangent is *a*, is added to the arch AH, of which the Tangent is $\frac{rr \times 2a}{rr-aa}$, or A, the Tangent of the arch AK (which is equal to the ſum of the two arches AH and HK) will be $= \frac{rr \times \overline{A+a}}{rr-Aa}$. But $rr \times \overline{A+a}$ is $(= rrA+rra = rr \times \frac{rr \times 2a}{rr-aa} + rra = \frac{2r^4a+r^4a-rra^3}{rr-aa}) = \frac{3r^4a-rra^3}{rr-aa}$; and $rr-Aa$ is $(= rr-a \times \frac{rr \times 2a}{rr-aa} = \frac{r^4-rraa-2rraa}{rr-aa}) = \frac{r^4-3rraa}{rr-aa}$. Therefore $\frac{rr \times A+a}{rr-Aa}$ will be (= the fraction $\frac{3r^4a-rra^3}{rr-aa}$ divided by the fraction $\frac{r^4-3rraa}{rr-aa}$ = the fraction $\frac{3r^4a-rra^3}{rr-aa}$ multiplied into the fraction $\frac{rr-aa}{r^4-3rraa}$ = the fraction $\frac{3r^4a-rra^3}{r^4-3rraa}) = \frac{3rra-a^3}{rr-3aa}$. Therefore the Tangent of the arch AK (which is triple of the arch AG, of which *a* is the Tangent,) will be $= \frac{3rra-a^3}{rr-3aa}$; or, if we ſubſtitute *t* inſtead of *a* for the tangent of the ſingle arch AG, and T for the Tangent of the arch AK, which is triple of the arch AG, we ſhall have $T = \frac{3rrt-t^3}{rr-3tt}$. Therefore $rr\,T - 3\,T\,tt$ will be $= 3rrt-t^3$, and $rr\,T$ will be $= 3rrt+3\,T\,tt-t^3$, or $3rr \times t+3\,T \times t^2-t^3$ will be $= rr\,T$; which is a cubick equation by the reſolution of which, if the Tangent

T of the triple arch AK is known, we may find t, the Tangent of the ſingle arch AG. This is called *triſecting* the arch AK by means of it's Tangent.

Art. 6. Coroll. 3. In like manner we may find the relation between the Tangent of the Single Arch and the Tangent of the Quadruple Arch.

For, if the Tangent of the triple arch (which has been ſhewn to be equal to $\frac{3rra-a^3}{rr-3aa}$,) be called c, and the Tangent of the exceſs of the quadruple arch above the triple arch be called d, the Tangent of the quadruple arch (which is the ſum of the triple arch and the ſaid exceſs,) will (by the foregoing Problem,) be $= \frac{rr \times \overline{c+d}}{rr-cd}$. But the exceſs of the quadruple arch above the triple arch is, equal to the ſingle arch, and conſequently it's Tangent d will be equal to a, the Tangent of the ſingle arch. Therefore the Tangent of the quadruple arch, to wit, $\frac{rr \times \overline{c+d}}{rr-cd}$, will be $= \frac{rr \times \overline{c+a}}{rr-ca}$.

But $c+a$ is $\left(= \frac{3rra-a^3}{rr-3aa} + a = \frac{3rra-a^3}{rr-3aa} + \frac{rra-3a^3}{rr-3aa}\right) = \frac{4rra-4a^3}{rr-3aa}$, and conſequently $rr \times \overline{c+a}$ will be $\left(= rr \times \frac{4rra-4a^3}{rr-3aa}\right) = \frac{4r^4a-4r^2a^3}{rr-3aa}$; and ca is $\left(= \frac{3rra-a^3}{rr-3aa} \times a\right) = \frac{3rraa-a^4}{rr-3aa}$, and conſequently $rr-ca$ is $\left(= rr - \frac{3rraa-a^4}{rr-3aa} = \frac{r^4-3rraa-3rraa+a^4}{rr-3aa}\right) = \frac{r^4-6rraa+a^4}{rr-3aa}$. Therefore the fraction $\frac{rr \times \overline{c+a}}{rr-ca}$ will be (= the fraction $\frac{4r^4a-4r^2a^3}{rr-3aa}$ divided by the fraction $\frac{r^4-6rraa+a^4}{rr-3aa}$ = the fraction $\frac{4r^4a-4r^2a^3}{rr-3aa}$ multiplied into the fraction $\frac{rr-3aa}{r^4-6rraa+a^4}$ = the fraction $\frac{4r^4a-4r^2a^3}{r^4-6rraa+a^4}$; that is, the Tangent of the quadruple arch will be equal to the fraction $\frac{4r^4a-4r^2a^3}{r^4-6rraa+a^4}$. Therefore, if t be put for a, or the Tangent of the ſimple arch, and T be put for the Tangent of the quadruple arch, we ſhall have $T = \frac{4r^4t-4r^2t^3}{r^4-6rrtt+t^4}$, and conſequently $r^4T-6rrTtt+Tt^4$ will be $= 4r^4t-4r^2t^3$, and (adding $6rrTtt$ to both ſides,) r^4T+Tt^4 will be $= 4r^4t+6rrTtt-4r^2t^3$

$4r^2t^3$, and (ſubtracting Tt^4 from both ſides,) r^4T will be $= 4r^4t+6rrTt^2-4r^2t^3-Tt^4$, or $4r^4t+6rrTt^2-4r^2t^3-Tt^4$ will be $= r^4T$, and (dividing all the terms by T,) $\frac{4r^4}{T}\times t+6rr\times t^3-\frac{4r^2}{T}\times t^3-t^4$ will be $= r^4$; which is a biquadratick equation, by the reſolution of which, if T, the Tangent of the quadruple arch, is known, we may find t, the Tangent of the ſingle arch. This is called *quadriſecting* a circular arch by means of it's Tangent.

Art. 7. Coroll. 4. In like manner we may find the relation between the Tangent of the Single Arch and the Tangent of the Quintuple Arch.

For, if the Tangent of the quadruple arch (which has been ſhewn to be $= \frac{4r^4a-4rra^3}{r^4-6rraa+a^4}$,) be called e, and the Tangent of the exceſs of the quintuple arch above the quadruple arch be called f, the tangent of the quintuple arch (which is the ſum of the quadruple arch and the ſaid exceſs,) will (by the foregoing Problem,) be $= \frac{rr\times\overline{e+f}}{rr-ef}$. But the exceſs of the quintuple arch above the quadruple arch is equal to the ſingle arch, and therefore it's Tangent f will be equal to a, the Tangent of the ſingle arch. Therefore the Tangent of the quintuple arch (which is equal to $\frac{rr\times\overline{e+f}}{rr-ef}$,) will be $= \frac{rr\times\overline{e+a}}{rr-ea}$.

But $e+a$ is $= \frac{4r^4a-4rra^3}{r^4-6rraa+a^4}+a\ (= \frac{4r^4a-4rra^3+r^4a-6rra^3+a^5}{r^4-6rraa+a^4}-) =$ $\frac{5r^4a-10rra^3+a^5}{r^4-6rraa+a^4}$, and conſequently $rr\times\overline{e+a}$ will be $(= rr\times\frac{5r^4a-10rra^3+a^5}{r^4-6rraa+a^4}) =$ $\frac{5r^6a-10r^4a^3+rra^5}{r^4-6rraa+a^4}$; and ea is $(= \frac{4r^4a-4rra^3}{r^4-6rraa+a^4}\times a) = \frac{4r^4aa-4rra^4}{r^4-6rraa+a^4}$, and conſequently $rr-ea$ will be $(= rr-\frac{4r^4aa-4rra^4}{r^4-6rraa+a^4} = \frac{r^6-6r^4aa+rra^4}{r^4-6rraa+a^4}-$ $\frac{4r^4aa-4rra^4}{r^4-6rraa+a^4} = \frac{r^6-6r^4aa-4r^4aa+rra^4+4rra^4}{r^4-6rraa+a^4}) = \frac{r^6-10r^4aa+5rra^4}{r^4-6rraa+a^4}$. Therefore $\frac{rr\times\overline{e+a}}{rr-ea}$ will be (= the fraction $\frac{5r^6a-10r^4a^3+rra^5}{r^4-6rraa+a^4}$ divided by the fraction $\frac{r^6-10r^4aa+5rra^4}{r^4-6rraa+a^4}$ = the fraction $\frac{5r^6a-10r^4a^3+rra^5}{r^4-6rraa+a^4}$ multiplied into the fraction $\frac{r^4-6rraa+a^4}{r^6-10r^4aa+5rra^4}$ = the fraction $\frac{5r^6a-10r^4a^3+rra^5}{r^6-10r^4aa+5rra^4}$) $= \frac{5r^4a-10r^2a^3+a^5}{r^4-10r^2aa+5a^4}$; that is,

that is, the Tangent of the quintuple arch will be $= \frac{5r^4a-10r^2a^3+a^5}{r^4-10r^2a^2+5a^4}$. Therefore, if t be put for a, (the Tangent of the ſingle arch AG,) and T be put for the tangent of the quintuple arch, or 5AG, (the ſaid quintuple arch being ſtill ſuppoſed to be leſs than the whole quadrantal arch AI,) we ſhall have $T = \frac{5r^4t-10r^2t^3+t^5}{r^4-10r^2t^2+5t^4}$, and conſequently $r^4T-10r^2T\times t^2+5T\times t^4 = 5r^4t-10r^2t^3+t^5$, and (adding $10r^2T\times t^2$ to both ſides,) $r^4T+5T\times t^4 = 5r^4t+10r^2T\times t^2-10r^2t^3+t^5$, and (ſubtracting $5T\times t^4$ from both ſides,) $r^4T = 5r^4t+10r^2T\times t^2-10r^2t^3-5T\times t^4+t^5$, or $t^5-5T\times t^4-10r^2\times t^3+10r^2T\times t^2+5r^4t = r^4T$; which is an equation of the fifth order, by the reſolution of which, if T, the tangent of the quintuple arch, is known, we may find t, the tangent of the ſingle arch. This is called *quinquiſecting* a circular arch by means of it's tangent.

Art. 8. Theſe ſeveral equations, obtained in the foregoing Corollaries for biſecting, triſecting, quadriſecting, and quinquiſecting, a circular arch of which the Tangent is given, or known, or that expreſs the relations between T, the Tangent of the given arch, and t, the Tangent of it's half, or it's third part, or it's fourth part, or it's fifth part, if they are ſet-down in order, ſeparately from the reaſonings by which they have been obtained, will be as follows; to wit.

For the biſection of the arch of which T is the Tangent,

$$t^2+\frac{2r^2}{T}\times t = r^2;$$

For the triſection of the ſame arch,

$$3rrt+3Tt^2-t^3 = rrT;$$

For the quadriſection of the ſame arch,

$$\frac{4r^3}{T}\times t+6rr\times t^2-\frac{4rr}{T}\times t^3-t^4 = r^4;$$

And for the quinquiſection of the ſame arch,

$$5r^4t+10rrT\times t^2-10rrt^3-5Tt^4+t^5 = r^4T.$$

And, if we ſuppoſe the radius of the Circle to be called 1 inſtead of r, the foregoing equations will be converted into the following equations; to wit,

For

For the Biſection of the arch of which T is the Tangent,

$$t^2 + \frac{2}{T} \times t = 1;$$

For the Triſection of the ſame arch,

$$3t + 3Tt^2 - t^3 = T;$$

For the Quadriſection of the ſame arch,

$$\frac{4}{T} \times t + 6t^2 - \frac{4}{T} \times t^3 - t^4 = 1;$$

And for the Quinquiſecton of the ſame arch,

$$5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T.$$

Art. 9. And in the ſame manner as theſe equations have been derived one from another by means of the foregoing Problem, it is evident that we might proceed to derive from them the like equations for expreſſing the relations, between T, the Tangent of the given arch, and the tangents of it's ſixth part, and it's ſeventh part, and it's eighth part, and it's ninth part, and other following leſſer aliquot parts of it, as far as we pleaſed to purſue the inquiry, But the Algebräical operations required for this purpoſe grow (as we have already ſeen in the foregoing Corollaries), continually more and more intricate and troubleſome the further we proceed in the inveſtigation. I ſhall therefore content myſelf with having obtained the foregoing four equations for the biſection, triſection, quadriſection, and quinquiſection, of a given arch of a circle, that is leſs than 90 degrees, or the arch of a quadrant; as theſe will be ſufficient to enable us to derive from the Tangent, of an arch of 45 degrees (which is known to be equal to the radius of the circle,) the Tangent of an arch of one minute, which is the object of the following computation performed by Dr. Mackay.

Art. 10. Nevertheleſs, as Mr. John Bernoulli, the celebrated Mathematician of the former part of the laſt, or eighteenth, century, has given us a general expreſſion of the value of T, the Tangent of the greater, or multiple, arch, in terms involving the powers of t, the Tangent of a leſſer arch that is any aliquot part of the former arch denoted by the whole number n, I will here ſet it down for the ſatisfaction of my readers, and likewiſe apply it to the biſection, triſection, quadriſection, and quinquiſection, of the ſaid given arch of which T, is the Tangent; whereby we ſhall obtain the very ſame equations, for thoſe ſeveral ſections of the

ſaid

ſaid greater arch, which have been obtained in the foregoing articles by means of the foregoing Problem of Mr. Machin. This general expreſſion of the value of the Tangent T may be deſcribed as follows.

A general Expreſſion of the value of T, *the Tangent of any given Arch of a Circle (that is leſs than* 90 *Degrees, or the fourth part of the whole Circumference,) in Terms involving the powers of* t, *or the Tangent of any Aliquot part of the ſaid former Arch that is denoted by any whole Number whatſoever called* n, *upon a Suppoſition that the Radius of the Circle is called* 1.

Invented by Mr. JOHN BERNOULLI, *formerly Profeſſor of Mathematicks at Baſle, or Baſil, in Switzerland, and publiſhed in the ſecond Volume of his Works in the Geneva Edition of them in four Volumes, Quarto, page* 532.

THE tangent, T, of the greater, or multiple, arch of a circle, which is equal to n times a leſſer arch of the ſame circle, of which the tangent is t, will, if the radius of the circle be called 1, be equal to the following fraction, to wit,

$$\frac{\frac{n}{1} \times t - \frac{n \times \overline{n-1} \times \overline{n-2}}{1.\ 2.\ 3} \times t^3 + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{1.\ 2.\ 3.\ 4.\ 5} \times t^5 - \&c}{1 - \frac{n \times \overline{n-1}}{1.\ 2} \times t^2 + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{1.\ 2.\ 3.\ 4} \times t^4 - \&c.}$$

In this fraction the Series which forms it's Numerator contains the ſeveral odd powers of the leſſer tangent t, to wit, t, t^3, t^5, t^7, t^9, t^{11}, t^{13}, &c, and the Series which forms it's Denominator has 1 for it's firſt term, and in it's ſecond

and

and third and other following terms contains the ſeveral even powers of the ſaid leſſer tangent t, to wit, t^2, t^4, t^6, t^8, t^{10}, t^{12}, &c. And the firſt Series, which forms the Numerator of this fraction, has, for the numeral co-efficients of the odd powers t, t^3, t^5, t^7, t^9, t^{11}, t^{13}, &c contained in it's terms, the correſponding terms of the Series $\frac{n}{1}$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, &c, or (if we put A for $\frac{n}{1}$, or the firſt term of this Series, and B for $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3}$, it's ſecond term, and C for $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5}$, it's third term, and D, E, F, G, H, &c for it's fourth, fifth, ſixth, ſeventh, eighth, and other following terms reſpectively,) the correſponding terms of the Series $\frac{n}{1}$, A $\times \frac{n-1}{2} \times \frac{n-2}{3}$, B $\times \frac{n-3}{4} \times \frac{n-4}{5}$, C $\times \frac{n-5}{6} \times \frac{n-6}{7}$, D $\times \frac{n-7}{8} \times \frac{n-8}{9}$, E $\times \frac{n-9}{10} \times \frac{n-10}{11}$, &c; and the ſecond Series, which forms the Denominator of the ſaid fraction, has for the numeral co-efficients of the even powers of t, to wit, t^2, t^4, t^6, t^8, t^{10}, t^{12}, &c, contained in it's ſecond, and third, and other following terms, the correſponding terms of the Series $\frac{n}{1} \times \frac{n-1}{2}$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6}$, &c, or (if we put a for $\frac{n}{1} \times \frac{n-1}{2}$, or the firſt term of this Series, and b for $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4}$, or the ſecond term of this Series, and c for $\frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} \times \frac{n-3}{4} \times \frac{n-4}{5} \times \frac{n-5}{6}$, or it's third term, and d, e, f, g, b, &c for it's fourth, fifth, ſixth, ſeventh, eighth, and other following terms, reſpectively,) the correſponding terms of the Series $\frac{n}{1} \times \frac{n-1}{2}$, $a \times \frac{n-2}{3} \times \frac{n-3}{4}$, $b \times \frac{n-4}{5} \times \frac{n-5}{6}$, $c \times \frac{n-6}{7} \times \frac{n-7}{8}$, $d \times \frac{n-8}{9} \times \frac{n-9}{10}$, $e \times \frac{n-10}{11} \times \frac{n-11}{12}$, &c; in which two Serieſes of the numeral co-efficients of the odd and the even powers of the tangent t, the laws of the continuation of the terms are very manifeſt. And, as to the ſigns, — and +, which are to be prefixed to the ſecond and other following terms of the two Serieſes which form the Numerator and Denominator of this fraction, they are the ſame in both Serieſes, the ſign — being prefixed in both Serieſes

to

to the ſecond, and fourth, and ſixth, and other following even terms, and the ſign + being prefixed to the third, and fifth, and ſeventh, and other following odd terms.

The Application of the foregoing General Expreſſion of the value of the Tangent T, *(given by Mr.* JOHN BERNOULLI,) *to the Biſection, the Triſection, the Quadriſection, and the Quinquiſection of a Circular Arch.*

Art. 11. In order to biſect, by means of the foregoing general expreſſion, the circular arch of which T is the tangent, we muſt proceed as follows.

In this caſe n is $= 2$. Therefore the foregoing fraction

$$\frac{\frac{n}{1} \times t - \frac{n \times \overline{n-1} \times \overline{n-2}}{1. \quad 2. \quad 3} \times t^3 + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3} \times \overline{n-4}}{1. \quad 2. \quad 3. \quad 4. \quad 5} \times t^5 - \&c}{1 - \frac{n \times \overline{n-1}}{1. \quad 2} \times t^2 + \frac{n \times \overline{n-1} \times \overline{n-2} \times \overline{n-3}}{1. \quad 2. \quad 3. \quad 4} \times t - \&c.}$$

will in this caſe become equal to the fraction

$$\frac{\frac{2}{1} \times t - \frac{2 \times \overline{2-1} \times \overline{2-2}}{1. \quad 2. \quad 3} \times t^3 + \frac{2 \times \overline{2-1} \times \overline{2-2} \times \overline{2-3} \times \overline{2-4}}{1. \quad 2. \quad 3. \quad 4. \quad 5} \times t^5 - \&c}{1 - \frac{2 \times \overline{2-1}}{1. \quad 2} \times t^2 + \frac{2 \times \overline{2-1} \times \overline{2-2} \times \overline{2-3}}{1. \quad 2. \quad 3. \quad 4} \times t^4 - \&c}$$

$= \frac{2t - 0 + 0 \&c}{1 - t^2 + 0 \&c} = \frac{2t}{1-t^2}$. Therefore T will be $= \frac{2t}{1-t^2}$, and $2t$ will be $=$ $T - Tt^2$, and $2t + Tt^2$ will be $= T$, and $\frac{2t}{T} + t^2$ will be $= 1$, or $t^2 + \frac{2}{T} \times t$ will be $= 1$; which is the ſame equation that was obtained for this purpoſe above in Art. 8.

Art. 12. For the Triſection of the Arch of which T is the Tangent, we muſt proceed as follows.

In

In this case n is $= 3$. Therefore the foregoing fraction will become $=$

$$\frac{\frac{3}{1} \times t - \frac{3 \times \overline{3-1} \times \overline{3-2}}{1. \quad 2. \quad 3} \times t^3 + \frac{3 \times \overline{3-1} \times \overline{3-2} \times \overline{3-3} \times \overline{3-4}}{1. \quad 2. \quad 3. \quad 4. \quad 5} \times t^5 - \&c}{1 - \frac{3 \times \overline{3-1}}{1. \quad 2} \times t^2 + \frac{3 \times \overline{3-1} \times \overline{3-2} \times \overline{3-3}}{1. \quad 2. \quad 3. \quad 4} \times t^4 - \&c.}$$

$$= \frac{3t - \frac{3 \times 2 \times 1}{1 \times 2 \times 3} \times t^3 + \frac{3 \times 2 \times 1 \times 0}{1 \times 2 \times 3 \times 4 \times 5} \times t^5 \&c}{1 - \frac{3 \times 2}{1 \times 2} \times t^2 + \frac{3 \times 2 \times 1 \times 0}{1 \times 2 \times 3 \times 4} \times t^4 - \&c}$$

$= \frac{3t - t^3 + 0}{1 - 3t^2 + 0} = \frac{3t - t^3}{1 - 3t^2}$. Therefore T will be $= \frac{3t - t^3}{1 - 3t^2}$, and consequently $3t - t^3$ will be $= T - 3Tt^2$, and $3t + 3T \times t^2 - t^3$ will be $= T$; which is the same equation that was obtained for this purpose above in Art. 8.

Art. 13. For the Quadrisection of the Arch of which T is the Tangent, we must proceed as follows.

In this case n is $= 4$. Therefore the foregoing fraction will become equal to the fraction

$$\frac{\frac{4}{1} \times t - \frac{4 \times \overline{4-1} \times \overline{4-2}}{1. \quad 2. \quad 3} \times t^3 + \frac{4 \times \overline{4-1} \times \overline{4-2} \times \overline{4-3} \times \overline{4-4}}{1. \quad 2. \quad 3. \quad 4. \quad 5} \times t^5 - \&c}{1 - \frac{4 \times \overline{4-1}}{1. \quad 2} \times t^2 + \frac{4 \times \overline{4-1} \times \overline{4-2} \times \overline{4-3}}{1. \quad 2. \quad 3. \quad 4} \times t^4 - 0}$$

$$= \frac{4t - \frac{4 \times 3 \times 2}{1 \times 2 \times 3} \times t^3 + 0}{1 - \frac{4 \times 3}{1 \times 2} \times t^2 + \frac{4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4} \times t^4 - 0} = \frac{4t - 4t^3}{1 - 6t^2 + t^4}$$. Therefore T will be $= \frac{4t - 4t^3}{1 - 6t^2 + t^4}$, and consequently $T - 6T \times t^2 + T \times t^4$ will be $= 4t - 4t^3$, and $T + T \times t^4$ will be $= 4t + 6T \times t^2 - 4t^3$, and $4t + 6T \times t^2 - 4t^3 - T \times t^4$ will be $= T$, and (dividing all the terms by T,) we shall have $\frac{4}{T} \times t + 6t^2 - \frac{4}{T} \times t^3 - t^4 = 1$; which is the same equation that was obtained for this purpose above in Art. 8.

Art. 14. Lastly, for the Quinquisection of the arch of which T is the tangent, we must proceed as follows.

In this case n is $= 5$. Therefore the foregoing fraction will become equal to the fraction

$$\frac{\frac{5}{1} \times t - \frac{5 \times \overline{5-1} \times \overline{5-2}}{1.\ 2.\ 3} \times t^3 + \frac{5 \times \overline{5-1} \times \overline{5-2} \times \overline{5-3} \times \overline{5-4}}{1.\ 2.\ 3.\ 4.\ 5} \times t^5 - \text{o} + \&\text{c}}{1 - \frac{5 \times \overline{5-1}}{1.\ 2} \times t^2 + \frac{5 \times \overline{5-1} \times \overline{5-2} \times \overline{5-3}}{1.\ 2.\ 3.\ 4} \times t^4 - \text{o} + \&\text{c}.}$$

$$= \frac{5t - \frac{5 \times 4 \times 3}{1 \times 2 \times 3} \times t^3 + \frac{5 \times 4 \times 3 \times 2 \times 1}{1 \times 2 \times 3 \times 4 \times 5} \times t^5 - \text{o}}{1 - \frac{5 \times 4}{1 \times 2} \times t^2 + \frac{5 \times 4 \times 3 \times 2}{1 \times 2 \times 3 \times 4} \times t^4 - \text{o}} = \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}.$$ Therefore T will be $= \frac{5t - 10t^3 + t^5}{1 - 10t^2 + 5t^4}$, and consequently $\text{T} - 10\text{T} \times t^2 + 5\text{T} \times t^4$ will be $= 5t - 10t^3 + t^5$, and $\text{T} + 5\text{T} \times t^4$ will be $= 5t + 10\text{T} \times t^2 - 10t^3 + t^5$, and T will be $= 5t + 10\text{T} \times t^2 - 10t^3 - 5\text{T} \times t^4 + t^5$, or (placing the known quantity T on the right-hand side of the equation, as it is usual to do in arranging the terms of equations,) $5t + 10\text{T} \times t^2 - 10t^3 - 5\text{T} \times t^4 + t^5$ will be $= \text{T}$; which is the same equation that was obtained for this purpose above in Art. 8.

Of the Manner of applying some of the foregoing Equations to the Computation of the Value of the Tangent of an Arch of one Minute of a Degree.

Art. 15. I will now proceed to shew how, by resolving a moderate number of equations of some of the four foregoing forms, we may, from the Tangent of an arch of 45 degrees (which is known to be equal to the radius of the circle,) find the length of the Tangent of an arch of one minute to as great a degree of exactness as we please. This process will be as follows:

In the first place let the arch of 45 degrees, (the tangent of which is equal to the radius of the circle, and which we will here suppose to be called 1,) be quinquisected by resolving the equation $5t + 10\text{T} \times t^2 - 10t^3 - 5\text{T} \times t^4 + t^5 =$ T, or

T, or (becauſe T is in this caſe equal to the radius, or 1,) by reſolving the equation $5t+10\times1\times t^2-10t^3-5\times1\times t^4+t^5=1$, or $5t+10t^2-10t^3-5t^4+t^5=1$. And we ſhall thereby obtain the value of t, or the tangent of the fifth part of 45 degrees, or the Tangent of an arch of 9 degrees.

In the ſecond place, let T be equal to the tangent of 9 degrees, the length of which is now known by means of the foregoing quinquiſection of an arch of 45 degrees; and let t repreſent the tangent of an arch of 3 degrees, or the third part of an arch of 9 degrees. And the relation between T and t will now be expreſſed by the equation $3t+3T\times t^2-t^3=T$. Let this equation therefore be reſolved. And we ſhall thereby obtain the value of t, or the tangent of an arch of 3 degrees.

In the third place, let T be equal to the tangent of an arch of 3 degrees, the length of which is now known by means of the foregoing triſection of an arch of 9 degrees; and let t repreſent the tangent of an arch of one degree; or the third part of an arch of 3 degrees. And the relation between T and t will be expreſſed by the equation $3t+3T\times t^2-t^3=T$. Let this equation therefore be reſolved. And we ſhall thereby obtain the value of t, or of the tangent of an arch of one degree.

In the 4th place, let T be equal to the Tangent of an arch of one degree, the length of which is now known by means of the foregoing triſection of an arch of 3 degrees; and let t repreſent the tangent of an arch that is the fifth part of an arch of one degree, that is, the tangent of an arch of 12 minutes of a degree. And the relation between T and t will be expreſſed by the equation $5t+10T\times t^2-10t^3-5T\times t^4+t^5=T$. Let this equation be reſolved. And we ſhall thereby obtain the value of t, or of the tangent of an arch of 12 minutes.

In the 5th place, let T be equal to the tangent of an arch of 12 minutes, the length of which is now known by means of the foregoing quinquiſection of an arch of one degree, or 60 minutes; and let t repreſent the tangent of an arch of four minutes, or the third part of an arch of 12 minutes. And the relation between T and t will be expreſſed by the equation $3t+3T\times t^2-t^3=T$. Let

T. Let this equation therefore be reſolved. And we ſhall thereby obtain the value of t, or the tangent of an arch of 4 minutes.

In the 6th place, let T be equal to the tangent of an arch of 4 minutes, the length of which is now known by means of the foregoing triſection of an arch of 12 minutes; and let t repreſent the tangent of an arch of 2 minutes, or of one half of an arch of 4 minutes. And the relation between T and t will be expreſſed by the quadratick equation $\frac{2}{T} \times t + t^2 = 1$, or $t^2 + \frac{2}{T} \times t = 1$. Let this equation therefore be reſolved. And we ſhall thereby obtain the value of t, or the tangent of an arch of 2 minutes.

In the 7th and laſt place, let T be equal to the tangent of an arch of 2 minutes, the length of which is now known by means of the foregoing biſection of an arch of 4 minutes; and let t repreſent the tangent of an arch of 1 minute, or of one half of an arch of 2 minutes. And the relation between the tangents T and t will be expreſſed by the quadratick equation $\frac{2}{T} \times t + t^2 = 1$, or $t^2 + \frac{2}{T} \times t = 1$. Let this equation therefore be reſolved. And we ſhall thereby obtain the value of t, or the tangent of an arch of 1 minute; which was the object of which we were in ſearch by means of the foregoing equations concerning the relations between the tangents of two circular arches of which the leſſer is an aliquot part of the greater.

Of the Manner of deriving the Length of the Sine of an Arch of one Minute from the Length of the Tangent of the ſame Arch.

Art. 16. When the length of the tangent of an arch of 1 minute has been thus obtained to the propoſed degree of exactneſs, the ſine of the ſame arch may be derived from it in the following manner.

Let the number expreſſing the length of the tangent of an arch of 1 minute be multiplied into itſelf, ſo as to obtain the value of its ſquare. Then add this

this ſquare to the ſquare of the radius, to wit, 1×1, or 1; and extract the ſquare-root of the ſum. This ſquare-root will be the value of the ſecant of the ſame arch of 1 minute. Laſtly, make this proportion, to wit, As the ſecant of this arch is to it's tangent, ſo will the radius of the circle, which is 1, be to the ſine of the ſame arch. And the fourth proportional, which will be hereby obtained, will be the value of the ſine of an arch of one minute; which was the ultimate object of the whole computation.

A Computation of the Tangent and the Sine of an Arch of one Minute of a Degree in the Manner described in this and the foregoing Articles, to eighteen places of decimal Figures.

Performed by Dr. ANDREW MACKAY, L. L. D.

LATE OF ABERDEEN IN SCOTLAND, A FELLOW OF THE ROYAL SOCIETY OF EDINBURGH.

All the calculations neceſſary to obtain the values of both the tangent and the ſine of an arch of one minute exact to eighteen decimal places of figures, reckoning from the place of units, have been performed by Dr. Andrew Mackay, L. L. D. and Fellow of the Royal Society of Edinburgh, with great ſkill and induſtry, in the manner following.

Dr. Mackay's Reſolution of the Equation $5t + 10t^2 - 10t^3 - 5t^4 + t^5 = 1$, *for the Quinquiſection of a Circular Arch of* 45 *Degrees, the radius of the Circle being called* 1.

Art. 17. Dr. Mackay obſerves in the firſt place, that, if the diameter of a circle be called 1, the circumference of it will be = 3.141,592,653,589,793, 238,462,6

238,462,6 &c, as was ſhewn by that induſtrious Dutch calculator, Van Ceulen, from the principles of Common Geometry, by computing the Perimeters of Polygons of a great number of ſides inſcribed in, and circumſcribed about, the circle, before Sir Iſaac Newton had invented the doctrine of Fluxions and infinite Serieſes. Therefore, when the radius of the circle is called 1, (as it is on the preſent occaſion,) the ſemi-circumference of the circle will be equal to the ſame number 3.141,592,653,589,793,238,462,6 &c. Conſequently an arch of 9 degrees, or the fifth part of an arch of 45 degrees, which is equal to the 20th part of the ſemi-circumference of the circle, will be $=$ $\frac{3.141,592,653,589,793,238,462,6\ \&c}{20} = 0.157,079,632,679,489,661,923,13$ &c.

But the tangent of any arch of a circle is always greater than the arch itſelf. Therefore t, the tangent of an arch of 9 degrees, (which we are now ſeeking,) muſt be greater than 0.157,079,632,679,489,661,923,13 &c. We will therefore conjecture that it is equal to 0.1571, and will ſubſtitute 0.1571 in it's ſtead in the quinquinomial quantity $5t + 10t^2 - 10t^3 - 5t^4 + t^5$, in order to diſcover whether the value of the ſaid quantity reſulting from ſuch ſubſtitution will be greater, or will be leſs, than 1, or the abſolute term of the equation, and conſequently whether 0.1571 is greater, or is leſs, than the true value of t, or the leaſt root of the ſaid equation. And for this purpoſe Dr. Mackay raiſes the powers of 0.1571 in the following manner.

$$\begin{array}{r} 0.1571 \\ \underline{0.1571} \\ 1571 \\ 10997 \\ 7855 \\ \underline{1571} \\ 0.1571^2 = 0.024,680,41 \\ \underline{0.1571} \\ 2\,468\,041 \\ 172\,762\,87 \\ 1\,234\,020\,5 \\ \underline{2\,468\,041} \\ 0.1571^3 = 0.003,877,192,411 \end{array}$$

$$\overline{0.1571}^{3} = 0.003,877,292,411$$

$$\begin{array}{r} 0.1571 \\ \hline 387\,729\,241\,1 \\ 27\,141\,046\,877 \\ 193\,864\,620\,55 \\ 387\,729\,241\,1 \end{array}$$

$$\overline{0.1571}^{4} = 0.000,609,122,637,768,1$$

$$\begin{array}{r} 0.1571 \\ \hline 60\,912\,263\,770\,81 \\ 4\,263\,858\,464\,376\,7 \\ 30\,456\,131\,888\,405 \\ 60\,912\,263\,776\,81 \end{array}$$

$$\overline{0.1571}^{5} = 0.000,095,693,166,393,368,51$$

Therefore, if 0.1571 be ſubſtituted inſtead of t in the quinquinomial quantity $5t+10t^2-10t^3-5t^4+t^5$, we ſhall have $5t$ $(=5\times0.1571)=0.785,50$, and $10t^2$ $(=10\times0.024,680,41)=0.246,804,1$, and $10t^3$ $(=10\times0.003,877,292,411)=0.038,772,924,11$, and $5t^4$ $(=5\times0.000,609,122,637,768,1)=0.003,045,613,188,840,5$. Therefore the quinquinomial quantity $5t+10t^2-10t^3-5t^4+t^5$ will be

$$\left.\begin{array}{rll} & 0.785,500,000,000,000,000 & -0.038,772,924,110,000,0 \\ +\ & 0.246,804,100,000,000,000 & -0.003,045,613,188,840,5 \\ +\ & 0.000,095,693,166,393,368,51 & \end{array}\right\} = \left\{\begin{array}{l} 1.032,399,793,166,393,368,51 \\ -\ \ .041,818,537,298,840,500,00 \end{array}\right\}$$

$$= 0.990,581,255,867,552,868,51$$

; which is leſs than 1, or the abſolute term of the equation $5t+10t^2-10t^3-10t^4+t^5=1$. Therefore 0.1571 is leſs than the true value of t, or the leaſt root of that equation.

A Process of Mr. Raphson's Method of Approximation.

Art. 18. Having thus diſcovered that 0.1571 is leſs than the true value of t, Dr. Mackay puts z for their unknown difference, and ſubſtitutes the binomial

mial quantity $0.1571+z$ inſtead of t in the propoſed equation $5t+10t^2-10t^3-5t^4+t^5=1$, omitting all the terms of the powers of the ſaid binomial quantity that involve any higher powers of the unknown quantity z than it's firſt, or ſimple, power, or z itſelf, agreeably to the directions of Mr. Raphſon in his excellent Tract on the Reſolution of all ſorts of Equations by the Method of Approximation. This he does in the following manner.

Since t is $= 0.1571+z$, we ſhall have

$t^2 = \overline{0.1571+z}^2$ $(= \overline{0.1571}^2 + 2\times 0.1571\times z + \&c = \overline{0.1571}^2 + 0.3142\times z + \&c)$
$= 0.024,680,41 + 0.3142\times z + \&c,$

and $t^3 = \overline{0.1571+z}^3$ $(= \overline{0.1571}^3 + \overline{3\times 0.1571}^2 \times z + \&c$
$= \overline{0.1571}^3 + 3\times 0.024,680,41\times z + \&c$
$= \overline{0.1571}^3 + 0.074,041,23\times z)$
$= 0.003,877,292,411 + 0.074,041,23\times z + \&c,$

and $t^4 = \overline{0.1571+z}^4$ $(= \overline{0.1571}^4 + 4\times\overline{0.1571}^3 \times z + \&c$
$= \overline{0.1574}^4 + 4\times 0.003,877,292,411\times z + \&c$
$= \overline{0.1574}^4 + 0.015,509,169,644 \times z + \&c)$
$= 0.000,609,122,637,768,1 + 0.015,509,169,644; \times z + \&c,$

and $t^5 = \overline{0.1571+z}^5$ $(=\overline{0.1571}^5 + 5\times\overline{0.1571}^4 \times z + \&c$
$= \overline{0.1571}^5 + 5\times 0.000,609,122,637,768,1\times z + \&c$
$= \overline{0.1571}^5 + 0.003,045,613,188,840,5\times z + \&c)$
$= 0.000,095,693,166,393,368,51 + 0.003,045,613,188,840,5 \times z + \&c.$

Therefore $5t$ will be $= 5\times\overline{0.1571+z}$ $(= 5\times 0.1571+5z) = 0.7855+5z;$
and $10t^2$ will be $= 10\times\overline{0.024,680,41+0.3142 \times z + \&c}$
$(= 10\times 0.024,680,41 + 10\times 0.3142\times z + \&c)$
$= 0.246,804,1 + 3.142\times z + \&c;$

and $10t^3$ will be $= 10\times\overline{0.003,877,292,411 + 0.074,041,23\times z + \&c}$
$(= 10\times 0.003,877,292,411 + 10\times 0.074,041,23\times z + \&c)$
$= 0.038,772,924,11 + 0.740,412,3\times z + \&c;$

and

and $5t^4$ will be $= 5\times0.000,609,122,637,768,1 + 0.015,509,169,644\times z +$ &c
$(= 5\times0.000,609,122,637,768,1 + 5\times0.015,509,169,644\times z$
$+$ &c)
$= 0.003,045,613,188,840,5 + 0.077,545,848,220\times z +$ &c.

Therefore the whole quinquinomial quantity $5t + 10t^2 - 10t^3 - 10^4 + t^5$ will be equal to the following multinomial quantity, to wit,

$$\left\{\begin{array}{ll} 0.785,500, & + 5z \\ + 0.246,804,1 & + 3.142\times z + \text{\&c} \\ - 0.038,772,924,11 & - 0.740,412,3\times z - \text{\&c} \\ - 0.003,045,613,188,840,5 & - 0.077,545,848,220\times z - \text{\&c} \\ + 0.000,095,693,166,393,368,51 & + 0.003,045,613,188,840,5\times z + \text{\&c} \end{array}\right\}$$

$$= \left\{\begin{array}{ll} 1.032,399,793,166,393,368,51 & + 8.145,045,613,188,840,5\times z + \text{\&c} \\ -0.041,813,537,298,840,500,00 & - 0.817,958,148,220,000,0\times z - \text{\&c} \end{array}\right\}$$

$$= 0.990,581,255,867,552,868,51 + 7.327,087,464,968,840,5\times z + \text{\&c}$$

But the quinquinomial quantity $5t + 10t^2 - 10t^3 - 10t^4 + t^5$ is $= 1$.

Therefore the quantity $0.990,581,255,867,552,868,51 + 7.327,087,464,968,840,5\times z +$ &c will also be $= 1$.

Therefore $7.327,087,464,968,840,5\times z$ will be $= 1 - 0.990,581,255,867,552,868,51 = 0.009,418,744,132,447,131,49$, and consequently z will be $= \frac{0.009,418,744,132,447,131,49}{7.327,087,464,968,840,5} = 0.001,28$. Therefore t, or $0.1571 + z$, will be $(= 0.1571 + 0.001,28) = 0.15838$, that is, the tangent of an arch of 9 degrees will be nearly equal to 0.15838. Q. E. I.

Art. 19. Having thus obtained the number 0.15838 for a second near value of t, or the tangent of an arch of 9 degrees, Dr. Mackay proceeds to substitute it instead of t in the quinquinomial quantity $5t + 10t^2 - 10t^3 - 5t^4 + t^5$, in order to discover whether the value of the said quantity resulting from this

ſubſtitution will be greater, or will be leſs, than 1, or the abſolute term of the propoſed equation $5t+10t^2-10t^3-10t^4+t^5=1$, and conſequently whether the ſaid number 0.15838 will be greater, or will be leſs, than the value of t, or the leaſt root of that equation. And for this purpoſe Dr. Mackay raiſes the powers of the number 0.15838 in the following manner.

$$\begin{array}{rl}
0.15838 & \\
\underline{0.15838} & \\
12\,670\,4 & \\
47\,514 & \\
1\,267\,04 & \\
7\,919\,0 & \\
\underline{15\,838} & \\
0.15838)^2 = & 0.025,084,224,4 \\
 & \underline{0.15838} \\
 & 2\,006\,737\,952 \\
 & 7\,525\,267\,32 \\
 & 200\,673\,795\,2 \\
 & 1\,254\,211\,220 \\
 & \underline{2\,508\,422\,44} \\
0.15838)^3 = & 0.003,972,839,460,472 \\
 & \underline{0.15838} \\
 & 317\,827\,156\,837\,76 \\
 & 1\,191\,851\,838\,141\,6 \\
 & 31\,782\,715\,683\,776 \\
 & 198\,641\,973\,023\,60 \\
 & \underline{397\,283\,946\,047\,2} \\
0.15838)^4 = & 0.000,629,218,313,749,555,36 \\
 & \underline{0.15838} \\
 & 0.000,062\,921\,831\,374\,955\,536 \\
 & 31\,460\,915\,687\,477\,768 \\
 & 5\,033\,746\,509\,996\,443 \\
 & 188\,765\,494\,124\,867 \\
 & \underline{50\,337\,465\,099\,964} \\
0.15838)^5 = & 0.000,099,655,596,531,654,578
\end{array}$$

Therefore, if 0.15838 be ſubſtituted inſtead of t in the quinquinomial quantity $5t+10t^2-10t^3-10t^4+t^5$, we ſhall have $5t$ $(=5\times0.15838)=$ 0.79190,

and

and $10t^2$ ($= 10 \times 0.025,084,224,4$) $= 0.250,842,244,$

and $10t^3$ ($= 10 \times 0.003,972,839,460,472$) $= 0.039,728,394,604,72,$

and $5t^4$ ($= 5 \times 0.000,629,218,313,749,555,36$) $= 0.003,146,091,568,$ $747,776,80$. Therefore $5t + 10t^2 + t^5$ will be $=$

$$\left\{\begin{array}{rl} & 0.791,900,000,000,000,000,000 \\ + & 0.250,842,244,000,000,000,000 \\ + & 0.000,099,655,596,531,654,578 \end{array}\right\}$$

$= 1.042,841,899,596,531,654,578$; and $10t^3 + 5t^4$ will be $=$

$$\left\{\begin{array}{rl} & 0.039,728,394,604,720,000,00 \\ + & 0.003,146,091,568,747,776,80 \end{array}\right\}$$

$= 0.042,874,486,173,467,776,80$; and consequently the whole quinquinomial quantity $5t + 10t^2 - 10t^3 - 5t^4 + t^5$ will be $=$

$$\left\{\begin{array}{rl} & 1.042,841,899,596,531,654,578 \\ - & 0.042,874,486,173,467,776,800 \end{array}\right\}$$

$= 0.999,967,413,423,063,877,778$; which is less than 1, or the absolute term of the equation $5t + 10t^2 - 10t^3 - 5t^4 + t^5 = 1$. Therefore 0.158,38 is less than the true value of t, or the least root of the said equation.

A Second Process of Mr. Raphson's Method of Approximation.

Art. 20. Dr. Mackay then puts z for the excess of t above 0.158,38, and in order to obtain a more exact value of t, he substitutes the binomial quantity $0.15838 + z$ instead of t in the equation $5t + 10t^2 - 10t^3 - 5t^4 + t^5 = 1$. This he does in the following manner.

Since t is $= 0.15838 + z$, we shall have

$t^2 = \overline{0.15838 + z}^2$ ($= \overline{0.15838}^2 + 2 \times 0.15838 \times z +$ &c

$= \overline{0.15838}^2$

$= \overline{0.15838}^2 + 0.31676 \times z + \&c)$

$= 0.025,084,224,4 + 0.31676 \times z + \&c;$

and $t^3 = \overline{0.15838+z}^3 \; (= \overline{0.15838}^3 + 3 \times \overline{0.15838}^2 \times z + \&c$

$= \overline{0.15838}^3 + 3 \times 0.025,084,224,4 \times z + \&c$

$= \overline{0.15838}^3 + 0.075,252,673,2 \times z + \&c)$

$= 0.003,972,839,460,472 + 0.075,252,673,2 \times z + \&c;$

and $t^4 = \overline{0.15838+z}^4 \; (= \overline{0.15838}^4 + 4 \times \overline{0.15838}^3 \times z + \&c$

$= \overline{0.15838}^4 + 4 \times 0.003,972,839,460,472 \times z + \&c$

$= \overline{0.15838}^4 + 0.015,891,357,841,888 \times z + \&c)$

$= 0.000,629,218,313,749,555,36 + 0.015,891,357,841,888 \times z + \&c;$

and $t^5 = \overline{0.15838+z}^5 \; (= \overline{0.15838}^5 + 5 \times \overline{0.15838}^4 \times z + \&c$

$= \overline{0.15838}^5 + 5 \times 0.000,629,218,313,749,555,36 \times z + \&c$

$= \overline{0.15838}^5 + 0.003,146,091,568,747,776,80 \times z + \&c$

$= 0.000,099,655,596,531,654,578 + 0.003,146,091,568,747,776,80 \times z + \&c.$

Therefore $5t$ will be $= 5 \times \overline{0.15838+z} \; (= 5 \times 0.15838 + 5 \times z) =$ $0.791,90 + 5z;$

and $10t^2$ will be $= 10 \times \overline{0.025,084,224,4 + 0.31676 \times z} + \&c$

$(= 10 \times 0.025,084,224,4 + 10 \times 0.31676 \times z + \&c)$

$= 0.250,842,244 + 3.1676 \times z + \&c;$

and $10t^3$ will be $= 10 \times \overline{0.003,972,839,460,472 + 0.075,252,673,2 \times z} + \&c$

$(= 10 \times 0.003,972,839,460,472 + 10 \times 0.075,252,673,2 \times z$

$+ \&c)$

$= 0.039,728,394,604,72 + 0.752,526,732 \times z + \&c$

and $5t^4$ will be $= 5 \times \overline{0.000,629,218,313,749,555,36 + 0.015,891,357,841,888 \times z} + \&c$

$(= 5 \times 0.000,629,218,313,749,555,36 + 5 \times 0.015,891,357,841,888 \times z + \&c,)$

$= 0.003,146,091,568,747,776,80 + 0.079,456,789,209,440, \times z + \&c.$

Therefore $5t + 10t^2 + t^3$ will be $=$

$0.791,900,000$

$$\left\{\begin{array}{lll} 0.791,900,000 & + & 5.000,000 \times z \\ + 0.250,842,244 & + & 3.167,600 \times z \\ + 0.000,099,655,596,531,654,578 & + & 0.003,146,091,568,747,776,80 \times z \end{array}\right\}$$

$$= 1.042,841,899,596,531,654,578 + 8.170,746,091,568,747,776,80 \times z$$
$$+ \&c;$$

and $10t^3 + 5t^4$ will be $=$

$$\left\{\begin{array}{lll} 0.039,728,394,604,72 & + & 0.752,526,732 \times z + \&c \\ + 0.003,146,091,568,747,776,80 & + & 0.079,456,789,209,440, \times z \end{array}\right\}$$

$$= 0.042,874,486,173,467,776,80 + 0.831,983,521,209,440 \times z,$$ and consequently the whole quinquinomial quantity $5t + 10t^2 - 10t^3 - 5t^4 + t^5$ will be $=$

$$\left\{\begin{array}{l} 1.042,841,899,596,531,654,578 + 8.170,746,091,568,747,776,80 \times z \\ - 0.042,874,486,173,467,776,800 - 0.831,983,521,209,440,000,00 \times z \end{array}\right\}$$

$$= 0.999,967,413,423,063,877,778 + 7.338,762,570,359,307,776,80 \times z.$$

But the quinquinomial quantity $5t + 10t^2 - 10t^3 - 5t^4 + t^5$ is $= 1$.

Therefore the quantity $0.999,967,413,423,063,877,778 + 7.338,762,570,359,307,776,80 \times z$ will alſo be $= 1$. And conſequently $7.338,762,570,359,307,776,80 \times z$ will be $=$

$$\left\{\begin{array}{l} 1.000,000,000,000,000,000,000 \\ - \underline{0.999,967,413,423,063,877,778} \end{array}\right\} = 0.000,032,586,576,936,122,222,$$

$$= 0.000,032,586,576,936,122,222$$

and z will be $= \frac{0.000,032,586,576,936,122,222}{7.338,762,570,359,307,776,80} = 0.000,004,440,3.$

Therefore t, or $0,15838 + z$, will be $= 0.158,38 + 0.000,004,440,3 = 0.158,384,440,3$; that is, the tangent of an arch of 9 degrees will be very nearly equal to 0.158,384,440,3. Q. E. I.

Art. 21. This value of t, or the tangent of 9 degrees, is probably exact in all it's ten figures. But, as Dr. Mackay in this valuable Computation has propoſed to obtain his final reſults exact to eighteen places of decimal figures, he makes uſe of another proceſs of Mr. Raphſon's Method of Approximation grounded on this laſt-found value of t, to wit, 0.158,384,440,3; and for that purpoſe he ſubſtitutes this value of t inſtead of t in the quinquinomial quantity

tity $5t+10t^2-10t^3-5t^4+t^5$, in order to discover whether the value of the said quantity resulting from such substitution will be greater, or will be less, than 1, or the absolute term of the equation $5t+10t^2-10t^3-5t^4+t^5 = 1$, and consequently whether the said number 0.158,384,440,3 is greater, or is less, than the true value of t, or the least root of the said equation. This he does in the following manner.

He first raises the powers of the number 0.158,384,440,3 as follows.

$t = 0.158,384,440,3$
$t = 0.158,384,440,3$

47 515 332 09
6 335 377 612 0
63 353 776 12
633 537 761 2
12 670 755 224
47 515 332 09
1 267 075 522 4
7 919 222 015
15 838 444 03

$t^2 = 0.025,085,630,929,144,264,09$
$t = 0.158,384,440,3$

.002 508 563 092 914 426 409 000 0
1 254 281 546 457 213 204 500 0
200 685 047 433 154 112 720 0
7 525 689 278 743 279 227 0
2 006 850 474 331 541 127 2
100 342 523 716 577 056 3
10 034 252 371 657 705 6
1 003 425 237 165 770 6
7 525 689 278 743 3

$t^3 = 0.003,973,173,614,284,883,225,850,0$
$t = 0.158,384,440,3$

0.000,397 317 361 428 488 322 585 0
198 658 680 714 244 161 292 5
31 785 388 914 279 065 806 8
1 191 952 084 285 464 967 7
317 853 889 142 790 658 1
15 892 694 457 139 532 9
1 589 269 445 713 953 3
158 926 944 571 395 3
1 191 952 084 285 5

$t^4 = 0.000,629,288,879,113,239,314,477,1$

$t^4 =$ 0.000,629,288,879,113,239,314,477,1
$t =$ 0.158,384,440,3

0.000,062 928 887 911 323 931 447 7
31 464 443 955 661 965 7239
5 034 311 032 905 914 515 8
188 786 663 733 971 794 3
50 343 110 329 059 145 2
2 517 155 516 452 957 2
251 715 551 645 295 7
25 171 555 164 529 6
188 786 663 734 0

$t^5 =$ 0.000,099,669,566,905,364,769,143,4

Therefore $5t$ will be $(= 5\times 0.158,384,440,3) = 0.791,922,201,5,$
and $10t^2$ will be $(= 10\times 0.025,085,630,929,144,264,09) = 0.250,856,309,$
$291,442,640,9,$
and $10t^3$ will be $(= 10\times 0.003,973,173,614,284,883,225,850,0) = 0.039,731,$
$736,142,848,832,258,500,$
and $5t^4$ will be $(= 5\times 0.000,629,288,879,113,239,314,477,1) = 0.003,146,$
$444,395,566,196,572,385,5.$

Therefore $5t + 10t^2 + t^5$ will be $=$

$$\left\{\begin{array}{l} 0.791,922,201,500,000,000,0 \\ +\ 0.250,856,309,291,442,640,9 \\ +\ 0.000,099,669,566,905,364,769,143,4 \end{array}\right\}$$

$= 1.042,878,180,358,348,005,669,143,4;$

and $10t^3 + 5t^4$ will be

$$= \left\{\begin{array}{l} 0.039,731,736,142,848,832,258,500,0 \\ +\ 0.003,146,444,395,566,196,572,385,5 \end{array}\right\}$$

$= 0.042,878,180,538,415,028,830,885,5;$ and consequently the whole quinquinomial quantity $5t + 10t^2 - 10t^3 - 5t^4 + t^5$ will be $=$

$$\left\{\begin{array}{l} 1.042,878,180,358,348,005,669,143,4 \\ -\ 0.042,878,180,538,415,028,830,885,5 \end{array}\right\}$$

$= 0.999,999,999,819,932,976,838,257,9;$ which is less than 1, or the absolute term of the equation $5t + 10t^2 - 10t^3 - 5t^4 + t^5 = 1$. Therefore the number 0.158,384,440,3 is less than the true value of t, or the least root of that equation, or of the tangent of an arch of 9 degrees. Q. E. I.

A Third Process of Mr. Raphson's Method of Approximation.

Art. 22. Having thus discovered that the number 0.158,384,440,3 is less than the true value of t, Dr. Mackay puts z for their unknown difference, and substitutes the binomial quantity $0.158,384,440,3+z$ instead of t in the terms of the quinquinomial quantity $5t+10t^3-10t^3+5t^4+t^5$. This he does in the following manner.

Since t is $= 0.158,384,440,3+z$, we shall have

$t^2 = \overline{0.158,384,440,3+z}^2 (= \overline{0.158,384,440,3}^2 + 2\times 0.158,384,440,3\times z + \&c)$
$= \overline{0.158,384,440,3}^2 + 0.316,768,880,6\times z + \&c)$
$= 0.025,085,630,929,144,264,09 + 0.316,768,880,6\times z + \&c,$

and $t^3 = \overline{0.158,384,440,3+z}^3 (= \overline{0.158,384,440,3}^3 + 3\times \overline{0.158,384,440,3}^2 \times z + \&c$
$= \overline{0.158,384,440,3}^3 + 3\times 0.025,085,630,929,144,264,09\times z + \&c$
$= \overline{0.158,384,440,3}^3 + 0.075,256,892,787,432,792,27\times z + \&c)$
$= 0.003,973,173,614,284,883,225,850,0 + 0.075,256,892,787,432,792,27\times z + \&c,$

and $t^4 = \overline{0.158,384,440,3+z}^4 (= \overline{0.158,384,440,3}^4 + 4\times \overline{0.158,384,440,3}^3 \times z + \&c$
$= \overline{0.158,384,440,3}^4 + 4\times 0.003,973,173,614,284,883,225,850,0 \times z + \&c$
$= \overline{0.158,384,440,3}^4 + 0.015,892,694,457,139,532,903,400,0\times z + \&c)$
$= 0.000,629,288,879,113,239,314,477,1 + 0.015,892,694,457,139,532,903,400,0\times z + \&c,$

and $t^5 = \overline{0.158,384,440,3+z}^5 (= \overline{0.158,384,400,3}^5 + 5\times \overline{0.158,384,440,3}^4 \times z + \&c$

=

$= \overline{0.158,384,440,3}^{5} + 5 \times 0.000,629,288,879,113,239,314,477,1 \times z$ + &c)

$= \overline{0.158,384,440,3}^{5} + 0.003,146,444,395,566,196,572,385,5 \times z$ + &c

$= 0.000,099,669,566,905,364,769,143,4 + 0.003,146,444,395,566,196,572,385,5 \times z$ + &c.

Therefore $5t$ will be $= 5 \times \overline{0.158,384,440,3 + z}$ $(= 5 \times 0.158,384,440,3 + 5 \times z) = 0.791,922,201,5 + 5z$;

and $10t^2$ will be $= 10 \times 0.025,085,630,929,144,264,09 + 0.316,768,880,6 \times z$ + &c

$(= 10 \times 0.025,085,630,929,144,264,09 + 10 \times 0.316,768,880,6 \times z$ + &c)

$= 0.250,856,309,291,442,640,9 + 3.167,688,806 \times z$ + &c;

and $10t^3$ will be $(= 10 \times 0.003,973,173,614,284,883,225,850,0 + 10 \times 0.075,256,892,787,432,792,27 \times z$ + &c)

$= 0.039,731,736,142,848,832,258,500 + 0.752,568,927,874,327,922,7 \times z$ + &c;

and $5t^4$ will be $(= 5 \times 0.000,629,288,879,113,239,314,477,1 + 5 \times 0.015,892,694,457,139,532,903,400,0 \times z$ + &c)

$= 0.003,146,444,395,566,196,572,385,5 + 0.079,463,472,285,697,664,517,000,0 \times z$ + &c.

Therefore the whole quinquinomial quantity $5t + 10t^2 - 10t^3 - 5t^4 + t^5$ will be equal to the following multinomial quantity, to wit,

$$\left\{\begin{array}{l} 0.791,922,201,5 + 5.000,000,000,000,000,000 \times z \\ + 0.250,856,309,291,442,640,9 \\ \qquad + 3.167,688,806,000,000,000 \times z + \&c \\ - 0.039,731,736,142,848,832,258,5 \\ \qquad - 0.752,568,927,874,327,922,7 \times z - \&c \\ - 0.003,146,444,395,566,196,572,385,5 \\ \qquad - 0.079,463,472,285,697,664,517 \times z - \&c \\ + 0.000,099,669,566,905,364,769,143,4 \\ \qquad + 0.003,146,444,395,566,196,572,385,5 \times z + \&c \end{array}\right\}$$

$$= \left\{\begin{array}{l} 1.042,878,180,358,348,005,669,143,4 + 8.170,835,250,395,566,196,572,385,5 \times z \\ - 0.042,878,180,538,415,028,830,885,5 + 0.832,032,400,160,025,587,217 \times z \end{array}\right\} =$$

$= 0.999,999,999,819,932,976,838,257,9 + 7.338,802,850,235,540,609,355,385,5 \times z +$ &c.

But the quinquinomial quantity $5t + 10t^2 - 10t^3 - 5t^4 + t^5$ is $= 1$.

Therefore the quantity $0.999,999,999,819,932,976,838,257,9 + 7.338,802,850,235,540,609,355,385,5 \times z +$ &c will be alſo $= 1$. And conſequently $7.338,802,850,235,540,609,355,385,5 \times z$ will be $(= 1 - 0.999,999,999,819,932,976,838,257,9) = 0.000,000,000,180,067,023,161,742,1$, and z will be $\left(= \frac{0.000,000,000,180,067,023,161,742,1}{7.338,802,850,235,540,609,355,385,5}\right) = 0.000,000,000,024,536,293,83$. Therefore t, or $0.158,384,440,3 + z$, will be $(= 0.158,384,440,3 + 0.000,000,000,024,536,293,83) = 0.158,384,440,324,536,293,83$; that is, the tangent of an arch of 9 degrees will be $= 0.158,384,440,324,536,293,83$. Q. E. I.

This value of t, or the tangent of 9 degrees, is exact in all it's twenty figures. But Dr. Mackay has purſued the inveſtigation ſtill further, and determined the value of this tangent exactly to 24 places of figures, which are $0.158,384,440,324,536,293,838,883$. This he does by employing a fourth proceſs of Mr. Raphſon's Method of Approximation in the following manner.

A fourth Process of Mr. Raphson's Method of Approximation.

Art. 23. He takes the number $0.158,384,440,324,536$, (conſiſting of the firſt fifteen figures of the number laſt-obtained, to wit, $0.158,384,440,324,536,293,83$,) and ſubſtitutes it inſtead of t in the quinquinomial quantity $5t + 10t^2 - 10t^3 - 5t^4 + t^5$, in order to diſcover whether the value of the ſaid quantity will be greater, or will be leſs, than 1, or the abſolute term of the propoſed equation $5t + 10t^2 - 10t^3 - 5t^4 + t^5 = 1$, and conſequently to determine whether the ſaid ſubſtituted number $0.158,384,440,324,536$, is greater, or is leſs, than the true value of t, or the leaſt root of that equation. And,

in

in order to make this ſubſtitution, he raiſes the powers of the ſaid number 0.158,384,440,324,536 in the following manner.

t = 0.158,384,440,324,536
t = 0.158,384,440,324,536

0.015 838 444 032 453 600 000 000 0
7 919 222 016 226 800 000 000 0
1 267 075 522 596 288 000 000 0
47 515 332 097 360 800 000 0
12 670 755 225 962 880 000 0
633 537 761 298 144 000 0
63 353 776 129 814 400 0
6 335 377 612 981 440 0
47 515 332 097 360 8
3 167 688 806 490 7
633 537 761 298 2
79 192 220 162 3
4751 533 209 7
950 306 641 9

t^2 = 0.025,085,630,936,916,505,345,003,6
t = 0.158,384,440,324,536

0.002 508 563 093 691 650 534 500 4
1 254 281 546 845 825 267 250 2
200 685 047 495 332 042 760 0
7 525 689 281 074 951 603 5
2 006 850 474 953 320 427 6
100 342 523 747 666 021 4
10 034 252 374 766 602 1
1 003 425 237 476 660 2
7 525 689 281 074 9
501 712 618 738 3
100 342 523 747 7
12 542 815 468 5
752 568 928 1
150 513 785 6

t^3 = 0.003,973,173,616,131,386,347,568,5

$t^3 =$ 0.003,973,173,616,131,386,347,568,5
$t =$ 0.158,384,440,324,536

0.000,397 317 361 613 138 634 756 9
198 658 680 806 569 317 378 4
31 785 388 929 051 090 780 5
1 191 952 084 839 415 904 3
317 853 889 290 510 907 8
15 892 694 464 525 545 4
1 589 269 446 452 554 5
158 926 944 645 255 5
1 191 952 084 839 4
79 463 472 322 6
15 892 694 464 5
1 986 586 808 1
119 195 208 5
23 839 041 7

$t^4 =$ 0.000,629,288,879,503,182,465,768,1
$t =$ 0.158,384,440,324,536

0.000,062,928,887,950,318,246,576,8
31 464 443 975 159 123 288 4
5 034 311 036 025 459 726 1
188 786 663 850 954 739 7
50 343 110 360 254 597 3
2 517 155 518 012 729 9
251 715 551 801 273 0
25 171 555 180 127 3
188 786 663 850 9
12 585 777 590 1
2 517 155 518 0
314 644 439 7
18 878 666 4
3 775 733 3

$t^5 =$ 0.000,099,669,566,982,565,928,856,9

Therefore $5t$ will be $(= 5 \times 0.158,384,440,324,536) = 0.791,922,201,622,680$, and $10t^2$ will be $(= 10 \times 0.025,085,630,936,916,505,345,003,6) = 0.250,856,309,369,165,053,450,036,0$, and consequently $5t + 10t^2 + t^5$ will be

$$= \left\{ \begin{array}{ll} & 0.791,922,201,622,680 \\ + & 0.250,856,309,369,165,053,450,036,0 \\ + & 0.000,099,669,566,982,565,928,856,9 \end{array} \right\}$$

$$= 1.042,878,180,558,827,619,378,892,9 ;$$

and

and $10t^3$ will be ($= 10\times0.003,973,173,616,131,386,347,568,5$) $= 0.039,731,736,161,313,863,475,685,0$,

and $5t^4$ will be ($= 5\times0.000,629,288,879,503,182,465,768,1$) $= 0.003,146,444,397,515,912,328,840,5$,

and consequently $10t^3+5t^4$ will be $=$

$$\left\{\begin{array}{ll} & 0.039,731,736,161,313,863,475,685,0 \\ + & 0.003,146,444,397,515,912,328,840,5 \end{array}\right\}$$

$= 0.042,878,180,558,829,775,804,525,5$. Therefore the whole quinquinomial quantity $5t+10t^2-10t^3-5t^4+t^5$ will be $=$

$$\left\{\begin{array}{ll} & 1.042,878,180,558,827,619,378,892,9 \\ - & 0.042,878,180,558,829,775,804,525,5 \end{array}\right\}$$

$= 0.999,999,999,999,997,843,574,367,4$, which is less than 1, or the absolute term of the equation $5t+10t^2-10t^3-5t^4+t^5 = 1$. Therefore 0.158,384,440,324,536, is less than the true value of t, or the least root of the said equation, or than the true value of the tangent of an arch of 9 degrees.

Art. 24. Having thus found that the true value of t, or of the tangent of an arch of 9 degrees, or of the least root of the equation $5t+10t^2-10t^3-5t+t^5 = 1$, is somewhat greater than 0.158,384,440,324,536, Dr. Mackay puts z for their difference, and then substitutes the binomial quantity 0.158,384,440,324,536 $+ z$ instead of t in the said equation, omitting, in the new, or transformed, equation, all the terms that involve any higher powers of the small quantity z than it's simple power or z itself, agreeably to the directions of Mr. Raphson's Method of Resolving High Equations by Approximation. This he does in the following manner.

To avoid the repetition of very long numbers, let us substitute the letter a for the number 0.158,384,440,324,536. And we shall then have $a+z$ ($=$ 0.158,384,440,324,536$+z$) $= t$. Therefore

t^2 will be ($= \overline{a+z}^2$) $= a^2+2az+$&c,

and t^3 will be ($= \overline{a+z}^3$) $= a^3+3a^2z+$&c,

and t^4 will be ($= \overline{a+z}^4$) $= a^4+4a^3z+$&c,

and t^5 will be ($= \overline{a+z}^5$) $= a^5+5a^4z+$&c;

and

and consequently $5t$ will be $(= 5 \times \overline{a+z}) = 5a+5z$,

and $10t^2$ will be $(= 10 \times \overline{a^2+2az+\&c}) = 10a^2+10\times 2az+\&c$,

and $10t^3$ will be $(= 10 \times \overline{a^3+3a^2z+\&c}) = 10a^3+10\times 3a^2z+\&c$,

and $5t^4$ will be $(= 5 \times \overline{a^4+4a^3z+\&c}) = 5a^4+5\times 4a^3z+\&c$, and the quinquinomial quantity $5t+10t^2-10t^3-5t^4+t^5$, will be $=$ the multinomial quantity

$$\left.\begin{array}{l} 5a+5z \\ +\ 10a^2+10\times 2az+\&c \\ -\ 10a^3-10\times 3a^2z-\&c \\ -\ 5a^4-5\times 4a^3z-\&c \\ +\ a^5+5a^4z+\&c \end{array}\right\} = \begin{array}{l} 5a+5z \\ +\ 10a^2+20a\times z+\&c \\ -\ 10a^3-30a^2\times z-\&c \\ -\ 5a^4-20a^3\times z-\&c \\ +\ a^5+5a^4\times z+\&c. \end{array}$$

But the quinquinomial quantity $5t+10t^2-10t^3-5t^4+t^5$ is $= 1$.

Therefore the multinomial quantity

$$\left.\begin{array}{l} 5a+5\times z \\ +\ 10a^2+20a\times z+\&c \\ -\ 10a^3-30a^2\times z-\&c \\ -\ 5a^4-20a^3\times z-\&c \\ +\ a^5+5a^4\times z+\&c \end{array}\right\} \text{ will also be } = 1.$$

But it has been shewn in the foregoing article that $5a+10a^2-10a^3-5a^4+a^5$ is $= 0.999,999,999,999,997,843,574,367,4$.

Therefore the multinomial quantity $0.999,999,999,999,997,843,574,367,4+5\times z+20a\times z-30a^2\times z-20a^3\times z+5a^4\times z$ &c will be $= 1$, and consequently the quantity $5\times z+20a\times z-30a^2\times z-20a^3\times z+5a^4\times z$ &c will be $(= 1-0.999,999,999,999,997,843,574,367,4) = 0.000,000,000,000,002,156,425,632,6$, and consequently z will be $= \frac{0.000,000,000,000,002,156,425,632,6}{5+20a-30a^2-20a^3+5a^4}$.

But, because a is $= 0.158,384,440,324,536$, we shall have

$20a\ (= 20\times 0.158,384,440,324,536) = 3.167,688,806,490,720$,

and $30a^2\ (= 30\times 0.025,085,630,936,916,505,345,003,6) = 0.752,568,928,107,495,160,350,108,0$,

and $20a^3\ (= 20\times 0.003,973,173,616,131,386,347,568,5) = 0.079,463,472,322,627,726,951,370,0$,

and

and $5a^4$ (= 5 × 0.000,629,288,879,503,182,465,768,1) = 0.003,146,444,397,515,912,328,840,5.

Therefore the quinquinomial quantity $5+20a-30a^2-20a^3+5a^4$ will be equal to the quantity

$$\left\{\begin{array}{l} \quad 5.000,000,000,000,000 \\ + \ 3.167,688,806,490,720 \\ - \ 0.752,568,928,107,495,160,350,108,0 \\ - \ 0.079,463,472,322,627,726,951,370,0 \\ + \ 0.003,146,444,397,515,912,328,840,5 \end{array}\right\} =$$

$$\left\{\begin{array}{l} \quad 8.170,835,250,888,235,912,328,840,5 \\ - \ 0.832,032,400,430,122,887,301,478,0 \end{array}\right\}$$

= 7.338,802,850,458,113,025,027,362,5. Therefore the fraction $\frac{0.000,000,000,000,002,156,425,632,6}{5+20a-30a^2-20a^3+5a^4}$ will be = $\frac{0.000,000,000,000,002,156,425,632,6}{7.338,802,850,458,113,025,027,362,5}$ = 0.000,000,000,000,000,293,838,883,06, and consequently z (which is equal to the fraction $\frac{0,000,000,000,000,002,156,425,632,6}{5+20a-30a^2-20a^3+5a^4}$) 0.000,000,000,000,000,293,838,883,06, and $a+z$, or t, will be = a + 0.000,000,000,000,000,293,838,883,06 = 0.158,384,440,324,536 + 0.000,000,000,000,000,293,838,883,06 = 0.158,384,440,324,536,293,838,883,06; that is, the tangent of an arch of 9 degrees will be equal to 0.158,384,440,324,536,293,838,883,06.

A Proof of the Truth of the foregoing Computation for quinquisecting an Arch of 45 *Degrees.*

Art. 25. In order to aſcertain the truth of the preceeding computation Dr. Mackay ſubſtitutes the twenty-five firſt figures, to the neareſt unit of the above-found value of the tangent of 9 degrees, to wit, 0.158,384,440,324,536,293,838,883,1, inſtead of t, in the quinquinomial quantity $5t+10t^2-10t^3-5t^4+t^5$, in order to diſcover how nearly the value of the ſaid quantity reſulting from ſuch ſubſtitution will approach to 1, or the abſolute term of the equation $5t+10t^2-10t^3-5t^4+t^5 = 1$, and conſequently to determine

termine how nearly the ſaid ſubſtituted number 0.158,384,440,324,536,293,838,883,1, will approach to the true value of t, or the leaſt root of that equation. In order to this he raiſes the powers of the number 0.158,384,440,324,536,293,838,883,1, as follows.

t = 0.158,384,440,324,536,293,838,883,1
t = 0.158,384,440,324,536,293,838,883,1

0.015 838 444 032 453 629 383 888 3
7 919 222 016 226 814 691 944 1
1 267 075 522 596 290 350 711 1
47 515 332 097 360 888 151 7
12 670 755 225 962 903 507 1
633 537 761 298 145 175 4
63 353 776 129 814 517 5
6 335 377 612 981 451 8
47 515 332 097 360 9
3 167 688 806 490 7
633 537 761 298 1
79 192 220 162 3
4 751 533 209 7
950 306 641 9
31 676 888 1
14 254 599 6
475 153 3
126 707 6
4 751 5
1 267 1
126 7
12 7
5

t^2 = 0.025,085,630,936,916,598,424,017,7

t^3 =

$t^2 = 0.025,085,630,936,916,598,424,017,7$
$t = 0.158,384,440,324,536,293,838,883,1$

0.002 508 563 093 691 659 842 401 8
1 254 281 546 845 829 921 200 9
200 685 047 495 332 787 392 1
7 525 689 281 074 979 527 2
2 006 850 474 953 327 873 9
100 342 523 747 666 393 7
10 034 252 374 766 639 4
1 003 425 237 476 663 9
7 525 689 281 075 0
501 712 618 738 3
100 342 523 747 7
12 542 815 468 4
752 568 928 1
150 513 785 6
5 017 126 2
2 257 706 8
75 256 9
20 068 5
752 6
200 7
20 1

$\frac{2\,q}{1} = 0.000,629,289,879,503,187,132,629,9$

$t^3 = 0.003,973,173,616,131,408,460,969,9$

$t^3 = 0.003,973,173,616,131,408,460,969,9$
$t = 0.158,384,440,324,536,293,838,883,1$

0.000 397 317 361 613 140 846 097 0
198 658 680 806 570 423 048 5
31 785 388 929 051 267 687 8
1 191 952 084 839 422 538 3
317 853 889 290 512 676 9
15 892 694 464 525 633 8
1 589 269 446 452 563 4
158 926 944 645 256 3
1 191 952 084 839 4
79 463 472 322 6
15 892 694 464 5
1 986 586 808 1
119 195 208 5
23 839 041 7
794 634 7
357 585 6
11 919 5
3 178 5
119 2
31 8
3 2
3

$t^5 = 0.000,629,288,879,503,187,135,659,6$
$t = 0.158,384,440,324,536,293,838,883,1$

0.000 062 928 887 930 318 713 566 0
31 464 443 975 159 356 783 0
5 034 311 036 025 497 085 3
188 786 663 850 956 140 7
50 343 110 360 254 970 8
2 517 155 518 012 748 5
251 715 551 801 2749
25 171 555 180 127 5
188 786 663 851 0
12 585 777 590 1
2 517 155 518 0
314 644 439 7
18 878 666 4
3 775 733 3
125 857 8
56 636 0
1 887 9
503 4
189
50
5

$t^5 = 0.000,099,669,566,982,566,853,404,7$

Therefore $5t$ will be $(= 5 \times 0.158,384,440,324,536,293,838,883,1) =$ 0.791,922,201,622,681,469,194,415,5;

and $10t^2$ will be $(= 10 \times 0.025,085,630,936,916,598,424,017,7) =$ 0.250, 856,309,369,165,984,240,177,0;

and $10t^3$ will be $(= 10 \times 0.003,973,173,616,131,408,460,969,9) =$ 0.039, 731,736,161,314,084,609,699,0;

and $5t^4$ will be $(= 5 \times 0.000,629,288,879,503,187,135,659,6) =$ 0.003, 146,444,397,515,935,678,298,0. Therefore $5t + 10t^2 + t^5$ will be

$$= \left\{ \begin{array}{l} 0.791,922,201,622,681,469,194,415,5 \\ 0.250,856,309,369,165,984,240,177,0 \\ 0.000,099,669,566,982,566,853,404,7 \end{array} \right\}$$

$$= 1.042,878,180,558,830,020,287,997,2;$$

and $10t^3 + 5t^4$ will be $=$

$$\left\{ \begin{array}{l} 0.039,731,736,161,314,084,609,699,0 \\ 0.003,146,444,397,515,935,678,298,0 \end{array} \right\}$$

$= 0.042,878,180,558,830,020,287,997,0$; and consequently the whole quinquinomial quantity $5t + 10t^2 - 10t^3 - 5t^4 + t^5$ will be equal to

$$\left\{ \begin{array}{ll} + & 1.042,878,180,558,830,020,287,997,2 \\ - & 0.042,878,180,558,830,020,287,997,0 \end{array} \right\}$$

$= 1.000,000,000,000,000,000,000,000,2$; which differs from 1, or the abſolute term of the equation $5t + 10t^2 - 10t^3 - 5t^4 + t^5 = 1$, only in the twenty-fifth decimal place. Therefore, the number 0.158,384,440,324, 536,293,838,883, is the true value of t, or the leaſt root of that equation, or of the tangent of 9 degrees, exact to twenty-four places of decimal figures.

Q. E. I.

End of the Quinquiſection of an Arch of 45 Degrees.

The Triſection of an Arch of 9 *Degrees by reſolving the Cubick Equation* $3t+3Tt^2-t^3=T$, *in which* T *is the Tangent of* 9 *Degrees, and is* = 0.158,384,440,324,536,293,838,883, *and* t, *or the leſſer Root of the Equation, is the Tangent of an Arch of* 3 *Degrees, the Radius of the Circle being called* 1.

Art. 26. Dr. Mackay obſerves that the tangent of one-third of an arch is leſs than one-third of the tangent of that arch, and greater than the length of the correſponding arch; but that it is neareſt to this laſt quantity. Therefore the tangent of 3 degrees is leſs than one-third of the tangent of 9 degrees, or than $\frac{0.15838}{3}$, or 0.05246; and greater than the length of an arch of 3 degrees, which is equal to one-ſixtieth part of the ſemi-circumference, or to one-third of the length of an arch of 9 degrees, or to $\frac{0.15707}{3}$, or 0.05236; but it will be neareſt to this laſt quantity. We will, therefore, conjecture that it is equal to 0.05240, and will ſubſtitute 0.0524 for t in the trinomial quantity $3t+3Tt^2-t^3$, which forms the firſt, or left-hand, ſide of the equation $3t+3Tt^2-t^3=T$, and compare the reſult of this ſubſtitution with the abſolute term T, or 0.158,384,440,324,536,293,838,883, in order thereby to diſcover whether 0.0524 is greater, or is leſs, than the true value of t, or the leaſt root of the ſaid equation. And for this purpoſe Dr. Mackay raiſes the ſecond and third powers of 0.0542 in the following manner, and computes the value of 3T, (which is the co-efficient of t^2 in the ſaid equation $3t+3Tt^2-t^3=T$,) and finds it to be (= 3×0.158,384,440,324,536,293,838,883) = 0.475,153,320,973,608,881,516,649, and then multiplies it into the value of t^2.

$$\begin{array}{rl} t = & 0.052,4 \\ t = & 0.052,4 \\ \hline & 2096 \\ & 1048 \\ & 2620 \\ \hline t^2 = & 0.002,745,76 \\ t = & 0.052,4 \\ \hline & 1098304 \\ & 549152 \\ & 13728800 \\ \hline t^3 = & 0.000,143,877,824 \end{array}$$

$$\begin{array}{rl} 3T = & 0.475,153,320,973,608,881,516,649 \\ t^3 = & 0.002,745,76 \\ \hline & 0.000\ 950\ 306\ 641\ 947\ 217\ 763\ 033 \\ & 332\ 607\ 324\ 681\ 526\ 217\ 062 \\ & 19\ 006\ 132\ 838\ 944\ 355\ 261 \\ & 2\ 375\ 766\ 604\ 868\ 044\ 408 \\ & 332\ 607\ 324\ 681\ 526\ 217 \\ & 28\ 509\ 199\ 258\ 416\ 533 \\ \hline 3Tt^2 = & 0.001,304,656,982,596,496,322,514 \end{array}$$

Therefore, if 0.0524 be fubſtituted inſtead of t in the trinomial quantity $3t+3Tt^2-t^3$, we ſhall have $3t$ $(= 3\times 0.0524) = 0.157,2$; and $3t+3Tt^2$ $(= 0.157,2+0.001,304,656,982,596,496,322,514) = 0.158,504,656,982,596,496,322,514$; and, conſequently the whole trinomial quantity $3t+3Tt^2-t^3$ will be equal to

$$\left\{\begin{array}{rl} & 0.158,504,656,982,596,496,322,514 \\ - & 0.000,143,877,824 \end{array}\right\}$$

$= 0.158,360,779,158,596,496,322,514$; which is leſs than $0.158,384,440,324,536,293,838,883$, or the abſolute term of the equation $3t+3Tt^2-t^3 = T$. Therefore 0.0524 is leſs than the true value of t, or the leaſt root of the ſaid equation.

A Process.

A Process of Mr. Raphson's Method of Approximation, in order to obtain a nearer Value of the lesser Root t *of the Equation* $3t+3Tt^2-t^3=T$.

Art. 27. Having discovered that 0.05240 is less than the true value of t, Dr. Mackay puts z for the excess of t above 0.0524; and, in order to obtain a more exact value of t, he substitutes the binomial quantity $0.0524+z$ instead of t, in the trinomial quantity $3t+3Tt^2-t^3$, which forms the first, or left-hand, side of the equation $3t+3Tt^2-t^3=T$. This he does in the following manner.

Since t is $= 0.01524+z$, we shall have

$$t^2 = \overline{0.0524+z}^2 \; (= \overline{0.0524}^2 + 2\times 0.0524\times z + \&c)\; \overline{0.0524}^2 + 0.1048\times z + \&c) = 0.002,745,76 + 0.1048\times z + \&c,$$

$$\text{and } t^3 = \overline{0.0524+z}^3 \; (= \overline{0.0524}^3 + 3\times\overline{0.0524}^2\times z + \&c = \overline{0.0524}^3 + 3\times 0.002,745,76\times z + \&c = \overline{0.0524}^3 + 0.008,237,28\times z + \&c) = 0.000,143,877,824 + 0.008,237,28\times z + \&c.$$

Therefore $3t$ will be equal to $3\times\overline{0.00524+z} = 0.1572+3\times z$, and $3T\times t^2$ will be $= 0.475,153,320,973,608,881,516,649 \times$

$$\overline{0.002,745,745,76+0.104,8\times z + \&c}$$

$$(= 0.475,153,320,973,608,881,516,649\times 0.002,745,76 + 0.049,796,068,038,034,210,782,945,\times z + \&c)$$

$$= 0.001,304,656,982,596,496,322,514 + 0.049,796,068,038,034,210,782,945\times z + \&c.$$

Therefore the whole trinomial quantity $3t+3Tt^2-t^3$ will be equal to the multinomial quantity

$$\left\{\begin{array}{lll} +\ 0.157,2 & & +\ 3.\times z + \&c \\ +\ 0.001,304,656,982,596,496,322,514 & +\ 0.049,796,068,038,034,210,782,945\times z + \&c & \\ -\ 0.000,143,877,824 & -\ 0.008,237,280\times z + \&c & \end{array}\right\}$$

$$= 0.158,360,779,158,596,496,322,514 + 3.041,558,788,038,034,210,782,945\times z + \&c.$$

But

But the trinomial quantity $3t+3Tt^2-t^3$ is $=$ 0.158,384,440,324,536,293,838,883. Therefore the quantity 0.158,360,779,158,596,496,322,514 + 3.041,558,788,038,034,210,782,945$\times z$, will alſo be $=$ 0.158,384,440,324,536,293,838,883. And conſequently 3.041,558,788,038,034,210,782,945$\times z$ will be $=$

$$\left\{\begin{array}{r} 0.158,384,440,324,536,293,838,883 \\ -\ 0.158,360,779,158,596,496,322,514 \end{array}\right\}$$

$=$ 0.000,023,661,165,939,797,516,369. And z will be $=$ $\frac{0.000,023,661,165,939,797,516,369}{3.041,558,788,038,034,210,782,945}$ $=$ 0.000,007,779,289,3. Therefore t, or 0.0524 + z, will be ($=$ 0.0524 + 0.000,007,779,289,3) $=$ 0.052,407,779,289,3; that is, the tangent of an arch of 3 degrees will be nearly equal to 0.052,407,779,289,3. Q. E. I.

Art. 28. Having thus obtained the number 0.052,407,779,289,3 for a ſecond near value of t, or the tangent of an arch of 3 degrees, of which it is probable the firſt nine figures, viz. 0,052,407,779, are correct, Dr. Mackay proceeds to ſubſtitute it inſtead of t in the trinomial quantity $3t+3Tt^2-t^3$, in order to diſcover whether the value of the quantity reſulting from this ſubſtitution will be greater, or will be leſs, than 0.158,384,440,324,536,293,838,883, or T, the abſolute term of the propoſed equation $3t+3Tt^2-t^3=T$, and conſequently whether the ſaid number 0.052,407,779 will be greater, or will be leſs, than the true value of t, or the leaſt root of that equation. And for this purpoſe Dr. Mackay raiſes the powers of the number 0,052,407,779, in the following manner.

$t = 0.052,407,779$

$t = 0.052,407,779$

471 670 011

3 668 544 53

36 685 445 3

366 854 453

20 963 111 60

104 815 558

2 620 388 95

$t^2 = 0,002,746,575,299,712,841$

$t^2 =$

$t^3 = 0.002,746,575,299,712,841$
$t = 0.052,407,779,$

0.000 137 328 764 985 642 05
5 493 150 599 425 682
1 098 630 119 885 136 4
19 226 027 097 989 887
1 922 602 709 798 989
192 260 270 979 899
24 719 177 697 415

$t^3 = 0.000,143,941,911,314,209,334,590$

$3T = 0.475,153,320,973,608,881,516,649$
$t^2 = 0.002,746,575,299,712,841,$

0.000 950 306 641 947 217 763 033
332 607 324 681 526 217 062
19 006 132 838 944 355 261
2 850 919 925 841 653 289
237 576 660 486 804 441
33 260 732 468 152 622
2 375 766 604 868 044
95 030 664 194 722
42 763 798 887 625
4 276 379 888 762
332 607 324 682
4 751 533 210
950 306 642
380 122 657
19 006 133
475 153

$3T \times t^2 = 0.001,305,044,374,962,641,553,338$

Therefore if 0.052,407,779 be ſubſtituted inſtead of t in the trinomial quantity $3t + 3Tt^2 - t^3$, we ſhall have $3t$ $(= 3 \times 0.052,407,779) = 0.157,223,337$, and $3t + 3Tt^2$ $(= 0.157,223,337 + 0.001,305,044,374,962,641,553,338)$ $= 0.158,528,381,374,962,641,553,338$; and conſequently the whole trinomial quantity $3t + 3Tt^2 - t^3$ will be $=$

0.158,

$$\left\{\begin{array}{l} +\ 0.158,528,381,374,962,641,553,338 \\ -\ 0.000,143,941,911,314,209,334,590 \end{array}\right\}$$

$= 0.158,384,439,463,648,432,218,748$; which is less than $0.158,384,440,324,536,293,838,883$, or the absolute term of the equation $3t+3Tt^2-t^3 = T$. Therefore 0.052,407,779 is less than the true value of t, or the root of the said equation.

A second Process of Mr. Raphson's Method of Approximation.

Art. 29. Dr. Mackay then puts z for the excess of t above 0.052,407,779; and, in order to obtain a more exact value of t, he substitutes the binomial quantity $0.052,407,779+z$ instead of t in the equation $3t+3Tt^2-t^3 = T$. This he does in the following manner.

Since t is equal to $0.052,407,779+z$, we shall have

$t^2 = \overline{0.052,407,799+z}^2$ $(= \overline{0.052,407,779}^2 + 2\times 0.052,407,779\times z$

$= \overline{0.052,407,779}^2 + 0.104,815,558\times z)$

$= 0.002,746,575,299,712,841 + 0.104,815,558\times z$,

and $t^3 = \overline{0.052,407,779+z}^3$ $(= \overline{0.052,407,779}^3 + 3\times\overline{0.052,407,779}^2\times z$

$= \overline{0.052,407,779}^3 + 3\times 0.002,746,575,299,712,841\times z$

$= \overline{0.052,407,779}^3 + 0.008,239,725,899,138,523\times z)$

$= 0.000,143,941,911,314,209,334,590 + 0.008,239,725,899,138,523\times z$.

Therefore $3t$ will be equal to $(3\times\overline{0.052,407,779+z})$

$= 0.157,223,337 + 3\times z$,

And $3Tt^2 = 0.475,153,320,973,608,881,516,649\times$

$0.002,746,575,299,712,841 + 0.104,815,558\times z$

$(= 0.475,153,320,973,608,881,516\times 0.002,746,575,299,712,$

$841 + 0.049,803,459,428,064,612,047,984\times z$,

$= 0.001,$

$$= 0.001,305,044,374,962,641,553,338 + 0.049,803,459,428,064,612,047,984 \times z.$$

Therefore the trinomial quantity $3t + 3Tt^2 - t^3$ will be equal to the multinomial quantity

$$\left\{\begin{array}{l} + 0.157,223,337, \qquad\qquad + 3.\times z + \&c \\ + 0.001,305,044,374,962,641,553,338 + 0.049,803,459,428,064,612, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 047,984 \times z \\ - 0.000,143,941,911,314,209,334,590 - 0.008,239,725,899,138,523 \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times z + \&c \end{array}\right\}$$

$$= 0.158,384,439,463,648,432,218,748 + 3.041,563,733,528,926,089, 047,984 \times z.$$

But the trinomial quantity $3t + 3Tt^2 - t^3$ is $= 0.158,384,440,324,536,293,838,883$. Therefore the quantity $0.158,384,439,463,648,432,218,748 + 3.041,563,733,528,926,089,047,984 \times z$, will alſo be $= 0.158,384,440,324,536,293,838,883$; and conſequently $3.041,563,733,528,926,089,047,984 \times z$, will be $=$

$$\left\{\begin{array}{l} 0.158,384,440,324,536,293,838,883 \\ - \; 0.158,384,439,463,648,432,218,748 \end{array}\right\}$$

$= 0.000,000,000,860,887,861,620,135$. And z will be $\left(= \frac{0.000,000,000,860,887,861,620,135}{3.041,563,733,528,926,089,047,984}\right) = 0.000,000,000,283,041,204$. Therefore t, or $0.052,407,779 + z$, will be $(= 0.052,407,779 + 0.000,000,000,283,041,204) = 0.052,407,779,283,041,204$; that is, the tangent of an arch of 3 degrees will be nearly equal to 0.052,407,779,283,041,204. Q. E. I.

Art. 30. This value of t, or the tangent of 3 degrees, is probably exact in all it's firſt ſeventeen figures. But, as Dr. Mackay, in this valuable computation, has propoſed to obtain his final reſults exact to eighteen places of decimals, he makes uſe of another proceſs of Mr. Raphſon's Method of Approximation, grounded on the firſt fifteen figures of the laſt-found value of t, to wit, 0.052,407,779,283,041. And for that purpoſe, he ſubſtitutes this value of t, inſtead of t, in the trinomial quantity $3t + 3Tt^2 - t^3$, in order to diſcover whether the value of the ſaid quantity reſulting from ſuch ſubſtitution, will be greater, or will be leſs, than 0.158,384,440,324,536,293,838,883, or the abſolute term of the equation $3t + 3Tt^2 - t^3 = T$, and conſequently whether the ſaid number 0.052,407, 779,

779,283,041, is greater, or is leſs, than the true value of t, or the leaſt root of the ſaid equation. This he does in the following manner. He firſt raiſes the powers of the number 0.052,407,779,283,041, as follows.

t = 0.052,407,779,283,041
t = 0.052,407,779,283,041

0.002 620 388 964 152 050 000 000
104 815 558 566 082 000 000
20 963 111 713 216 400 000
3 668 544 549 812 870
471 670 013 547 369
10 481 555 856 608
4 192 622 342 643
157 223 337 849
2 096 311 171
52 407 779

t^2 = 0.002,746,575,329,379,941,431,989
t = 0.052,407,779,283,041

0.000 137 328 766 468 997 071 599
5 493 150 658 759 882 864
1 098 630 131 751 976 573
19 226 027 305 659 590
1 922 602 730 565 959
192 260 273 056 596
24 719 177 964 419
549 315 065 876
219 726 026 350
8 239 725 988
109 863 013
2 746 575

t^3 = 0.000,143,941,913,646,389,605,402

$3T = 0.475,153,320,973.608,881,516,649$
$t^2 = 0.002,746,575,329,379,941,431,989$

0.000 950 306 641 947 217 763 033
332 607 324 681 526 217 062
19 006 132 838 944 355 261
2 850 919 925 841 653 289
237 576 660 486 804 441
33 260 732 468 152 622
2 375 766 604 868 044
142 545 996 292 083
9 503 066 419 472
4 276 379 888 762
142 545 996 292
33 260 732 468
4 276 379 889
427 637 989
19 006 133
475 153
190 061
14 255
475
428
38
4

$3Tt^2 = 0.001,305,044,389,059,062,847,254$

Therefore, if 0.052,407,779,283,041 be ſubſtituted inſtead of t in the trinomial quantity $3t+3Tt^2-t^3$, we ſhall have

$3t$ ($= 3\times 0.052,407,779,283,041$) $= 0.157,223,337,849,123,$

and $3t+3Tt^2$ ($= 0.157,223,337,849,123 + 0.001,305,044,389,059,062,847,254$)

$= 0.158,528,382,238,882,062,847,254$, and conſequently the whole trinomial quantity $3t+3Tt^2-t^3$ will be $=$

$$\left\{\begin{array}{l} + \ 0.158,528,382,238,182,062,847,254 \\ - \ 0.000,143,941,913,646,389,605,402 \end{array}\right\}$$

$= 0.158,384,440,324,535,673,241,852$, which is leſs than 0.158,384,440,324,536,293,838,883, or the abſolute term of the equation $3t+3Tt^2-t^3 = T$. Therefore 0.052,407,779,283,041, is leſs than the true value of t, or the leaſt root of the ſaid equation.

A third

A third Process of Mr. Raphson's Method of Approximation.

Art. 31. Dr. Mackay then puts z for the excefs of t above 0.052,407,779,283,041; and, in order to obtain a more exact value of t, he ſubſtitutes the binomial quantity $0.052,407,779,283,041 + z$, inſtead of t, in the equation $3t + 3Tt^2 - t^3 = T$. This he does in the following manner.

Since t is $= 0.052,407,779,283,041 + z$, we ſhall have

$$t^2 = \overline{0.052,407,779,283,041 + z}^2 \quad (= \overline{0.052,407,779,283,041}^2 + 2 \times 0.052,407,779,283,041 \times z + \&c$$
$$= \overline{0.052,407,779,283,041}^2 + 0.104,815,558,566,082 \times z + \&c)$$
$$= 0.002,746,575,329,379,941,431,989 + 0.104,815,558,566,082 \times z + \&c,$$

and $t^3 = \overline{0.052,407,779,283,041 + z}^3 \quad (= \overline{0.052,407,779,283,041}^3 + 3 \times \overline{0.052,407,779,283,041}^2 \times z + \&c$
$$= \overline{0.052,407,779,283,041}^3 + 3 \times 0.002,746,575,329,379,941,431,989 \times z + \&c$$
$$= \overline{0.052,407,779,283,041}^3 + 0.008,239,725,988,139,824,295,967 \times z + \&c)$$
$$= 0.000,143,941,913,646,389,605,402 + 0.008,239,725,988,139,824,295,967 \times z + \&c.$$

Therefore $3t$ will be $= 3 \times \overline{0.052,407,779,283,041 + z}\ \&c$
$$= 0.157,223,337,849,123 + 3 \times z + \&c,$$

and $3T \times t^2 = 0.475,153,320,973,608,881,516,649 \times \overline{0.002,746,575,329,379,941,431,989 + 0.104,815,558,566,082 \times z + \&c}$
$$(= 0.475,153,320,973,608,881,516,649 \times 0.002,746,575,329,379,941,431,989 + 0.049,803,460,742,377,660,433,306 \times z + \&c)$$
$$= 0.001,305,044,389,059,062,847,254 + 0.049,803,460,742,377,660,433,306 \times z + \&c.$$

Therefore

Therefore the whole trinomial quantity $3t + 3Tt^2 - t^3$ will be equal to the multinomial quantity

$$\left\{\begin{array}{ll} + \text{ 0.157,223,337,849,123} & + 3 \times z, \\ + \text{ 0.001,305 044,389,059,062,847,254} & + \text{ 0.049,803,460,742,377,660,} \\ & \quad 433,306 \times z \\ - \text{ 0.000,143,941,913,646,389,605,402} & - \text{ 0.008,239,725 988,139,824,} \\ & \quad 295,967 \times z \end{array}\right\}$$

$$= 0.158,384,440,324,535,673,241,852 + 3.041,563,734,754,237,836,$$
$$137,339 \times z + \&c.$$

But the trinomial quantity $3t + 3Tt^2 - t^3$ is $= 0.158,384,440,324,536,293,838,883$, the absolute term of the equation $3t + 3Tt^2 - t^3 = T$. Therefore the quantity $0.158,384,440,324,535,673,241,852 + 3.041,563,734,754,237,836,137,339 \times z + \&c$, will also be equal to $0.158,384,440,324,536,293,838,883$, and consequently $3.041,563,734,754,237,836,137,339 \times z$ will be $=$

$$\left\{\begin{array}{ll} & 0.158,384,440,324,536,293,838,883 \\ - & 0.158,384,440,324,535,673,241,852 \end{array}\right\}$$

$= 0.000,000,000,000,000,620,597,031$. And z will be $=$

$$\frac{0.000,000,000,000,000,620,597,031}{3.041,563,734,754,237,836,137,339} = 0.000,000,000,000,000,204,038,805,4.$$

Therefore t, or $0.052,407,779,283,041 + z$, will be $(= 0.052,407,779,283,041 + 0.000,000,000,000,000,204,038,805,4) = 0.052,407,779,283,041,204,038,805,4$; that is, the tangent of an arch of 3 degrees will be nearly equal to $0.052,407,779,283,041,204,038,805,4$. Q. E. I.

This value of t, or the tangent of 3 degrees, is exact in the first twenty-four places of decimals, to wit, in the figures $0.052,407,779,283,041,204,038,805$; as is shewn by Dr. Mackay in the following manner.

A Proof

A Proof of the Exactness of the foregoing Number 0.052,407,779,283,041,204,038,805, *found in the foregoing Article for the Value of the Tangent of an Arch of* 3 *Degrees.*

Art. 32. Dr. Mackay takes the above-found number to all its twenty-four places of decimals, viz. 0.052,407,779,283,041,204,038,805, and ſubſtitutes it, inſtead of t, in the trinomial quantity $3t+3Tt^2-t^3$, in order to diſcover whether the value of the ſaid quantity ariſing from ſuch ſubſtitution will be greater, or will be leſs, than 0.158,384,440,324,536,293,838,883, or the abſolute term of the propoſed equation $3t+3Tt^2-t^3 = T$; and conſequently to determine whether the ſaid ſubſtituted number 0.052,407,779,283,041,204,038,805, is greater, or is leſs, than the true value of t, or the leaſt root of that equation. And in order to make this ſubſtitution, he raiſes the powers of the ſaid number 0.052,407,779,283,041,204,038,805 in the following manner.

t = 0.052,407,779,283,041,204,038,805
t = 0.052,407,779,283,041,204,038,805

0.002 620 388 964 152 060 201 940
104 815 558 566 082 408 078
20 963 111 713 216 481 615
366 854 454 981 288 428
36 685 445 498 128 843
3 668 544 549 812 884
471 670 013 547 371
10 481 555 856 608
4 192 622 342 643
157 223 337 849
2 096 311 171
52 407 779
10 481 556
209 631
1 572
419
42

---

t^2 = 0.002,746,575,329,379,962,818,429

t =

$$t^2 = 0.002,746,575,329,379,962,818,429$$
$$t = 0.052,407,779,283,041,204,038,805$$

0.000,137 328 766 468 998 140 921
5 493 150 658 759 925 637
1 098 630 131 751 985 127
19 226 027 305 659 740
1 922 602 730 565 974
192 260 273 056 597
24 719 177 964 420
549 315 065 876
219 726 026 350
8 239 725 988
109 863 013
2 746 575
549 315
10 986
82
22
2

$$t^3 = 0.000,143,941,913,646,391,286,625$$

$$3T = 0.475,153,320,973,608,881,516,649$$
$$t^2 = 0.002,746,575,329,379,962,818,429$$

0.000 950 306 641 947 217 763 033
332 607 324 681 526 217 062
19 006 132 838 944 355 261
2 850 919 925 841 653 289
237 576 660 486 804 441
33 260 732 468 152 622
2 375 766 604 868 044
142 545 996 292 083
9 503 066 419 472
4 276 379 888 762
142 545 996 292
33 260 732 468
4 276 379 889
427 637 989
28 509 199
950 307
380 123
4 752
3 801
190
9
4

$$3Tt^2 = 0.001,305,044,389,059,073,009,092$$

Therefore, if 0.052,407,779,283,041,204,038,805, be ſubſtituted inſtead of t in the trinomial quantity $3t+3Tt^2-t^3$, we ſhall have

$3t$ (= 3 × 0.052,407,779,283,041,204,038,805) = 0.157,223,337,849,123,612,116,415;

and $3t+3Tt^2$ (= 0.157,223,337,849,123,612,116,415 + 0.001,305,044,389,059,073,009,092) = 0.158,528,382,238,182,685,125,507; and, conſequently the whole trinomial quantity $3t+3Tt^2-t^3$ will be =

$$\left\{\begin{array}{ll} & 0.158,528,382,238,182,685,125,507 \\ - & 0.000,143,941,913,646,391,286,625 \end{array}\right\}$$

= 0.158,384,440,324,536,293,838,882; which is leſs than 0.158,384,440,324,536,293,838,883, or the abſolute term of the equation $3t+3Tt^2-t^3 = T$, by the very ſmall difference of 0.000,000,000,000,000,000,000,001, or an unit in the 24th place of decimal fractions. Therefore 0.052,407,779,283,041,204,038,805 is leſs than the true value of t, or the leſſer root of the ſaid equation, or than the tangent of an arch of 3 degrees.

Q. E. I.

Art. 33. But, though this number 0.052,407,779,283,041,204,038,805 is leſs than the true value of t, or the leſſer root of the ſaid equation $3t+3Tt^2-t^3 = T$, or than the tangent of 3 degrees, yet it will be nearer to the ſaid true value than any greater number expreſſed in the ſame number of places of decimal fractions. For, if we increaſe this number by only a ſingle unit in the laſt, or 24th, place of figures, or by the very ſmall decimal fraction 0.000,000,000,000,000,000,001, the number thence ariſing, to wit, 0.052,407,779,283,041,204,038,806, will be greater than the ſaid true value. This Dr. Mackay proves in the following manner.

To avoid the repetition of the great number of figures contained in the number 0.052,407,779,283,041,204,038,805, he denotes it by the letter a; and then the next greater number conſiſting of the ſame number of figures, to wit, the number 0.052,407,779,283,041,204,038,806, will be = a + 0.000,000,000,000,000,000,000,001, or $a + \frac{1}{10^{24}}$. Now, if this binomial quantity be ſubſtituted inſtead of t in the trinomial quantity $3t+3T\times t^2-t^3$, we ſhall have

$$3t\left(=3\times\overline{a+\frac{1}{10|^{24}}}\right)=3a+\frac{3}{10|^{24}},$$

$$\text{and } t^2\left(=\overline{a+\frac{1}{10|^{24}}}^2\right)=a^2+2a\times\frac{1}{10|^{24}}+\frac{1}{10|^{48}},$$

$$\text{and } t^3\left(=\overline{a+\frac{1}{10|^{24}}}^3\right)=a^3+3a^2\times\frac{1}{10|^{24}}+3a\times\frac{1}{10|^{48}}+\frac{1}{10|^{72}},$$

$$\text{and } 3\mathrm{T}\times t^2\left(=3\mathrm{T}\times a^2+2a\times\frac{1}{10|^{24}}+\frac{1}{10|^{48}}\right)=3\mathrm{T}\times a^2+3\mathrm{T}\times 2a\times\frac{1}{10|^{24}}+3\mathrm{T}\times\frac{1}{10|^{48}}.$$

Therefore the whole trinomial quantity $3t+3\mathrm{T}\times t^2-t^3$ will be equal to the following multinomial quantity, to wit,

$$\left\{\begin{array}{l} 3a+\frac{3}{10|^{24}} \\ +\ 3\mathrm{T}\times a^2+3\mathrm{T}\times 2a\times\frac{1}{10|^{24}}+3\mathrm{T}\times\frac{1}{10|^{48}} \\ -\ a^3-3a^2\times\frac{1}{10|^{24}}-3a\times\frac{1}{10|^{48}}-\frac{1}{10|^{72}}. \end{array}\right\}$$

Now of theſe ſeveral terms there are only four that will enter into the 24th, or any higher, place of decimal fractions; which are the four terms $3a+3\mathrm{T}\times a^2-a^3+\frac{3}{10|^{24}}$. For, ſince a is $=0.052,407$, &c, and conſequently is leſs than 0.06, and $3\mathrm{T}$ is $=0.475,153$, &c, and conſequently is leſs than 0.5, it follows that $3\mathrm{T}\times 2a\times\frac{1}{10|^{24}}$ will be leſs than $0.5\times 2\times 0.06\times\frac{1}{10|^{24}}$ (or than $1\times 0.06\times\frac{1}{10|^{24}}$, or than $\frac{6}{100}\times\frac{1}{10|^{24}}$, or than $\frac{6}{10|^{2}}\times\frac{1}{10|^{24}}$, or than $\frac{6}{10|^{26}}$,) or than 6 in the 26th place of decimal fractions; and that $3\mathrm{T}\times\frac{1}{10|^{48}}$ will be leſs than $0.5\times\frac{1}{10|^{48}}$, (or than $\frac{5}{10|}\times\frac{1}{10|^{48}}$, or than $\frac{5}{10|^{49}}$,) or than 5 in the 49th place of decimal fractions; and that $3a^2\times\frac{1}{10|^{24}}$ will be leſs than $3\times 0.0036\times\frac{1}{10|^{24}}$, (or than $0.0108\times\frac{1}{10|^{24}}$, or than $\frac{108}{10,000}\times\frac{1}{10|^{24}}$, or than $\frac{108}{10|^{4}}\times\frac{1}{10|^{24}}$,) or than $\frac{108}{10|^{28}}$, of which the higheſt figure 1 would

would enter only in the 26th place of decimal fractions; and that $3a \times \frac{1}{10|^{48}}$ will be leſs than $3 \times 0.06 \times \frac{1}{10|^{48}}$,) or than $0.18 \times \frac{1}{10|^{48}}$, or than $\frac{18}{100} \times \frac{1}{10|^{48}}$, or than $\frac{18}{10|^{2}} \times \frac{1}{10|^{48}}$, (or than $\frac{18}{10|^{50}}$, of which the higheſt figure 1 would enter only in the 49th place of decimal fractions; and, laſtly, it is evident $\frac{1}{10|^{72}}$ will enter only in the 72d place of decimal fractions. All theſe terms therefore will not affect the figures of the firſt 24 places of decimal fractions, and conſequently may, on the preſent occaſion, be neglected. Therefore the trinomial quantity $3t+3T\times t^2-t^3$, reſulting from this ſubſtitution of $a+\frac{1}{10|^{24}}$, or 0.052,407,779,283,041,204,038,806, inſtead of t, in the ſaid quantity, may be conſidered as being equal to the four terms $3a+3T\times a^2-a^3 + \frac{3}{10|^{24}}$. But it has been ſhewn already in Art. 32, that $3a+3T\times a^2-a^3$ is $=$ 0.158,384,440,324,536,293,838,882. Therefore the four terms $3a+3T \times a^2-a^3+\frac{3}{10|^{24}}$ will be $=$ 0.158,384,440,324,536,293,838,882 $+ \frac{3}{10|^{24}}$ ($=$ 0.158,384,440,324,536,293,838,882 + 0.000,000,000,000,000,000,000,003) $=$ 0.158,384,440,324,536,293,838,885; that is, the value of the trinomial quantity $3t+3T\times t^2-t^3$ reſulting from the ſubſtitution of 0.052,407,779,283,041,204,038,806 in its terms inſtead of t, will be equal to 0.158,384,440,324,536,293,838,885; which is greater than 0.158,384,440,324,536,293,838,883, or T, or the abſolute term of the equation $3t+3T\times t^2-t^3=T$. And conſequently the ſaid number 0.052,407,779,283,041,204,038,806 is greater than the true value of t, or the leſſer root of the equation $3t+3T\times t^2-t^3=T$, or than the true value of the tangent of an arch of 3 degrees. Therefore the ſaid tangent is of an intermediate magnitude between 0.052,407,779,283,041,204,038,805, and 0.052,407,779,283,041,204,038,806, and all the 24 figures of the former of theſe numbers are conſequently exact. Q. E. D.

End of the Triſection of an Arch of 9 *Degrees by reſolving the Cubick Equation* $3t+3T\times t^2-t^3=T$, *in which* T *is the tangent of* 9 *degrees, and is* $=$ 0.158,384,440,324,536,293,838,883, *and* t, *or the leſſer Root of the ſaid Equation, is the Tangent of an Arch of* 3 *Degrees, the Radius of the Circle being called* 1.

The Trisection of an Arch of 3 *Degrees by resolving the Cubick Equation* $3t+3T\times t^2-t^3=T$, *in which* T *is the Tangent of* 3 *Degrees, and is* = 0.052,407,779,283, 041,204,038,805, *and* t, *or the lesser Root of the said Equation, is the Tangent of an Arch of* 1 *Degree.*

Art. 34. Dr. Mackay begins this computation by obſerving that the tangent of an arch of one degree is leſs than the third part of the tangent of an arch of three degrees; but greater than the length of an arch of one degree, which is the 180th part of the ſemi-circumference of the circle, or (when the radius is called 1, as it is in the preſent computation,) the 180th part of the number 3.141,592, &c; but that it will be nearer to this latter quantity than to the former. Therefore this tangent will be leſs than $\frac{0.052,407}{3}$, or 0.017,469, but greater than $\frac{3.141,592}{180}$, or 0.017,453, and leſs than $\frac{\frac{0.017,469+0.017,453}{2}}{2}$, or than $\frac{0.034,922}{2}$, or 0.017,461. And therefore Dr. Mackay conjectures that the tangent of an arch of one degree will be = 0.017,455, and he ſubſtitutes this number for t in the trinomial quantity $3t+3T\times t^2-t^3$, which forms the firſt, or left-hand, ſide of the equation $3t+3T\times t^2-t^3=T$, which is to be reſolved; and he compares the reſult of this ſubſtitution with the abſolute term T, or 0.052,407,779,283,041,204,038,805, of the ſaid equation, in order to diſcover whether the ſaid reſult will be greater, or will be leſs, than the ſaid abſolute term, and conſequently whether the ſaid quantity 0.017,455 is greater, or is leſs, than the true value of t, or the leſſer root of the ſaid equation. And for this purpoſe Dr. Mackay raiſes the powers of the number 0.017,455, and likewiſe computes the value of 3T, or the co-efficient of t^2 in the ſecond term of the ſaid equation $3t+3T\times t^2-t^3=T$, and multiplies it into the ſquare of 0.017,455, or t. This he does in the following manner.

Since T is = 0.052,407,779,283,041,204,038,805, it follows that 3T will be (= 3×0.052,407,779,283,041,204,038,805) = 0.157,223,337,849,123, 612,116,415.

t =

$t = 0.017,455$
$t = 0.017,455$

87 275
87275
69820
122185
17455

$t^2 = 0.000,304,677,025$
$t = 0.017,455$

1 523 385 125
15 233 851 25
121 870 8100
2 132 739 175
3 046 770 25

$t^3 = 0.000,005,318,137,471,375$

$3T = 0.157,223,337,849,123,612,116,415$
$t^2 = 0.000,304,677,025$

0.000 047 167 001 354 737 083 635
628 893 351 396 494 448
94 334 002 709 474 167
11 005 633 649 438 653
1 100 563 364 943 865
3 144 466 756 983
786 116 689 246

$3Tt^2 = 0.000,047,902,338,836,440,880,997$

Therefore if 0.017,455 be fubſtituted, inſtead of t, in the trinomial quantity $3t + 3Tt^2 - t^3$; we ſhall have

$3t\ (= 3 \times 0.017,455) = 0.052,365$;

and $3t + 3Tt^2\ (= 0.052,365 + 0.000,047,902,338,836,440,880,997)$
$= 0.052,412,902,338,836,440,880,997$; and therefore the whole trinomial quantity $3t + 3Tt^2 - t^3$ will be equal to

$$\left\{ \begin{array}{l} + \ 0.052,412,902,338,836,440,880,997 \\ - \ 0.000,005,318,137,471,375, \end{array} \right\}$$

$= 0.052,407,584,201,365,065,880,997$; which is leſs than 0.052,407,779,283,041,204,038,805, or the abſolute term of the equation $3t + 3Tt^2 - t^3 = T$; and therefore 0.017,455 is leſs than the true value of t, or the leſſer root of the ſaid equation. Q. E. I.

A Process

A Process of Mr. Raphson's Method of Approximation.

Art. 35. Having thus discovered that 0.017,455 is less than the true value of t, Dr. Mackay puts z for the excess of t above 0.017,455, and, in order to obtain a more exact value of t, he substitutes the binomial quantity $0.017,455+z$ instead of t, in the equation $3t+3Tt^2-t^3 = T$. This he does in the following manner.

Since t is $= 0.017,455+z$, we shall have

$t^2 = \overline{0.017,455+z}^2$ $(= \overline{0.017,455}^2 + 2\times 0.017,455\times z +$ &c
$= \overline{0.017,455}^2 + 0.034,910\times z +$ &c$) = 0.000,304,677,025 +$
$0.034,910\times z +$ &c,

and $t^3 = \overline{0.017,455+z}^3$ $(= \overline{0.017,455}^3 + 3\times\overline{0.017,455}^2\times z +$ &c
$= \overline{0.017,455}^3 + 3\times 0.000,304,677,025\times z +$ &c
$= \overline{0.017,455}^3 + 0.000,914,031,075\times z +$ &c$)$
$= 0.000,005,318,137,471,375 + 0.000,914,031,075\times z +$ &c.

Therefore $3t$ will be $(= 3\times\overline{0.017,455+z}) = 0.052,365+3\times z +$ &c,
and $3T\times t^2$ will be $(= 0.157,223,337,849,123,612,116,415 \times$
$\overline{0.000,304,677,025+0.034,910\times z +}$ &c
$(= 0.157,223,337,849,123,612,116,415 \times 0.000,304,$
$677,025 + 0.005,488,666,724,312,905,298,984\times z$
$+$ &c$)$
$= 0.000,047,902,338,836,440,880,997 + 0.005,488,$
$666,724,312,905,298,984\times z +$ &c.

Therefore the whole trinomial quantity $3t+3Tt^2-t^3$ will be equal to the quantity

$$\left\{\begin{array}{l} +\ 0.052,365, \\ +\ 0.000,047,902,338,836,440,880,997 \\ -\ 0.000,005,318,137,471,375 \end{array}\right. \left\{\begin{array}{l} +\ 3\times z +\ \&c \\ +\ 0.005,488,666,724,312,905, \\ \quad 298,984\times z +\ \&c \\ -\ 0.000,914,031,075\times z +\ \&c \end{array}\right.$$

$= 0.052,407,584,201,365,065,880,997 + 3.004,574,635,649,312,905,$
$298,984\times z.$

But

But the trinomial quantity $3t+3Tt^2-t^3$ is equal to 0.052,407,779,283,041, 204,038,805, or T, the abſolute term of the equation $3t+3T\times t^2-t^3 = T$, Therefore the quantity 0.052,407,584,201,365,065,880,997 + 3.004,574, 635,649,312,905,298,984 $\times z$ will alſo be equal to 0.052,407,779,283,041, 204,038,805, and conſequently 3.004,574,635,649,312,905,298,984 $\times z$ will be equal to

$$\left\{\begin{array}{l} + \ 0.052,407,779,283,041,204,038,805 \\ - \ 0.052,407,584,201,365,065,880,997 \end{array}\right\}$$

$= 0.000,000,195,081,676,138,157,808$, and z will be $= \frac{0.000,000,195,081,676,138,157,808}{3.004,574,635,649,312,905,298,984} = 0.000,000,064,928,217,733$. Therefore t, or $0.017,455 + z$, will be equal to $(0.017,455 + 0.000,000,064,928,217,733)$ $= 0.017,455,064,928,217,733$, that is, the tangent of 1 degree, or the leaſt root of the equation $3t+3Tt^2-t^3 = T$, will be nearly equal to 0.017,455, 064,928,217,733. Q. E. I.

Art. 36. Having thus obtained the number 0.017,455,064,928,217,773 for a ſecond near value of t, or the tangent of an arch of 1 degree, of which it is probable the firſt twelve figures, viz. 0.017,455,064,928, are correct, Dr. Mackay proceeds to ſubſtitute thoſe twelve figures inſtead of t in the trinomial quantity $3t+3Tt^2-t^3$, in order to diſcover whether the value of the quantity reſulting from this ſubſtitution will be greater, or will be leſs, than 0.052,407, 779,283,041,204,038,805, or T, the abſolute term of the propoſed equation $3t+3T\times t^2-t^3 = T$; and conſequently whether the ſaid number 0.017,455,064, 928, will be greater, or will be leſs, than the true value of t, or the leſſer root of that equation. And for this purpoſe Dr. Mackay raiſes the powers of the number 0.017,455,064,928 in the following manner.

$t =$

$t = 0.017,455,064,928$
$t = 0.017,455,064,928$

139 640 519 424
349 101 298 56
15 709 558 435 2
69 820 259 712
1 047 303 895 68
87 275 324 640 0
872 753 246 40
6 982 025 971 2
122 185 454 496
174 550 649 28

$t^2 = 0.000,304,679,291,640,695,645,184$
$t = 0.017,455,064,928,$

0.000 003 046 792 916 406 956 452
2 132 755 041 484 869 516
121 871 716 656 278 258
15 233 964 582 034 782
1 523 396 458 203 478
18 280 757 498 442
1 218 717 166 563
274 211 362 477
6 093 585 833
2 437 434 333

$t^3 = 0.000,005,318,196,817,805,390,134$

$= \frac{1}{}$

$3T = 0.157,223,337,849,123,612,116,415$
$t^2 = 0.000,304,679,291,640,695,645,184$

0.000 047 167 001 354 737 083 635
628 893 351 396 494 448
94 334 002 709 474 167
11 005 633 649 438 653
1 415 010 040 642 112
31 444 667 569 825
14 150 100 406 421
157 223 337 849
94 334 002 709
6 288 933 514
94 334 003
14 150 100
786 117
94 334
6 289
786
16
13
1

$3T \times t^2 = 0.000,047,902,695,205,256,754,992$

Therefore, if the number 0.017,455,064,928 be ſubſtituted inſtead of t in the trinomial quantity $3t + 3Tt^2 - t^3$, we ſhall have

$3t$ $(= 3 \times 0.017,455,064,928) = 0.052,365,194,784,$

and $3t + 3Tt^2$ $(= 0.052,365,194,784 + 0.000,047,902,695,205,256,754,992)$
$= 0.052,413,097,479,205,256,754,992$; and therefore the whole trinomial quantity $3t + 3Tt^2 - t^3$ will be $=$

$$\left\{\begin{array}{l} + \ 0.052,413,097,479,205,256,754,992 \\ - \ 0.000,005,318,196,817,805,390,134 \end{array}\right\}$$

$= 0.052,407,779,282,387,451,364,858$, which is leſs than 0.052,407,779,283,041,204,038,805, or the abſolute term of the equation $3t + 3Tt^2 - t^3 = T$. And therefore 0.017,455,064,928 is leſs than the true value of t, or the leſſer root of the ſaid equation. Q. E. I.

A Second Process of Mr. Raphson's Method of Approximation.

Art. 37. Having discovered that 0.017,455,064,928, is less than the true value of t, Dr. Mackay puts z for the excess of t above 0.017,455,064,928; and, in order to obtain a more exact value of t, he substitutes the binomial quantity $0.017,455,064,928 + z$, instead of t, in the equation $3t + 3T \times t^2 - t^3 = T$. This he does in the following manner.

Since t is $= 0.017,455,064,928 + z$, we shall have

$t^2 = \overline{0.017,455,064,928 + z}^2$ $(= \overline{0.017,455,064,928}^2 + 2 \times 0.017,455,064,928 \times z + \&c$

$= \overline{0.017,455,064,928}^2 + 0.034,910,129,856 \times z + \&c)$

$= 0.000,304,679,291,640,695,645,184 + 0.034,910,129,856 \times z + \&c,$

and $t^3 = \overline{0.017,455,064,928 + z}^3$ $(= \overline{0.017,455,064,928}^3 + 3 \times \overline{0.017,455,064,928}^2 \times z$

$= \overline{0.017,455,064,928}^3 + 3 \times 0.000,304,679,291,640,695,645,184 \times z$

$= \overline{0.017,455,064,928}^3 + 0.000,914,037,874,922,086,935,552 \times z + \&c)$

$= 0.000,005,318,196,817,805,390,134 + 0.000,914,037,874,922,086,935,552 \times z + \&c.$

Therefore $3t$ will be $= 3 \times \overline{0.017,455,064,928 + z}$,

$= 0.052,365,194,784 + 3 \times z + \&c,$

and $3T \times t^2$ will be $= 0.157,223,337,849,123,612,116,415 \times \overline{0.000,304,679,291,640,695,645,184 + 0.034,910,129,856 \times z + \&c}$

$(= 0.157,223,337,849,123,612,116,415 \times 0.000,304,679,291,640,695,645,184 + 0.005,488,687,140,706,665,034,780 \times z)$

$= 0.000,047,902,695,205,256,754,992 + 0.005,488,687,140,706,665,034,780 \times z.$

Therefore

Therefore the whole trinomial quantity $3t+3T\times t^2-t^3$ will be equal to the quantity

$$\left\{\begin{array}{ll} +\ 0.052,365,194,784 & +\ 8\times z, \\ +\ 0.000,047,902,695,205,256,754,992 & +\ 0.005,488,687,140,706,665,034,780\times z \\ -\ 0.000,005,318,196,817,805,390,134 & -\ 0.000,914,037,874,922,086,935,552\times z \end{array}\right\}$$

$= 0.052,407,779,282,387,451,364,858 + 3.004,574,649,275,784,578,099,228\times z$

But the trinomial quantity $3t+3Tt^2-t^3$ is equal to 0.052,407,779,283,041,204,038,805, or T, the abſolute term of the equation $3t+3Tt^2-t^3 = T$. Therefore the quantity $0.052,407,779,282,387,451,364,858 + 3.004,574,649,275,784,578,099,228\times z$, will alſo be equal to 0.052,407,779,283,041,204,038,805, and conſequently $3.004,574,649,275,784,578,099,228\times z$ will be equal to

$$\left\{\begin{array}{ll} +\ 0.052,407,779,283,041,204,038,805 \\ -\ 0.052,407,779,282,387,451,364,858 \end{array}\right\}$$

$= 0.000,000,000,000,653,752,673,947$. And z will be $=$

$\frac{0.000,000,000,000,653,752,673,947}{3.004,574,649,275,784,578,099,228} = 0.000,000,000,000,217,585,765,126$. Therefore t, or $0.017,455,064,928+z$, will be equal to $(0.017,455,064,928 + 0.000,000,000,000,217,585,765,126) = 0.017,455,064,928,217,585,765,126$; that is, the tangent of one degree, or the leſſer root of the equation $3t+3Tt^2-t^3 = T$, will be nearly equal to 0.017,455,064,928,217,585,765,126.

Q. E. I.

Art. 38. Having thus obtained the number 0.017,455,064,928,217,585,765,126, for a third near value of t, or the tangent of an arch of one degree, of which it is probable that all the figures are correct, Dr. Mackay proceeds to ſubſtitute it, inſtead of t, in the trinomial quantity $3t+3Tt^2-t^3$, in order to diſcover whether the value of that quantity reſulting from the ſaid ſubſtitution, will be greater, or will be leſs, than 0.052,407,779,283,041,204,038,805, or the abſolute term of the equation $3t+3Tt^2-t^3 = T$, and conſequently to determine, whether the ſaid ſubſtituted number, 0.017,455,064,928,217,585,765,126, will be greater, or will be leſs, than the true

value of t, or the leſſer root of that equation. In order to this he raiſes the powers of 0.017,455,064,928,217,585,765,126, in the following manner.

t = 0.017,455,064,928,217,585,765,126
t = 0.017,455,064,928,217,585,765,126

0.000 174 550 649 282 175 857 651
122 185 454 497 523 100 356
6 982 025 971 287 034 306
872 753 246 410 879 288
87 275 324 641 087 929
1 047 303 895 693 055
69 820 259 712 870
15 709 558 435 396
349 101 298 564
139 640 519 426
3 491 012 986
174 550 649
122 185 454
8 727 532
1 396 405
87 275
12 219
1 047
87
2

t^3 = 0.000,304,679,291,648,291,592,497

$t^2 =$ 0.000,304,679,291,648,291,592,497
$t =$ 0.017,455,064,928,217,585,765,126

0.000 003 046 792 916 482 915 925
2 132 755 041 538 041 147
121 871 716 659 316 637
15 233 964 582 414 586
1 523 396 458 241 458
18 280 757 498 897
1 218 717 166 593
274 211 362 483
6 093 585 833
2 437 434 333
60 935 858
3 046 793
2 132 755
152 340
24 374
1 523
213
18
2

$t^3 =$ 0.000,005,318,196,818,004,271,768

$3T =$ 0.157,223,337,849,123,612,116,415
$t^2 =$ 0.000,304,679,291,648,291,592,497

0.000 047 167 001 354 737 083 635
628 893 351 396 494 448
94 334 002 709 474 167
11 005 633 649 438 653
1 415 010 040 642 112
31 444 667 569 825
14 150 100 406 421
157 223 337 849
94 334 002 709
6 288 933 514
1 257 786 703
31 444 668
14 150 100
157 223
78 612
14 150
314
63
14
1

$3T \times t^2 =$ 0.000,047,902,695,206,451,015,181

Therefore if 0.017,455,064,928,217,585,765,126 be ſubſtituted inſtead of t, in the trinomial quantity $3t + 3Tt^2 - t^3$, we ſhall have

$3t$ ($= 3 \times 0.017,455,064,928,217,585,765,126$) $=$ 0.052,365,194,784,652,757,295,378;

and $3t + 3Tt^2$ ($=$ 0.052,365,194,784,652,757,295,378 + 0.000,047,902,695,206,451,015,181) $=$ 0.052,413,097,479,859,208,310,559. And therefore the whole trinomial quantity $3t + 3Tt^2 - t^3$ will be equal to

$$\left\{\begin{array}{ll} + & 0.052,413,097,479,859,208,310,559 \\ - & 0.000,005,318,196,818,004,271,768 \end{array}\right\}$$

$=$ 0.052,407,779,283,041,204,038,791; which is leſs than 0.052,407,779,283,041,204,038,805, or the abſolute term of the equation $3t + 3Tt^2 - t^3 = T$; and therefore 0.017,455,064,928,217,585,765,126 will be leſs than the true value of t, or the leſſer root of the ſaid equation.

Q. E. I.

A Third Process of Mr. Raphson's Method of Approximation.

Art. 39. Since t, or the leſſer root of the equation $3t + 3T \times t^2 - t^3 = T$, or $3t + 3T \times t^2 - t^3 =$ 0.052,407,779,283,041,204,038,805, is greater than 0.017,455,064,928,217,585,765,126, let z be put for their difference. And we ſhall then have $t =$ 0.017,455,064,928,217,585,765,126 $+ z$; or, if, to avoid the repetition of this long number, we denote it by the letter a, we ſhall have $t = a + z$.

Therefore t^2 will be ($= \overline{a+z}^2$) $= a^2 + 2az$ + &c,

and t^3 will be ($= \overline{a+z}^3$) $= a^3 + 3a^2z$ + &c,

and $3T \times t^2$ will be ($= 3T \times \overline{a^2 + 2az + \&c}$) $= 3T \times a^2 + 3T \times 2az +$ &c, and the whole trinomial quantity $3t + 3T \times t^2 - t^3$ will be equal to the multinomial quantity

$$\left\{\begin{array}{lll} & 3a & + \ 3z \\ + & 3Ta^2 & + \ 3T \times 2az + \&c \\ - & a^3 & - \ 3a^2 \times z - \&c \end{array}\right\}$$

But

But the trinomial quantity $3t+3T\times t^2-t^3$ is $=$ 0.052,407,779,283,041,204,038,805. Therefore the multinomial quantity $3a+3Ta^2-a^3+3z+3T\times 2az+$ &c $-3a^2z-$ &c will alſo be $=$ 0.052,407,779,283,041,204,038,805.

But it has been ſhewn in the laſt article that the trinomial quantity $3a+3Ta^2-a^3$ is $=$ 0.052,407,779,283,041,204,038,791.

Therefore the multinomial quantity 0.052,407,779,283,041,204,038,791 $+3z+3T\times 2az+$ &c $-3a^2z-$ &c will be $=$ 0.052,407,779,283,041,204,038,805. And conſequently $3z+3T\times 2az+$ &c $-3a^2z-$ &c will be $(=$ 0.052,407,779,283,041,204,038,805 $-$ 0.052,407,779,283,041,204,038,791) $=$ 0.000,000,000,000,000,000,000,014; that is, $z\times 3+3T\times 2a-3a^2$ &c will be $=$ 0.000,000,000,000,000,000,000,014, and therefore z will be $=$ $\frac{0.000,000,000,000,000,000,000,014}{3+3T\times 2a-3a^2}$.

But a is $=$ 0.017,455,&c, and a^2 is $=$ 0.000,304,&c, and $3T$ is $=$ 0.157,223,&c. Therefore $3a^2$ is $(=3\times$ 0.000,304,&c$)=$ 0.000,912,&c, and $3T\times 2a$ is $(=$ 0.157,223,&c $\times$ 2$\times$0.017,455,&c $=$ 0.157,223,&c $\times$ 0.034,910$)$ $=$ 0.005,488,654,930, and $3+3T\times 2a-3a^2$ is $(=3+$ 0.005,488,654,930 $-$ 0.000,912 $=$ 3.005,488,654,930 $-$ 0.000,912 $=$ 3.004,576,654,930$)=$ 3.004,576,&c. Therefore the fraction $\frac{0.000,000,000,000,000,000,000,014}{3+3T\times 2a-3a^2}$ will be $=$ $\frac{0.000,000,000,000,000,000,000,014}{3.004,576}=$ 0.000,000,000,000,000,000,000,004,659.

Therefore z will be $=$ 0.000,000,000,000,000,000,000,004,659, and conſequently $a+z$, or t, will be $=a+$ 0.000,000,000,000,000,000,000,004,659 $(=$ 0.017,455,064,928,217,585,765,126 $+$ 0.000,000,000,000,000,000,000,004,659$)=$ 0.017,455,064,928,217,585,765,130,659, or the true value of t, or the leſſer root of the equation $3t+3T\times t^2-t^3=T$, or $3t+3T\times t^2-t^3=$ 0.052,407,779,283,041,204,038,805, or the true value of the tangent of an arch of 1 degree, will be $=$ 0.017,455,064,928,217,585,765,130,659; which (if no miſtake has been made in the operations of the calculation,) will be exact in all its twenty-ſeven places of figures, and may therefore be depended-upon as exact in the firſt 24 places of figures, to wit, in the figures 0.017,455,064,928,217,585,765,130, which are all that Dr. Mackay makes uſe of in the following part of this extenſive computation. Q. E. I.

End of the Triſection of an Arch of 3 Degrees.

The Quinquisection of an Arch of 1 *Degree, or* 60 *Minutes, by resolving the Equation* $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$, *in which* T *denotes the Tangent of an Arch of* 1 *Degree, and* t *denotes the Tangent of an Arch that is the fifth Part of* 1 *Degree, or an Arch of* 12 *Minutes, the Tangent* T *of the Arch of* 1 *Degree, being equal to the number* 0.017,455,064,928,217,585,765,13, *when the Radius of the Circle is called* 1.

Art. 40. Dr. Mackay begins his reſolution of this equation by obſerving that the tangent of twelve minutes, or the fifth part of an arch of 1 degree, is leſs than one fifth part of the tangent of one degree, and greater than the length of a circular arch of 12 minutes, but that it is neareſt to this laſt quantity; that is, the tangent of 12 minutes is leſs than $\frac{0.017,455,06}{5}$ or 0.003,491,01; and greater than $\frac{3.141,592,65}{180 \times 5}$, or 0.003,490,65; but, being neareſt to this laſt quantity, it is conſequently leſs than $\frac{0.003,491,01 + 0.003,490,65}{2}$, or than $\frac{0.006,981,66}{2}$, or 0.003,490,83. And he therefore conjectures that the tangent of twelve minutes is = 0.003,490,67; and ſubſtitutes this number for t, in the equation $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$, and compares the reſult with the abſolute term T, or 0.017,455,064,928,217,585,765,13, in order to diſcover whether 0.003,490,67 is greater, or is leſs, than the true value of t, or the leaſt root of the ſaid equation. And, for this purpoſe, Dr. Mackay raiſes the powers of 0.003,490,67, in the following manner.

$t = 0.003,490,67$
$t = 0.003,490,67$

244 346 9
2 094 402
314 160 30
1 396 268
10 472 01

$t^2 = 0.000,012,184,777,048,9$
$t = 0.003\ 490\ 67$

$852\ 934\ 393\ 42^3$
$7\ 3^{10}\ 866\ 229\ 34$
$1\ 096\ 629\ 934\ 401$
$4\ 873\ 910\ 819\ 56$
$36\ 554\ 331\ 146\ 7$

$t^3 = 0.000,000,042,533,035,701,283,763$
$t = 0.003,490,67$

$0.000\ 000\ 000\ 127\ 599\ 107\ 103\ 851$
$17\ 013\ 214\ 280\ 513$
$3\ 827\ 973\ 213\ 116$
$25\ 519\ 821\ 421$
$2\ 977\ 312\ 499$

$t^4 = 0.000,000,000,148,468,791,731,400$
$t = 0.003,490,67$

$0.000\ 000\ 000\ 000\ 445\ 406\ 375\ 194$
$59\ 387\ 516\ 693$
$13\ 362\ 191\ 256$
$89\ 081\ 275$
$10\ 392\ 815$

$t^5 = 0.000,000,000,000,518,255,557,233$

$$\begin{array}{rl} 10T = & 0.174,550,649,282,175,857,651,3 \\ t^2 = & 0.000,012,184,777,048,9 \\ \hline & 0.000\,001\,745\,506\,492\,821\,758\,576\,5 \\ & 349\,101\,298\,564\,351\,715\,3 \\ & 17\,455\,064\,928\,217\,585\,8 \\ & 13\,964\,051\,942\,574\,068\,6 \\ & 698\,202\,597\,128\,703\,4 \\ & 122\,185\,454\,497\,523\,1 \\ & 12\,218\,545\,449\,752\,3 \\ & 1\,221\,854\,544\,975\,2 \\ & 6\,982\,025\,971\,3 \\ & 1\,396\,405\,194\,3 \\ & 157\,095\,584\,3 \\ \hline 10T \times t^2 = & 0.000,002,126,860,745,244,049,651,0 \end{array}$$

$$\begin{array}{rl} 5T = & 0.087,275,324,641,087,928,825,65 \\ t^4 = & 0.000,000,000,148,468,791,731,4 \\ \hline & 0.000\,000\,000\,008\,727\,532\,464\,108\,8 \\ & 3\,491\,012\,985\,643\,5 \\ & 698\,202\,597\,128\,7 \\ & 34\,910\,129\,856\,4 \\ & 5\,236\,519\,478\,4 \\ & 698\,202\,597\,1 \\ & 61\,092\,727\,2 \\ & 7\,854\,779\,2 \\ & 872\,753 \\ & 61\,092\,7 \\ & 2\,618\,3 \\ & 873 \\ & 349 \\ \hline 5T \times t^4 = & 0.000,000,000,012,957,661,997,427,8 \end{array}$$

Therefore if 0.003,490,67 be fubſtituted, inſtead of t, in the quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$, we ſhall have

$5t$ (= 5×0.003,490,67) = 0.017,453,35;

and $10t^3$ (= 10×0.000,000,042,533,035,701,283,763)

= 0.000,000,425,330,357,012,837,63. Therefore the quantity $5t + 10T \times t^2 + t^5$, will be equal to

$$\left\{\begin{array}{l} \text{0.017,453·35} \\ +\ \text{0.000,002,126,860,745,244,049,651} \\ +\ \text{0.000,000,000,000,518,255,557,233} \end{array}\right\}$$

$= 0.017,455,476,861,263,499,606,884.$ And the quantity $10t^3+5T\times t^4$ will be equal to

$$\left\{\begin{array}{l} \text{0.000,000,425,330,357,012,837,63} \\ \text{0.000,000,000,012,957,661,997,43} \end{array}\right\}$$

$= 0.000,000,425,343,314,674,835,06.$ Therefore the whole quinquinomial quantity $5t+10T\times t^2-10t^3-5T\times t^4+t^5$, will be equal to $0.017,455,476,861,263,499,606,88 - 0.000,000,425,343,314,674,835,06 = 0.017,455,051,517,948,824,771,82$; which is leſs than $0.017,455,064,928,217,585,765,13$, or the abſolute term of the equation $5t+10T\times t^2-10t^3-5T\times t^4+t^5 = T$. Therefore 0.003,490,67 is leſs than the true value of t, or the leaſt root of that equation. Q. E. I.

A Process of Mr. Raphson's Method of Approximation.

Art. 41. Having thus diſcovered that 0.003,490,67 is leſs than the true value of t, Dr. Mackay puts z for their unknown difference, and ſubſtitutes the binomial quantity $0.003,490,67+z$, inſtead of t, in the propoſed equation $5t+10T\times t^2-10t^3-5T\times t^4+t^5 = T$, omitting all the terms of the powers of the ſaid binomial quantity, that involve any higher powers of the unknown quantity z, than its firſt, or ſimple power, or z itſelf, agreeably to Mr. Raphſon's directions for reſolving all ſorts of equations by the method of approximation. This he does in the following manner.

Since t is $= 0.003,490,67+z$, we ſhall have

$t^2 = \overline{0.003,490,67+z}^2 (= \overline{0.003,490,67}^2 + 2\times 0.003,490,67\times z +$ &c
$= \overline{0.003,490,67}^2 + 0.006,981,34\times z +$ &c) $= 0.000,012,184,777,048,9 + 0.006,981,34\times z +$ &c,

and $t^3 = \overline{0.003,490,67+z}^3 (= \overline{0.003,490,67}^3 + 3\times\overline{0.003,490,67}^2\times z$ + &c

$= 0.003,$

$= \overline{0.003,490,67}|^3 + 3 \times 0.000,012,184,777,048\,9 \times z + \&c$
$= \overline{0.003,490,67}|^3 + 0.000,036,554,331,146,7 \times z + \&c)$
$= 0.000,000,042,533,035,701,283,763 + 0.000,036,554,331,146,7 \times z + \&c,$

and $t^4 = \overline{0.003,490,67+z}|^4$ $(= \overline{0.003,490,67}|^4 + 4 \times \overline{0.003,490,67}|^3 \times z + \&c$
$= \overline{0.003,490,67}|^4 + 4 \times 0.000,000,042,533,035,701,283,763 \times z + \&c$
$= \overline{0.003,490,67}|^4 + 0.000,000,170,132,142,805,135,052 \times z + \&c)$
$= 0.000,000,000,148,468,791,731,400 + 0.000,000,170,132,142,805,135,052 \times z + \&c,$

and $t^5 = \overline{0.003,490,67+z}|^5$ $(= \overline{0.003,490,67}|^5 + 5 \times \overline{0.003,490,67}|^4 \times z + \&c$
$= \overline{0.003,490,67}|^5 + 5 \times 0.000,000,000,148,468,791,731,400 \times z + \&c$
$= \overline{0.003,490,67}|^5 + 0.000,000,000,742,343,958,657,000 \times z + \&c)$
$= 0.000,000,000,000,518,255,557,233 + 0.000,000,000,742,343,958,657,000 \times z + \&c.$

Therefore $5t$ will be $= 5 \times \overline{0.003,490,67+z}$ $(= 5 \times 0.003,490,67 + 5z)$
$= 0.017,453,35 + 5z + \&c,$

and $10T \times t^2$ $(= 10 \times 0.017,455,064,928,217,585,765,13 \times \overline{0.000,012,184,777,048,9 + 0.006,981,34 \times z + \&c}$
$= 0.174,550,649,282,175,857,651,3 \times 0.000,012,184,777,048,9 + 0.001,218,597,429,859,625,602,055 \times z)$
$= 0.000,002,126,860,745,244,049,651,0 + 0.001,218,597,429,859,625,602,055 \times z + \&c.$

and $10t^3$ $(= 10 \times 0.000,000,042,533,035,701,283,763 + 10 \times 0.000,036,554,331,146,7 \times z + \&c)$
$= 0.000,000,425,330,357,012,837,63 + 0.000,365,543,311,467 \times z + \&c,$

and $5T \times t^4$ $(= \overline{5 \times 0.017,455,064,928,217,585,765,13 \times \overline{0.000,000,000,148,468,791,731,400 + 0.000,000,170,132,142,805,135,052 \times z + \&c}}$
$= 0.087,$

$$= 0.087,275,324,641,087,928,825,65 \times \frac{0.000,000,000,148,468,791,731,400 + 0.000,000,170,132,142,805,135,052 \times z + \&c}{}$$

$$= 0.087,275,324,641,087,928,825,65 \times 0\,000,000,000,148,468,791,731,400 + 0.000,000,014,848,337,995,202,093,6 \times z + \&c)$$

$$= 0.000,000,000,012,957,661,997,427,8 + 0.000,000,014,848,337,995,202,093,6 \times z + \&c.$$

Therefore the whole quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$, will be equal to the following multinomial quantity, to wit,

$$\left\{\begin{array}{l} + 0.017,453,35 + 5. \times z \\ + 0.000,002,126,860,745,244,049,65 + 0.001,218,597,429,859,625,602,055 \times z + \&c \\ - 0.000,000,425,330,357,012,837,63 - 0.000,365,543,311,467,000,000,000 \times z - \&c \\ - 0.000,000,000,012,957,661,997,43 - 0.000,000,014,848,337,995,202,094 \times z - \&c \\ + 0.000,000,000,000,518,255,557,23 + 0.000,000,000,742,343,958,657,000 \times z + \&c \end{array}\right\} =$$

$$\left\{\begin{array}{l} + 0.017,455,476,861,263,499,606,88 + 5.001,218,598,172,203,584,259,055 \times z + \&c \\ - 0.000,000,425,343,314,674,835,06 - 0.000,365,558,159,804,995,202,094 \times z - \&c \end{array}\right\}$$

$$= 0.017,455,051,517,948,824,771,82 + 5.000,853,040,012,398,589,056,961 \times z.$$

But the quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$ is $(= T) = 0.017,455,064,928,217,585,765,13$. Therefore the quantity 0.017,455,051,517,948,824,771,82 + 5.000,853,040,012,398,589,056,961 $\times z$ + &c, will also be $= 0.017,455,064,928,217,585,765,13$. Hence 5.000,853,040,012,398,589,056,961 $\times z$ will be $= 0.017,455,064,928,217,585,765,13 - 0.017,455,051,517,948,824,771,82 = 0.000,000,013,410,268,760,993,31$, and consequently z will be $(= \frac{0.000,000,013,410,268,760,993,31}{5.000,853,040,012,398,589,056,96} = 0.000,000,002,681,596,250,401$. Therefore t, or $0.003,490,67 + z$, will be $(= 0.003,490,67 + 0.000,000,002,681,596,250,401) = 0.003,490,672,681,596,250,401$; that is, the tangent of an arch of 12 minutes will be nearly equal to 0.003,490,672,681,596,250,401. Q. E. I.

Art. 42. Having thus obtained the number 0.003,490,672,681,596,250,401 for a ſecond near value of t, or the tangent of 12 minutes, of which it is probable the firſt fifteen figures, to wit, 0.003,490,672,681,596 are correct, Dr. Mackay proceeds to ſubſtitute thoſe fifteen figures, inſtead of t, in the quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$, in order to diſcover whether the value of the ſaid quantity reſulting from this ſubſtitution will be greater, or will be leſs, than 0.017,455,064,928,217,585,765,13, or the abſolute term of the propoſed equation $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 - t^5 = T$; and conſequently whether the ſaid number 0.003,490,672,681,596, will be greater, or will be leſs, than the true value of t, or the leſſer root of that equation. And for this purpoſe Dr. Mackay raiſes the powers of the number 0.003,490,672,681,596, in the following manner.

$t = 0.003,490,672,681,596$
$t = 0.003,490,672,681,596$

0.000 010 472 018 044 788
1 396 269 072 638 4
314 160 541 343 64
2 094 403 608 957 6
244 347 087 711 72
6 981 345 363 192
2 094 403 608 957 6
279 253 814 527 7
3 490 672 681 6
1 745 336 340 8
314 160 541 3
20 944 036 1

$t^2 = 0.000,012,184,795,770,040,609,597,1$

$t^2 = 0.000,012,184,795,770,040,609,597,1$
$t = 0.003,490,672,681,596$

0,000 000 036 534 387 310 121 828 8
4 873 918 308 016 243 8
1 096 631 619 303 654 9
7 310 877 462 024 4
852 935 703 902 8
24 369 591 540 1
7 310 877 462 0
974 783 661 6
12 184 795 8
6 092 397 9
1 096 631 6
73 108 8

$t^3 = 0.000,000,042,533,133,725,307,252,5$
$t = 0.003,490,672,681,596$

0,000 000 000 127 599 401 175 921 8
17 013 253 490 122 9
3 827 982 035 277 6
255 19 880 235 2
2 977 319 360 8
85 066 267 4
25 519 880 2
3 402 650 7
42 533 1
21 266 6
3 828 0
255 2

$t^4 = 0.000,000,000,148,469,247,957,599,5$

$t^4 = 0.000,000,000,148,469,247,957,599,5$
$t = 0.003,490,672,681,596$

0.000 000 000 000 445 407 743 872 8
59 387 699 183 0
13 362 232 316 2
89 081 548 8
10 392 847 4
296 938 5
89 081 5
11 877 5
148 5
74 2
13 4
9

$t^5 = 0.000,000,000,000,518,257,547,902,7$

$10T = 0.174,550,649,282,175,857,651,3$
$t^2 = 0.000,012,184,795,770,040,609,597,1$

0.000 001 745 506 492 821 758 576 5
349 101 298 564 351 715 3
17 455 064 928 217 585 8
13 964 051 942 574 068 6
698 202 597 128 703 4
122 185 454 497 523 1
15 709 558 435 395 8
872 753 246 410 9
122 185 454 497 5
12 218 545 449 8
6 982 026 0
104 730 4
1 571 0
87 3
15 7
2

$10T \times t^2 = 0.000,002,126,864,013,031,298,357,2$

$5T = 0.087,275,324,641,087,928,825,65$.
$t^4 = 0.000,000,000,148,469,247,957,599,5$

0.000 000 000 008 727 532 464 108 8
3 491 012 985 643 5
698 202 597 128 7
34 910 129 856 4
5 236 519 478 4
785 477 921 8
17 455 064 9
3 491 013 0
610 927 3
78 547 8
4 363 8
610 9
43 6
7 8
8

$5T \times t^4 = 0.000,000,000,012,957,701,814,717,6$

Therefore, if 0.003,490,672,681,596, be ſubſtituted inſtead of t, in the quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$, we ſhall have

$5t$ $(= 5 \times 0.003,490,672,681,596) = 0.017,453,363,407,980,$

and $10t^3$ $(= 10 \times 0.000,000,042,533,133,725,307,252,5)$

$= 0.000,000,425,331,337,253,072,525$. Therefore the quantity $5t + 10T \times t^2 + t^5$ will be equal to

$$\left\{\begin{array}{l} 0.017,453,363,407,980 \\ 0.000,002,126,864,013,031,298,358 \\ 0.000,000,000,000,518,257,547,903 \end{array}\right\}$$

$= 0.017,455,490,272,511,288,846,261$, and the quantity $10t^3 + 5T \times t^4$ will be equal to

$$\left\{\begin{array}{l} 0.000,000,425,331,337,253,072,525 \\ 0.000,000,000,012,957,701,814,718 \end{array}\right\}$$

$= 0.000,000,425,344,294,954,887,243$. Therefore the whole quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$, will be equal to $0.017,455,490,272,511,288,846,261 - 0.000,000,425,344,294,954,887,243 = 0.017,455,064,928,216,333,959,018$; which is leſs than 0.017,455,064,928,217,585,765,130, or the abſolute term of the equation $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$; and therefore 0.003,490,672,681,596 is leſs than the true value of t, or the leaſt root of that equation.

A Second Process of Mr. Raphson's Method of Approximation.

Art. 43. Dr. Mackay then puts z for the excefs of t above 0.003,490,672,681,596; and, in order to obtain a more exact value of t, he fubftitutes the binomial quantity 0.003,490,672,681,596+z, inftead of t, in the equation $5t+10T\times t^2-10t^3-5T\times t^4+t^5 = T$. This he does in the following manner.

Since t is = 0.003,490,672,681,596+z, we fhall have

$t^2 = \overline{0.003,490,672,681,596+z}^2$ (= $\overline{0.003,490,672,681,596}^2$ + $2\times 0.003,490,672,681,596\times z$ + &c

$= \overline{0.003,490,672,681,596}^2 + 0.006,981,345,363,192\times z$ + &c)

$= 0.000,012,184,795,770,040,609,597,1 + 0.006,981,345,363,192\times z$ + &c,

and $t^3 = \overline{0.003,490,672,681,596+z}^3$ (= $\overline{0.003,490,672,681,596}^3$ + $3\times\overline{0.003,490,672,681,596}^2\times z$ + &c

$= \overline{0.003,490,672,681,596}^3 + 3\times 0.000,012,184,795,770,040,609,597,1\times z$ + &c

$= \overline{0.003,490,672,681,596}^3 + 0.000,036,554,387,310,121,828,791,3\times z$ + &c)

$= 0.000,000,042,533,133,725,307,252,5 + 0.000,036,554,387,310,121,828,791,3\times z$ + &c,

and $t^4 = \overline{0.003,490,672,681,596+z}^4$ (= $\overline{0.003,490,672,681,596}^4$ + $4\times\overline{0.003,490,672,681,596}^3\times z$ + &c

$= \overline{0.003,490,672,681,596}^4 + 4\times 0.000,000,042,533,133,725,307,252,5\times z$

$= \overline{0.003,490,672,681,596}^4 + 0.000,000,170,132,534,901,229,010,0\times z$ + &c)

$= 0.000,000,000,148,469,247,957,599,5 + 0.000,000,170,132,534,901,229,010,0\times z$ + &c,

and

and $t^5 = \overline{0.003,490,672,681,596+z}^5$ $(= \overline{0.003,490,672,681,596}^5 +$
$5\times\overline{0.003,490,672,681,596}^4 \times z + \&c$
$= \overline{0.003,490,672,681,596}^5 + 5\times 0.000,000,000,148,469,247,957,$
$599,5\times z + \&c$
$= \overline{0.003,490,672,681,596}^5 + 0.000,000,000,742,346,239,787,997,$
$5\times z + \&c)$
$= 0.000,000,000,000,518,257,547,902,7 + 0.000,000,000,742,346,$
$239,787,997,5\times z + \&c.$

Therefore $5t$ will be $= 5\times\overline{0.003,490,672,681,596+z}$
$(= 5\times 0.003,490,672,681,596 + 5\times z$
$= 0.017,453,363,407,980+5\times z + \&c,$

and $10T\times t^2$ will be $(= 10\times 0.017,455,064,928,217,585,765,13 \times$
$\overline{0.000,012,184,795,770,040,609,597,1} + 0.006,981,$
$\overline{345,363,192}\times z + \&c$
$(= 0.174,550,649,282,175,857,651,3 \times 0.000,012,184,$
$795,770,040,609,597,1 + 0.001,218,598,366,008,$
$271,427,026,5\times z + \&c)$
$= 0.000,002,126,864,013,031,298,358,3 + 0.001,218,$
$598,366,008,271,427,026,5\times z + \&c,$

and $10t^3$ will be $(= 10\times 0.000,000,042,533,133,725,307,252,5 + 10\times$
$0.000,036,554,387,310,121,828,791,3\times z + \&c)$
$= 0.000,000,425,331,337,253,072,525 + 0.000,365,$
$543,873,101,218,287,913\times z + \&c,$

and $5T\times t^4$ will be $(= 5\times 0.017,455,064.928,217,585,765,13 \times$
$\overline{0.000,000,000,148,469,247.957,599,5} + 0.000,000,$
$\overline{170,132,534,901,229,010}\times z + \&c$
$= 0.087,275,324,641,087,928,825,65 \times$
$\overline{0.000,000,000,148,469,247,957\ 599,5} + 0.000,000,$
$\overline{170,132,534,901,229,010}\times z + \&c$
$= 0.087,275,324,641,087,928,825,65 \times 0.000,000,$
$000,148,469,247,957,599,5 + 0.000,000,014,848,$
$372,215,515,983,8\times z + \&c)$
$= 0.000,$

$$= 0.000,000,000,012,957,701,814,717,6 + 0.000,000,014,848,372,215\ 515,983,8 \times z + \&c.$$

Therefore the whole quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$, will be equal to the following multinomial quantity, viz.

$$\left\{\begin{array}{l} + 0.017,453,363,407,980 + 5 \times z \\ + 0.000,002,126,864,013,031,298,353 + 0.001,218,598,366,008,271,427,026 \times z + \&c \\ - 0.000,000,425,331,337,253,072,525 - 0.000,365,543,873,101,218,287,913 \times z + \&c \\ - 0.000,000,000,012,957,701,814,718 - 0.000,000,014,848,372,215,515,983 \times z + \&c \\ + 0.000,000,000,000,518,257,547,903 + 0.000,000,000,742,346,239,787,997 \times z + \&c \end{array}\right\}$$

$$= \left\{\begin{array}{l} + 0.017,455,490,272,511,288,846,261 + 5.001,218,599,108,354,511,215,023 \times z \\ - 0.000,000,425,344,294,954,887,243 - 0.000,365,558,721,473,433,803,896 \times z \end{array}\right\}$$

$$= 0.017,455,064,928,216,333,959,018 + 5.000,853,040,386,881,077,411,127 \times z + \&c.$$

But the quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$ is equal to T, or 0.017,455,064,928,217,585,765,130. Therefore the quantity 0.017,455,064,928,216,333,959,018 + 5.000,853,040,386,881,077,411,127 × z, will alſo be equal to 0.017,455,064,928,217,585,765,130. Therefore 5.000,853,040,386,881,077,411,127 × z will be equal to 0.017,455,064,928,217,585,765,130 − 0.017,455,064,928,216,333,959,018 = 0.000,000,000,000,001,251,806,112, and conſequently z will be equal to $\frac{0.000,000,000,000,001,251,806,112}{5.000,853,040,386,881,077,411,127}$ = 0.000,000,000,000,000,250,318,516. Therefore *t*, or 0.003,490,672,681,596 + z, will be (= 0.003,490,672,681,596 + 0.000,000,000,000,000,250,318,516) = 0.003,490,672,681,596,250,318,516; that is, the tangent of an arch of 12 minutes will be 0.003,490,672,681,596,250,318,516. Q. E. I.

A Proof of the Truth of the preceeding Computation.

Art. 44. In order to confirm the truth of the preceeding computation, Dr. Mackay ſubſtitutes the above-found value of *t*, or the tangent of twelve minutes

minutes, namely, 0.003,490,672,681,596,250,318,516, inſtead of t, in the quinquinomial quantity $5t+10T\times t^2-10t^3-5T\times t^4+t^5$, that he may thereby diſcover whether the value of the ſaid quantity reſulting from ſuch ſubſtitution, will be equal to, greater than, or leſs than, the number, 0.017,455,064,928, 217,585,765,130, or the abſolute term of the equation $5t+10T\times t^2-10t^3-5T\times t^4+t^5 = T$, and conſequently whether the ſaid ſubſtituted number, 0.003,490,672,681,596,250,318,516, will be equal to, greater than, or leſs than, the true value of t, or the leaſt root of that equation. And for this purpoſe he raiſes the powers of 0.003,490,672,681,596,250,318,516, in the following manner.

$t =$ 0.003,490,672,681,596,250,318,516
$t =$ 0.003,490,672,681,596,250,318,516

	0.000 010 472 018 044 788 750 955 5
	1 396 269 072 638 500 127 4
	314 160 541 343 662 528 7
	2 094 403 608 957 750 2
	244 347 087 711 737 5
	6 981 343 563 192 5
	2 094 403 608 957 7
	279 253 814 527 7
	3 490 672 681 6
	1 745 336 340 8
	314 160 541 3
	20 944 036 1
	698 134 5
	174 533 6
	1 047 2
	349
	279
	17
	1

$t^5 =$ 0.000,012,184,795,770,042,357,156,9

$= t$

$t^5 =$

$t^2 =$ 0.000,012,184,795,770,042,357,156,9
$t =$ 0.003,490,672,681,596,250,318,516

0.000 000 036 554 387 310 127 071 5
4 873 918 308 016 942 9
1 096 631 619 303 812 1
7 310 877 462 025 4
852 935 703 903 0
24 369 591 540 1
7 310 877 462 0
974 783 661 6
12 184 795 8
6 092 397 9
1 096 631 6
73 108 8
2 437 0
609 2
37
1
1

$t^3 =$ 0.000,000,042,533,133,725,316,402,8
$t =$ 0.003,490.672,681,596,250,318,516

0.000 000 000 127 599 401 175 949 2
17 913 253 490 126 6
3 827 982 035 278 4
25 519 880 235 2
2 977 319 360 8
85 066 267 4
25 519 880 2
3 402 650 7
42 533 1
22 266 6
3 828 0
255 2
85
21

$t^4 =$ 0.000,000,000,148,469,247,957,642,0

$t^4 = 0.000,000,000,148,469,247,957,642,0$
$t = 0.003,490,672,681,596,250,318,516$

0.000 000 000 000 445 407 743 872 9
59 387 699 183 1
13 362 232 316 2
89 081 548 8
10 392 847 4
296 938 5
89 081 5
11 877 5
148 5
74 2
13 4
9

$t^5 = 0.000,000,000,000,518,257,547,902,9$

$10T = 0.174,550,649,282,175,857,651,30$
$t^2 = 0.000,012,184,795,770,042,357,156,9$

0.000 001 745 506 490 821 758 576 5
349 101 298 564 351 715 3
17 455 064 928 217 585 8
13 964 051 942 574 068 6
698 202 597 128 703 4
122 185 454 497 523 1
15 709 558 435 395 8
872 753 246 410 9
122 185 454 497 5
12 218 545 449 8
6 982 026 0
349 101 3
52 365 2
8 727 5
1 221 9
175 5
87
10
2

$10T \times t^2 = 0.000,002,126,864,013,031,603,396,0$

$5T = 0.087,275,324,641,087,928,825,65$
$t^4 = 0.000,000,000,148,469,247,957,764,2$

0.000 000 000 008 727 532 464 108 8
3 491 012 985 643 5
698 202 597 128 7
34 910 129 856 4
5 236 519 478 4
785 477 921 8
17 455 064 9
3 491 013 0
610 927 3
78 547 8
4 363 8
610 9
52 4
3 5
2

$5T \times t^4 = 0.000,000,000,012,957,701,814,721,5$

Therefore if 0.003,490,672,681,596,250,318,516 be ſubſtituted, inſtead of t, in the quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$, we ſhall have

$5t$ (= $5 \times 0.003,490,672,681,596,250,318,516$) = 0.017,453,363,407,981,251,592,580;

and $10t^3$ (= $10 \times 0.000,000,042,533,133,725,316,402,8$) = 0.000,000,425,331,337,253,164,028. Therefore the quantity $5t + 10T \times t^2 + t^5$ will be equal to

$$\left\{\begin{array}{l} 0.017,453,363,407,981,251,592,580 \\ + \quad 0.000,002,126,864,013,031,603,396 \\ + \quad 0.000,000,000,000,518,257,547,903 \end{array}\right\}$$

= 0.017,455,490,272,512,540,743,879. And the quantity $10t^3 + 5T \times t^4$ will be equal to

$$\left\{\begin{array}{l} 0.000,000,425,331,337,253,164,028 \\ + \quad 0.000,000,000,012,957,701,814,721 \end{array}\right\}$$

= 0.000,000,425,344,294,954,978,749.

Therefore the whole quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$ will be equal to 0.017,455,490,272,512,540,743,879 — 0.000,000,425,

425,344,294,954,978,749 = 0.017,455,064,928,217,585,765,130; which is equal to 0.017,455,064,928,217,585,765,130, or the abſolute term of the equation $5t+10T\times t^2-10t^3-5T\times t^4+t^5$; and therefore 0.003,490,672,681,596,250,318,516 is the true value of t, or the leaſt root of that equation, or the true value of the tangent of twelve minutes, exact in all its figures.

Q. E. I.

End of the Quinquiſection of an Arch of 1 *Degree.*

The Triſection of an Arch of 12 *Minutes by reſolving the Cubick Equation* $3t+3Tt^2-t^3=T$, *in which* T *is the Tangent of an Arch of* 12 *Minutes, and is* = 0.003,490,672,681,596,250,318,516, *and t, or the leſſer Root of the Equation, is the Tangent of an Arch of* 4 *Minutes.*

Art. 45. Dr. Mackay obſerves that the tangent of four minutes is leſs than one-third part of the tangent of 12 minutes, and greater than the length of a circular arch of four minutes; but that it is neareſt to this laſt quantity; that is, the tangent of four minutes is leſs than $\frac{0.003,490,672,6}{3}$, or 0.001,163,557,5; and greater than $\frac{3.141,592,653,5}{180\times5\times3}$, or 0.001,163,552,8; but, being neareſt to this laſt quantity, it is conſequently leſs than $\frac{0.001,163,557,5+0.001,163,552,8}{2}$, or than $\frac{0.002,327,110,3}{2}$, or 0.001,163,555,1: and therefore, Dr. Mackay conjectures that the tangent of four minutes is = 0.001,163,553, and then ſubſtitutes that number for t, in the trinomial quantity $3t+3T\times t^2-t^3$, (which forms the firſt, or left-hand, ſide of the equation $3T+3Tt^2-t^3=T$, or $3t+3\times0.003,490,672,681,596,250,318,516\times t^2-t^3=0.003,490,672,881,596,250,318,516$)

and compares the result of that substitution with the absolute term T, (of which the first twenty-two figures are 0.003,490,672,681,596,250,318,5,) in order to discover whether the said result is greater, or is less, than the said absolute term, and consequently whether 0.001,163,553 is greater, or is less, than the true value of t, or the lesser root of the said equation. And for this purpose Dr. Mackay raises the powers of 0.001,163,553, in the following manner.

$t = 0.001,163,553$

3 490 659
58 177 65
581 776 5
3 490 659
69 813 18
116 355 3
1 163 553

$t^2 = 0.000,001,353,855,583,809$
$t = 0.001,163,553$

0.000 000 001 353 855 583 809
135 385 558 380 9
81 231 335 028 54
4 061 566 751 427
676 927 791 904 5
67 692 779 190 4
4 061 566 751 4

$t^3 = 0.000,000,001,575,282,726,107,713,3$

T =

$$T = 0.003,490,672,681,596,250,318,5$$

$$3$$

$$3T = 0.010,472,018,044,788,750,955,5$$
$$t^2 = 0.000,001,353,855,583,809$$

0.000 000 010 472 018 044 788 751 0
3 141 605 413 436 625 3
523 600 902 239 437 5
31 416 054 134 366 3
8 377 614 435 831 0
523 600 902 239 4
52 360 090 223 9
5 236 009 022 4
837 761 443 6
31 416 054 1
8 377 614 4
94 248 2

$$3T \times t^2 = 0.000,000,014,177,600,103,685,857,1$$

Therefore, if 0.001,163,553 be fubſtituted, inſtead of t, in the trinomial quantity $3t+3T\times t^2-t^3$, we ſhall have

$3t$ (= 3×0.001,163,553) = 0.003,490,659;

and $3t+3Tt^2$ (= 0.003,490,659 + 0.000,000,014,177,600,103,685,86)
= 0.003,490,673,177,600,103,685,86; and, therefore, the whole trinomial quantity $3t+3T\times t^2-t^3$, will be equal to

$$\left\{\begin{array}{l} +\ 0.003,490,673,177,600,103,685,86 \\ -\ 0.000,000,001,575,282,726,107,71 \end{array}\right\}$$

= 0.003,490,671,602,317,377,578,15; which is leſs than 0.003,490,672,681,596,250,318,5, or the abſolute term of the equation $3t+3T\times t^2-t^3 = T$; and therefore 0.001,163,553 is leſs than the true value of t, or the leſſer root of the ſaid equation. Q. E. I.

A Process of Mr. Raphson's Method of Approximation.

Art. 46. Having thus discovered that 0.001,163,553, is less than the true value of t, Dr. Mackay puts z for the excess of t above 0.001,163,553; and, in order to obtain a more exact value of t, he substitutes the binomial quantity $0.001,163,553 + z$, instead of t, in the equation $3t + 3T \times t^2 - t^3 = T$. This he does in the following manner.

Since t is $= 0.001,163,553 + z$, we shall have

$t^2 = \overline{0.001,163,553 + z}^2$ $(= \overline{0.001,163,553}^2 + 2 \times 0.001,163,553 \times z$ $+ \&c)$

$= 0.000,001,353,855,583,809 + 0.002,327,106 \times z + \&c,$

and $t^3 = \overline{0.001,163,553 + z}^3$ $(= \overline{0.001,163,553}^3 + 3 \times \overline{0.001,163,553}^2 \times z$

$= \overline{0.001,163,553}^3 + 3 \times 0.000,001,353,855,583,809 \times z$

$= \overline{0.001,163,553}^3 + 0.000,004,061,566,751,427 \times z + \&c)$

$= 0.000,000,001,575,282,726,107,71 + 0.000,004,061,566,751,427 \times z + \&c.$

Therefore $3t$ will be $= 3 \times \overline{0.001,163,553 + z} = 0.003,490,659 + 3 \times z$ $+ \&c,$

and $3T \times t^2$ will be $(= 0.010,472,018,044,788,750,955,5 \times \overline{0.000,001,353,855,583,809} + 0.002,327,106 \times z + \&c$

$= 0.010,472,018,044,788,750,955,5 \times 0.000,001,353,855,583,809 + 0.000,024,369,496,024,131,671,08 \times z)$

$= 0.000,000,014,177,600,103,685,86 + 0.000,024,369,496,024,131,671,08 \times z.$

Therefore the whole trinomial quantity $3t + 3T \times t^2 - t^3$, will be equal to the quantity

$+ 0.003,$

$$\left\{\begin{array}{l} +\ 0.003,490,659 \qquad\qquad +\ 3.\times z \\ +\ 0.000,000,014,177,600,103,685,86 + 0.000,024,369,496,024,131, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 671,08\times z \\ -\ 0.000,000,001,575,282,726,107,71 - 0.000,004,061,566,751,427, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \times z \end{array}\right\}$$

$= 0.003,490,671,602,317,377,578,15 + 3.000,020,307,929,272,704,671,$
$08\times z.$

But the trinomial quantity $3t+3T\times t^2-t^3$, is equal to 0.003,490,672,681,596,250,318,5, or T, the abſolute term of the equation $3t+3T\times t^2-t^3 = T$. Therefore the quantity 0.003,490,671,602,317,377,578,15 + 3.000,020,307,929,272,704,671,08$\times z$ will alſo be equal to 0.003,490,672,681,596,250,318,5, and conſequently 3.000,020,307,929,272,704,671,08 $\times$ z will be equal to

$$\left\{\begin{array}{l} +\ 0.003,490,672,681,596,250,318,5 \\ -\ 0.003,490,671,602,317,377,578,1 \end{array}\right\}$$

$= 0.000,000,001,079,278,872,740,4$; and therefore z will be $= \frac{0.000,000,001,079,278,872,740,4}{3.000,020,307,929,272,704,671,08} = 0.000,000,000,359,757,188,938,9$. Therefore t, or 0.001,163,553 + z, will be ($=$ 0.001,163,553 + 0.000,000,000,359,757,188,938,9) $=$ 0.001,163,553,359,757,188,938,9; that is, the tangent of four minutes, or the leſſer root of the equation $3t+3T\times t^2-t^3 = T$, will be nearly equal to 0.001,163,553,359,757,188,938,9.

Q. E. I.

Art. 47. Having thus obtained the number 0.001,163,553,359,757,188,938,9 for a ſecond near value of t, or the tangent of an arch of four minutes, of which it is probable the firſt ſixteen figures, to wit, 0.001,163,553,359,757,1 are correct, Dr. Mackay proceeds to ſubſtitute this number, inſtead of t, in the trinomial quantity $3t+3T\times t^2-t^3$, in order to diſcover whether the value of that quantity reſulting from this ſubſtitution will be greater, or will be leſs, than 0.003,490,672,681,596,250,318,516, or T, the abſolute term of the propoſed equation $3t+3T\times t^2-t^3 = T$; and conſequently whether the ſaid number 0.001,163,553,359,757,1 will be greater, or will be leſs, than the true value of t, or the leſſer root of that equation. And for this purpoſe Dr. Mackay raiſes the powers of the number 0.001,163,553,359,757,1 in the following manner.

$$t = 0.001,163,553,359,757,1$$
$$t = 0.001,163,553,359,757,1$$

0.000 001 163 553 359 757 1
116 355 335 975 71
69 813 201 585 426
3 490 660 079 271 3
581 776 679 878 55
58 177 667 987 855
3 490 660 079 271 3
349 066 007 927 1
58 177 667 987 9
10 471 980 237 8
814 487 351 8
58 177 668 0
8 144 873 5
116 355 3

$$= 0.000,001,353,856,421,002,035,377,7$$
$$= 0.001,163,553,359,757,1$$

0.000 000 001 353 856 421 002 035 4
135 385 642 100 203 5
81 231 385 260 122 1
4 061 569 263 006 1
676 928 210 501 0
67 692 821 050 1
4 061 569 263 0
406 156 926 3
67 692 821 0
12 184 707 8
947 699 5
67 692 8
9 477 0
1354

$$t^3 = 0.000,000,001,575,284,187,285,641,0$$

$3T$ = 0.010,472,018,044,788,750,955,5
t^2 = 0.000,001,353,856,421,002,035,377,7

0.000 000 010 472 018 044 788 751 0
3 141 605 413 436 625 3
523 600 902 239 437 5
31 416 054 134 366 3
8 377 614 435 831 0
523 600 902 239 4
62 832 108 268 7
4 188 807 217 9
209 440 360 9
10 472 018 0
20 944 0
3142
524
31
7
1

$3T \times t^2$ = 0.000,000,014,177,608,870,786,430,5

Therefore if 0.001,163,553,359,757,1 be ſubſtituted inſtead of t in the trinomial quantity $3t + 3T \times t^2 - t^3$, we ſhall have

$3t$ (= 3×0.001,163,553,359,757,1) = 0.003,490,660,079,271,3 ;

and $3t + 3T \times t^2$ (= 0.003,490,660,079,271,3 + 0.000,000,014,177,608,870,786,430,5)

= 0.003,490,674,256,880,170,786,430,5 ; and therefore the whole trinomial quantity $3t + 3T \times t^2 - t^3$, will be equal to

{ + 0.003,490,674,256,880,170,786,43
{ − 0.000,000,001,575,284,187,285,64

= 0.003,490,672,681,595,983,500,79, which is leſs than 0.003,490,672,681,596,250,318,516, or the abſolute term of the equation $3t + 3T \times t^2 - t^3 = T$; and therefore 0.001,163,553,359,757,1 is leſs than the true value of t, or the leſſer root of theſ aid equation. Q. E. I.

A Second

A Second Process of Mr. Raphson's Method of Approximation.

Art. 48. Dr. Mackay then puts z for the excefs of t above 0.001,163,553,359,757,1 ; and in order to obtain a more exact value of t, he fubftitutes the binomial quantity 0.001,163,553,359,757,1+z, inftead of t, in the equation $3t+3T\times t^2-t^3 = T$. This he does in the following manner.

Since t is = 0.001,163,553,359,757,1+z, we fhall have

$$t^2 = \overline{0.001,163,553,359,757,1+z}^2 \; (= \overline{0.001,163,553,359,757,1}^2 + 2\times 0.001,163,553,359,757,1\times z + \&c$$
$$= \overline{0.001,163,553,359,757,1}^2 + 0.002,327,106,719,514,2\times z+\&c)$$
$$= 0.000,001,353,856,421,002,035,38 + 0.002,327,106,719,514,2\times z$$

$$\text{and } t^3 = \overline{0.001,163,553,359,757,1+z}^3 \; (= \overline{0.001,163,553,359,757,1}^3 + 3\times\overline{0.001,163,553,359,757,1}^2 \times z$$
$$= \overline{0.001,163,553,359,757,1}^3 + 3\times 0.000,001,353,856,421,002,035,38\times z$$
$$= 0.001,163,553,359,757,1^3 + 0.000,004,061,569,263,006,106,14\times z)$$
$$= 0.000,000,001,575,284,187,285,64 + 0.000,004,061,569,263,006,106,14\times z + \&c.$$

Therefore $3t$ will be equal to $3\times 0.001,163,553,359,757,1+z$
$$= 0.003,490,660,079,271,3+3\times z+\&c,$$

and $3T\times t^2$ will be $(= \overline{0.010,472,018,044,788,750,955,5} \times \overline{0.000,001,353,856,421,002,035,38} + 0.002,327,106,719,514,2\times z+\&c$
$$= 0.010,472,018,044,788,750,955,5 \times 0.000,001,353,856,421,002,035,38 + 0.000,024,369,503,558,901,856,96\times z)$$
$$= 0.000,000,014,177,608,870,786,43 + 0.000,024,369,503,558,901,856,96\times z.$$

Therefore

Therefore the whole trinomial quantity $3t+3T\times t^2-t^3$, will be equal to the quantity

$$\left\{\begin{array}{l} +\ 0.003,490,660,079,271,3 \qquad\qquad +\ 3\times z \\ +\ 0.000,000,014,177,608,870,786,43 + 0.000,024,369,503,558,901,856,96\times z \\ -\ 0.000,000,001,575,284,187,285,64 - 0.000,004,061,569,263,006,106,14\times z \end{array}\right\}$$

$$= 0.003,490,672,681,595,983,500,79 + 3.000,020,307,934,295,895,750,82\times z$$

But the trinomial quantity $3t+3T\times t^2-t^3$ is equal to 0.003,490,672,681,596,250,318,516, or T, the abſolute term of the equation $3t+3T\times t^2-t^3 = T$. Therefore the quantity 0.003,490,672,681,595,983,500,79 + 3.000,020,307,934,295,895,750,82×z, will alſo be equal to 0.003,490,672,681,596,250,318,516, and conſequently 3.000,020,307,934,295,895,750,82×z will be equal to

$$\left\{\begin{array}{l} +\ 0.003,490,672,681,596,250,318,516 \\ -\ 0.003,490,672,681,595,983,500,790 \end{array}\right\}$$

$= 0.000,000,000,000,000,266,817,726$; and therefore z will be $= \frac{0.000,000,000,000,000,266,817,726}{3.000,020,307,934,295,895,750,82} = 0.000,000,000,000,000,088,938,65$.

Therefore t, or 0.001,163,553,359,757,1+z, will be (equal to 0.001,163,553,359,757,1 + 0.000,000,000,000,000,088,938,65) = 0.001,163,553,359,757,188,938,65; that is, the tangent of four minutes, or the leaſt root of the equation $3t+3T\times t^2-t^3 = T$, will be equal to 0.001,163,553,359,757,188,938,65, which is true in all its figures. Q. E. I.

A Proof of the Truth of the foregoing Computation.

Art. 49. In order to prove that the number 0.001,163,553,359,757,188,938,65, is the true value of the tangent of four minutes, Dr. Mackay ſubſtitutes it for t, in the trinomial quantity $3t+3Tt^2-t^3$, which forms the firſt, or left-hand, ſide of the equation $3t+3T\times t^2-t^3 = T$, and thereby ſhews that the final reſult will agree with 0.003,490,672,681,596,250,318,516, the abſolute term of the ſaid equation. For this purpoſe he raiſes the powers of the number 0.001,163,553,359,757,188,938,65, in the following manner.

$t = 0.001,163,553,359,757,188,938,65$

$t = 0.001,163,553,359,757,188,938,65$

0.000 001 163 553 359 757 188 938 65
116 355 335 975 718 893 9
69 813 201 585 431 336 3
3 490 660 079 271 566 8
581 776 679 878 594 5
58 177 667 987 859 4
3 490 660 079 271 6
349 066 007 927 2
58 177 667 987 9
10 471 980 237 8
814 487 351 8
58 177 668 0
8 144 873 5
116 355 3
93 084 3
9 308 4
1 047 2
349
93
7
1

$t^2 = 0.000,001,353,856,421,002,242,347,5$

$t = 0.001,163,553,359,757,188,938,65$

0.000 000 001 353 856 421 002 242 3
135 385 642 100 224 2
81 231 385 260 134 5
4 061 569 263 006 7
676 928 210 501 1
67 692 821 050 1
4 061 569 263 0
406 156 926 3
67 692 821 0
12 184 707 8
947 699 5
67 692 8
9 477 0
1 354
108 3
10 8
1 2
4
1

$t^3 = 0.000,000,001,575,284,187,286,002,5$

$3T = 0.010,472,018,044,788,750,955,5$
$t^2 = 0.000,001,353,856,421,002,242,347,5$

0.000 000 010 472 018 044 788 751 0
3 141 605 413 436 625 3
523 600 902 239 437 5
31 416 054 134 366 3
8 377 614 435 831 0
523 600 902 239 4
62 832 108 268 7
4 188 807 217 9
209 440 360 9
10 472 018 0
20 944 0
2 094 4
418 9
209
3 1
4
1

$3T \times t^2 = 0.000,000,014,177,608,870,788,597,8$

Therefore, if 0.001,163,553,359,757,188,938,65 be fubſtituted, inſtead of t, in the trinomial quantity $3t + 3T \times t^2 - t^3$, we ſhall have

$3t$ (= 3×0.001,163,553,359,757,188,938,65) = 0.003,490,660,079,271,566,815,95;

and $3t + 3T \times t^2$ (= 0.003,490,660,079,271,566,815,95 + 0.000,000,014,177,608,870,788,597,8)

= 0.003,490,674,256,880,437,604,55; and therefore the whole trinomial quantity $3t + 3T \times t^2 - t^3$, will be equal to

$$\left\{\begin{array}{ll} + & 0.003,490,674,256,880,437,604,55 \\ - & 0.000,000,001,575,284,187,286,00 \end{array}\right\}$$

= 0.003,490,672,681,596,250,318,55; which agrees with the number 0.003,490,672,681,596,250,318,516, or the abſolute term of the equation $3t + 3T \times t^2 - t^3 = T$ in the firſt 22 figures, to wit, 0.003,490,672,681,596,250,318,5, and exceeds the ſaid abſolute term by only the very ſmall quantity 0.000,000,000,000,000,000,000,034, and is therefore conſidered by Dr. Mackay as being equal to the ſaid abſolute term. And hence he concludes that the number 0.001,163,553,359,757,188,938,65, is the true value of t, or the leaſt root of the ſaid equation, or of the tangent of an arch of four minutes.

Q. E. D.

End of the Triſection of an Arch of 12 *Minutes.*

The Bisection of an Arch of 4 *Minutes of a Degree, by resolving the Quadratick Equation* $t^2 + \frac{2}{T} \times t = 1$, *in which* T *is the Tangent of an Arch of* 4 *Minutes, and is* = 0.001,163,553,359,757,188,938,65, *and* t, *or the Root of the said Equation, is the Tangent of an Arch of* 2 *Minutes.*

Art. 50. Dr. Mackay begins this Bifection by reducing the vulgar fraction $\frac{2}{T}$, or $\frac{2}{0.001,163,553,359,757,188,938,65}$, to a mixt number conſiſting of a whole number and a decimal fraction, by dividing it's numerator by it's denominator, and finds the mixt number that reſults from this diviſion to be 1718.872,609,690,509,838,486,820,225,3. Therefore the equation $t^2 + \frac{2}{T} \times t = 1$ will, when this mixt number is ſubſtituted in it inſtead of $\frac{2}{T}$, be $t^2 + 1718.872,609,690,509,838,486,820,225,3 \times t = 1$. This diviſion is performed as follows.

The

The Quotient.

(1718.872,609,690,509,838,486,820,225,3

The Divisor.

0.001,163,553,359,757,188,938,65)

The Dividend.

2.000,000,000,000,000,000,000,000,000,00

1.163,553,359,757,188,938,65

836 446 640 242 811 061 350

814 487 351 830 032 257 055

21 959 288 412 778 804 295 0

11 635 533 597 571 889 386 5

10 323 754 815 206 914 908 50

9 308 426 878 057 511 509 20

1 015 327 937 149 403 399 300

930 842 687 805 751 150 920

84 485 249 343 652 248 380 0

81 448 735 183 003 225 705 5

3 036 514 160 649 022 674 50

2 327 106 719 514 377 877 30

709 407 441 134 644 797 200

698 132 015 854 313 363 190

11 275 425 280 331 434 010 00

10 471 980 237 814 700 447 85

803 445 042 516 733 562 15

698 132 015 854 313 363 19

105 313 026 662 420 198 96

104 719 802 378 147 904 48

593 224 284 273 194 48

581 776 679 878 594 47

11 447 604 394 600 01

10 471 980 237 814 70

975 624 156 785 31

930 842 687 805 75

44 781 468 979 56

34 906 600 792 72

9 874 868 186 84

9 308 426 878 06

566 441 308 78

465 421 343 90

101 019 964 88

101 019 964 88

93 084 268 78

7 935 696 10

6 981 320 16

954 375 94

930 842 69

23 533 25

23 271 07

262 18

232 71

29 47

23 27

6 20

5 82

38

35

3

The

The abridged method of performing this long operation of diviſion, which has been here employed by Dr. Mackay, conſiderably leſſens the labour of it, without at all diminiſhing the exactneſs of the quotient, when carried only to the 25th place of decimal fractions, as will appear from the following exhibition of the ſame operation performed at full length, which produces the very ſame quotient as has been here obtained by Dr. Mackay's abridged method.

The Quotient.

(1718.872,609,690,509,838,486,820,225,3

The Divisor.

0.001,163,553,359,757,188,938,65)

The Dividend.

2.000,000,000,000,000,000,000,000,000,000,000,000,000,000, &c

1 163 553 359 757 188 938 65

·836 446 640 242 811 061 350
814 487 351 830 032 257 055

·21 959 288 412 778 804 295 0
11 635 533 597 571 889 386 5

10 323 754 815 206 914 908 50
9 308 426 878 057 511 509 20·

1 015 327 937 149 403 399 300
930 842 687 805 751 150 920

··84 485 249 343 652 248 380 0
81 448 735 183 003 225 705 5

·3 036 514 160 649 022 674 50
2 327 106 719 514 377 877 30

·709 407 441 134 644 797 200
698 132 015 854 313 363 190

·11 275 425 280 331 434 010 00
10 471 980 237 814 700 447 85

··803 445 042 516 733 562 150
698 132 015 854 313 363 190

105 313 026 662 420 198 960 0
104 719 802 378 147 004 478 5

··· 593 224 284 273 194 481 500
581 776 679 878 594 469 325

·11 447 604 394 600 012 175 00
10 471 980 237 814 700 447 85

··975 624 156 785 311 727 15

975 624 156 785 311 727 150
930 842 687 805 751 150 920

·44 781 468 979 560 576 230 0
34 906 600 792 715 668 159 5

·9 874 868 186 844 908 070 50
9 308 426 878 057 511 509 20

·566 441 308 787 396 561 300
465 421 343 902 875 575 460

101 019 964 884 520 985 840 0
93 084 268 780 575 115 092 0

··7 935 696 103 945 870 748 00
6 981 320 158 543 133 631 90

·954 375 945 402 737 116 100
930 842 687 805 751 150 920

·23 533 257 596 985 965 180 0
23 271 067 195 143 778 773 0

··262 190 401 842 186 407 000
232 710 671 951 437 787 730

·29 479 729 890 748 619 270 0
23 271 067 195 143 778 773 0

·6 208 662 695 604 840 497 00
5 817 766 798 785 944 693 25

·390 895 896 818 895 803 750
349 066 007 927 156 681 595

·41 829 888 891 739 122 155

Art. 51. Having thus obtained the value of $\frac{2}{T}$, the co-efficient of t in the equation $t^2 + \frac{2}{T} \times t = 1$, and thereby converted the ſaid equation into the equation $t^2 + 1718.872,609,690,509,838,486,820,225,3 \times t = 1$, Dr. Mackay proceeds to reſolve this laſt equation in the following manner.

He begins by obſerving that the tangent of an arch of 2 minutes is leſs than half the tangent of an arch of 4 minutes, (or than $\frac{0.001,163,553,359,359,757,188,938,65}{2}$,) or than 0.000,581,776,67, &c; and that it is greater than an arch of 2 minutes, (or than $\frac{3.141,592,653,59}{180 \times 30}$, or than $\frac{3.141,592,653,59}{5400}$,) or than 0.000,581,776,41; and that it is nearer to it's leſſer limit, 0.000,581,776,41, than to it's greater limit, 0.000,581,776,67, and conſequently that it will be leſs than half their

ſum,

ſum, (or than $\frac{0.001,163,553,08}{2}$,) or than 0.000,581,776,54. He therefore conjectures that this tangent will be equal to 0.000,581,776,42, or that the root t of the equation $t^2 + 1718.872,609,690,509,838,486,820,225,3 \times t = 1$ will be $=$ 0.000,581,776,42, and he ſubſtitutes this number for t in the binomial quantity $t^2 + 1718.872,609,690,509,838,486,820,225,3 \times t$, which forms the firſt, or left-hand, ſide of the ſaid equation, and then compares the value of the ſaid binomial quantity, reſulting from ſuch ſubſtitution, with the abſolute term 1, of the ſaid equation, in order to diſcover whether the ſaid value will be greater, or will be leſs, than the ſaid abſolute term, and thereby to determine whether the ſaid number 0.000,581,776,42 will be greater, or will be leſs, than the true value of t, or the root of the ſaid equation. And for this purpoſe Dr. Mackay finds the values of t^2 and of $\frac{2}{T} \times t$, or 1718.872,609,690,509,838,486,820,225,3 $\times t$, in the following manner.

$t =$ 0.000,581,776,42
$t =$ 0.000,581,776,42

11 635 528 4
232 710 568
3 490 658 52
40 724 349 4
407 243 494
581 776 42
46 542 113 6
290 888 210

$t^2 =$ 0.000,000,338,463,802,868,016,4

$\frac{2}{T} =$ 1718.872,609,690,509,838,486,820,225

$t =$ 0.000,581,776,42

0.859 436 304 845 254 919 243 410 1
137 509 808 775 240 787 078 945 6
1 718 872 609 690 509 838 486 8
1 203 210 826 783 356 886 940 8
120 321 082 678 335 688 694 1
10 313 235 658 143 059 030 9
687 549 043 876 203 935 4
34 377 452 193 810 196 8

$\frac{2}{T} \times t =$ 0.999,999,553,301,802,121,809,640,5

Therefore,

Therefore, if 0.000,581,776,42 be ſubſtituted, inſtead of t, in the binomial quantity $t^2 + \frac{2}{T} \times t$, the value of that quantity reſulting from ſuch ſubſtitution, will be (= 0.000,000,338,463,802,868,016,4 + 0.999,999,553,301,802,121,809,640,5) = 0.999,999,891,765,604,989,826,040,5; which is leſs than 1, or the abſolute term of the equation $t^2 + \frac{2}{T} \times t = 1$, and conſequently 0.000,581,776,42 is leſs than the true value of t in the ſaid equation, or than the tangent of an arch of 2 minutes.

Q. E. I.

A Process of Mr. Raphson's Method of Approximation.

Art. 52. Having thus diſcovered that 0.000,581,776,42 is leſs than the true value of t, or the root of the equation $t^2 + \frac{2}{T} \times t = 1$, Dr. Mackay puts z for the exceſs of t above 0.000,581,776,42; and, in order to obtain a more exact value of t, he ſubſtitutes the binomial quantity 0.000,581,776,42 + z, inſtead of t, in the equation $t^2 + \frac{2}{T} \times t = 1$. This he does in the following manner.

Since t is = 0.000,581,776,42 + z, we ſhall have

$t^2 = \overline{0.000,581,776,42 + z}^2$ (= $\overline{0.000,581,776,42}^2$ + 2 × 0.000,581,776,42 × z + &c

= $\overline{0.000,581,776,42}^2$ + 0.001,163,552,84 × z + &c)

= 0.000,000,338,463,802,868,016,4 + 0.001,163,552,84 × z + &c,

and $\frac{2}{T} \times t = \frac{2}{T} \times \overline{0.000,581,776,42 + z}$ (= $\frac{2}{T}$ × 0.000,581,776,42 + $\frac{2}{T} \times z$) = 0.999,999,553,301,802,121,809,640,5 + 1718.872,609,690,509,838,486,820,225 × z.

Therefore the binomial quantity $t^2 + \frac{2}{T} \times t$ will be equal to the multinomial quantity

$$\left\{\begin{array}{l} 0.000,000,338,463,802,868,016,4 \quad + 0.001,163,552,84 \times z \\ + 0.999,999,553.301,802,121,809,640,5 \quad + 1718.872,609,690,509,838, \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad 486,820,225 \times z \end{array}\right\}$$

$$= 0.999,999,891,765,604,989,826,040,5 + 1718.873,773,243,349,838, 486,820,225 \times z.$$

But the binomial quantity $t^2 + \frac{2}{T} \times t$ is $= 1$, the absolute term of the equation $t^2 + \frac{2}{T} \times t = 1$.

Therefore the quantity 0.999,999,891,765,604,989,826,040,5 + 1718.873,773,243,349,438,486,820,225 $\times z$ will also be $= 1$. And consequently 1718.873,773,243,349,838,486,820,225 $\times z$ will be $=$

$$\left\{\begin{array}{l} 1.000,000,000,000,000,000,000,000,0 \\ - 0.999,999,891,765,604,989,826,040,5 \end{array}\right\}$$

$= 0.000,000,108,234,395,010,173,959,5$; and z will be $= \frac{0.000,000,108,234,395,010,173,959,5}{1718.873,773,243,349,838,486,820,225} = 0.000,000,000,062,968,204,352$. Therefore t, or $0.000,581,776,42 + z$, will be $(= 0.000,581,776,42 + 0.000,000,000,062,968,204,352) = 0.000,581,776,482,968,204,352$; that is, the tangent of an arch of two minutes will be nearly equal to 0.000,581,776,482,968,204,352. Q. E. I.

Art. 53. Having thus obtained the number 0.000,581,776,482,968,204,352 for a second near value of t, or the tangent of an arch of two minutes, Dr. Mackay proceeds to substitute this number, instead of t, in the binomial quantity $t^2 + \frac{2}{T} \times t$, in order to discover whether the value of this quantity resulting from such substitution will be greater, or will be less, than 1, or the absolute term of the equation $t^2 + \frac{2}{T} = 1$; and consequently whether the said number will be greater, or will be less, than the true value of t, or the root of that equation. And for this purpose Dr. Mackay proceeds in the following manner.

$t = 0.000,581,776,482,968,204,352$
$t = 0.000,581,776,482,968,204,352$

0.000 000 290 888 241 484 102 176 0
46 542 118 637 456 348 2
581 776 482 968 204 4
407 243 538 077 743 0
40 724 353 807 774 3
3 490 658 897 809 2
232 710 593 187 3
46 542 118 637 5
1 163 552 965 9
523 598 834 7
34 906 589 0
4 654 211 9
116 355 3
2 327 1
174 5
29 1
1 2

$t^2 = 0.000,000,338,463,876,134,853,368,6$

$\frac{2}{T} = 1718.872,609,690,509,838,486,820,225$

$t = 0.000,581,776,482,968,204,352$

0.859 436 304 845 254 919 243 410 1
.137 509 808 775 240 787 078 945 6
1 718 872 609 690 509 838 486 8
1 203 210 826 783 356 886 940 8
120 321 082 678 335 688 694 1
10 313 235 658 143 059 030 9
687 549 043 876 203 935 4
137 509 808 775 240 787 1
3 437 745 219 381 019 7
1 546 985 348 721 458 9
103 132 356 581 430 6
13 750 980 877 524 1
343 774 521 938 1
6 875 490 438 8
515 661 782 9
85 943 630 5
3 437 745 2

$\frac{2}{T} \times t = 0.999,999,661,536,123,863,857,199,6$

Therefore,

Therefore, if 0.000,581,776,482,968,204,352 be ſubſtituted, inſtead of t, in the binomial quantity $t^2 + \frac{2}{T} \times t$, the value of that quantity reſulting from ſuch ſubſtitution will be (= 0.000,000,338,463,876,134,853,368,6 + 0.999,999,661,536,123,863,857,199,6) = 0.999,999,999,999,999,998,710,568,2; which is leſs than 1, or the abſolute term of the equation $t^2 + \frac{2}{T} \times t = 1$; and conſequently 0.000,581,776,482,968,204,352, is leſs than the true value of t, in the ſaid equation, or than the tangent of an arch of two minutes.

Q. E. I.

A Second Process of Mr. Raphson's Method of Approximation.

Art. 54. Having thus diſcovered that 0.000,581,776,482,968,204,352, is leſs than the true value of t, or the root of the equation $t^2 + \frac{2}{T} \times t = 1$, Dr. Mackay puts z for the exceſs of t above 0.000,581,776,482,968,204,352, and, in order to obtain a more exact value of t, he ſubſtitutes the binomial quantity 0.000,581,776,482,968,204,352 + z, inſtead of t, in the equation $t^2 + \frac{2}{T} \times t = 1$. This he does in the following manner.

Since t is = 0.000,581,776,482,968,204,352 + z, we ſhall have

$$t^2 = \overline{0.000,581,776,482,968,204,352 + z}^2$$

$$(= \overline{0.000,581,776,482,968,204,352}^2 + 2 \times 0.000,581,776,482,968,204,352 \times z$$

$$= \overline{0.000,581,776,482,968,204,352}^2 + 0.001,163,552,965,936,408,704 \times z)$$

$$= 0.000,000,338,463,876,134,853,368,6 + 0.001,163,552,965,936,408,704 \times z + \&c,$$

and $\frac{2}{T} \times t = \frac{2}{T} \times \overline{0.000,581,776,482,968,204,352 + z}$

$$(= \frac{2}{T} \times 0.000,581,776,482,968,204,352 + \frac{2}{T} \times z + \&c)$$

$$= 0.999,999,661,536,123,863,857,199,6 + 1718.872,609,690,509,838,486,820,225 \times z + \&c.$$

Therefore,

Therefore, the binomial quantity $t^2 + \frac{2}{T} \times t$, will be equal to the multinomial quantity

$$\left\{\begin{array}{l} 0.000,000,338,463,876,134,853,368,6 + 0.001,163,552,965,936,408,704 \times z \\ + 0.999,999,661,536,123,863,857,199,6 + 1718.872,609,690,509,838,486,820,225 \times z \end{array}\right\}$$

$$= 0.999,999,999,999,999,998,710,568,2 + 1718.873,773,243,475,774,895,524,225 \times z$$

But the binomial quantity $t^2 + \frac{2}{T} \times t$ is $= 1$, the abfolute term of the equation $t^2 + \frac{2}{T} \times t = 1$. Therefore the quantity 0.999,999,999,999,999,998,710,568,2 + 1718.873,773,243,475,774,895,524,225 $\times z$, will alfo be $= 1$. And conſequently 1718.873,773,243,475,774,895,524,225$\times z$ will be $=$

$$\left\{\begin{array}{l} 1.000,000,000,000,000,000,000,000,0 \\ - 0.999,999,999,999,999,998,710,568,2 \end{array}\right\}$$

$= 0.000,000,000,000,000,001,289,431,8$; and z will be equal to the fraction $\frac{0.000,000,000,000,000,001,289,431,8}{1718.873,773,243,475,774,895,524,225} = 0.000,000,000,000,000,000,000,750,16$. Therefore t, or $0.000,581,776,482,968,204,352 + z$, will be $(= 0.000,581,776,482,968,204,352 + 0.000,000,000,000,000,000,000,750,16) = 0.000,581,776,482,968,204,352,750,16$; that is, the tangent of an arch of two minutes will be $= 0.000,581,776,482,968,204,352,750,16$.

Q. E. I.

As this method of approximation generally doubles the number of figures at every operation, and there were twenty-one places of decimals uſed in the above proceſs, it is evident that the additional five places of figures muſt be correct: and conſequently the above-found value of t, or the tangent of two minutes, to wit, 0.000,581,776,482,968,204,352,750,16, muſt be exact in all its twenty-ſix places of decimals. This would alſo appear by ſubſtituting the above-found value of the tangent of an arch of two minutes, inſtead of t, in the equation $t^2 + \frac{2}{T} \times t = 1$: but this ſeems to be altogether unneceſſary, ſince only the twenty-three firſt places of decimals, to wit, 0.000,581,776,482,968,204,352,75, are uſed in the remaining part of the computation.

End of the Bisection of an Arch of 4 Minutes, by resolving the Equation $t^2 + \frac{2}{T} \times t = 1$, *by Mr. Raphson's Method of Approximation.*

Art. 55. But this equation $t^2 + \frac{2}{T} \times t = 1$, being a quadratick equation, may alſo be reſolved to any propoſed degree of exactneſs, in a direct manner, or by only one ſet of arithmetical operations, without having recourſe to the repeated proceſſes of Mr. Raphſon's Method of Approximation. And Dr. Mackay has furniſhed me likewiſe with ſuch a reſolution of it, which is as follows.

The Biſection of an Arch of 4 Minutes, performed by reſolving in the common way, or by the extraction of the ſquare-root, the Quadratick Equation $t^2 + \frac{2}{T} \times t = 1$, *in which* T *is the Tangent of an Arch of 4 Minutes, and is* = 0.001,163,553,359,757,188,938,65, *and* t, *or the Root of the ſaid Equation, is the Tangent of an Arch of 2 Minutes.*

The co-efficient $\frac{2}{T}$ of the unknown quantity t in the ſecond term of this equation is $= \frac{2}{0.001,163,553,359,757,188,938,65} =$ the mixt number 1718.872, 609,690,509,838,486,820,225; which, being ſubſtituted for $\frac{2}{T}$ in this equation $t^2 + \frac{2}{T} \times t = 1$, will convert it into the equation $t^2 + 1718.872,609,690,509,838,486,820,225 \times t = 1$, or (neglecting the three laſt figures 225 of the value of $\frac{2}{T}$,) into the equation $t^2 + 1718.872,609,690,509,838,486,820 \times t = 1$.

Now this equation may be reſolved by the common method of reſolving quadratick equations of th s form, to wit, by adding to both ſides of it the ſquare of half the co-efficient of the unknown quantity t, that is, of half the number 1718. 872,609,690,509,838,486,820, or the ſquare of the number 859.436,304,845, 254,919,243,410, and extracting the ſquare-roots of the oppoſite ſides of the equation

tion thereby obtained. The ſquare of this number 859.436,304,845,254,919,243,410, Dr. Mackay finds to be = 738,630.762,086,065,944,179,205,412,018; which, being added to both ſides of the equation $t^2 + 1718.872,609,690,509,838,486,820 \times t = 1$, produces the equation $t^2 + 1718.872,609,690,509,838,486,820 \times t + \overline{859.436,304,845,254,919,243,410}^2 = 738,631.762,086,065,944,179,205,412,018$. Therefore, by extracting the ſquare-roots of both ſides, we ſhall have $t + 859.436,304,845,254,919,243,410 (= \sqrt{738,631.762,086,065,944,179,205,412,018}) = 859.436,886,621,737,887,447,762,7$, and (by ſubtracting 859.436,304,845,254,919,243,410 from both ſides,) we ſhall have $t =$

$$\left\{\begin{array}{l} 859.436,886,621,737,887,447,762,7 \\ -\ 859.436,304,845,254,919,243,410,0 \end{array}\right\}$$

$$= \quad 0.000,581,776,482,968,204,352,7;$$

that is, the root of the equation $t^2 + \frac{2}{T} \times t = 1$, or the tangent of an arch of 2 minutes, will be = 0.000,581,776,482,968,204,352,7. Q. E. I.

This number agrees with the number 0.000,581,776,482,968,204,352,750,16, found above in Art. 54, by Mr. Raphſon's Method of Approximation, in all it's twenty-two figures 0.000,581,776,482,968,204,352,7.

The operations of ſquaring the number 859.436,304,845,254,919,243,410, and of extracting the ſquare-root of the number 738,631.762,086,065,944,179,205,412,018, are performed by Dr. Mackay in the following manner.

859.

859.436,304,845,254,919,243,410
859.436,304,845,254,919,243,410

687,549.043 876 203 935 394 728
42,971.815 242 262 745 962 170 5
7,734.926 743 607 294 273 190 69
343.774 521 938 101 967 697 364
25.783 089 145 357 647 577 302 3
5.156 617 829 071 529 515 460 46
.257 830 891 453 576 475 773 023
3 437 745 219 381 019 676 974
687 549 043 876 203 935 395
34 377 452 193 810 196 770
4 297 181 524 226 274 596
171 887 260 969 050 984
42 971 815 242 262 746
3 437 745 219 381 020
773 492 674 360 729
8 594 363 048 452
7 734 926 743 607
171 887 260 969
34 377 452 194
2 578 308 915
343 774 521
8 594 363

738,630.762,086,065,944,179,205,412,018
1

738,631.762,086,065,944,179,205,412,018

Root. 859.436,886,621,737,887,447,762,7

73 86 31.76 20 86 06 59 44 17 92 05 41 20 18

64

165 } 986
    } 825

1709 } 16131
     } 15381

17184 } 75076
      } 68736

171883 } 634020
       } 515649

1718866 } 11837186
        } 10313196

17188728 } 152399006
         } 137509824

171887368 } 1488918259
          } 1375098944

1718873766 } 11381931544
           } 10313242596

17188737726 } 106868894817
            } 103132426356

171887377322 } 3736846846192
              } 3437747546 44

1718873773241 } 29872091548 05
              } 17188737732 41

17188737732427 } 126833538156441
               } 120321164126989

171887377324343 } 6512374029452 20
                 } 5156621319730 29

1718873773243467 } 13557527097219118
                 } 12032116412704269

152541068451 4849

152541068451 4849
137509901859 4780
150311665920069
137509901859478
12801764060591
12032116412704
769647647887
687549509297
82098138590
68754950930
13343187660
12032116413
1311071247
1203211641
107859606
103132426
4727180
3437745
1289435
1203212
86223

End of the Bisection of an Arch of 4 Minutes, by resolving the Quadratick Equation $t^2 + \frac{2}{T} \times t = 1$, *(in which* T *is the Tangent of an Arch of 4 Minutes, and is =* 0.001,163,553,359,757,188,938,65,) *in a direct way, or by adding to both sides the Square of* $\frac{1}{T}$, *and extracting the Square-Roots of the Quantities thence arising.*

The Bisection of an Arch of 2 *Minutes, by resolving the Quadratick Equation* $t^2 + \frac{2}{T} \times t = 1$, *in which* T *is the Tangent of an Arch of* 2 *Minutes, and is equal to* 0.000,581,776,482,968,204,352,750, *(as it has been found to be above in Art.* 54,) *and* t, *or the Root of the said Equation, is the Tangent of half the said Arch, or of an Arch of* 1 *Minute.*

Art. 56. In order to prepare this equation for reſolution, Dr. Mackay, in the firſt place, reduces the fraction $\frac{2}{T}$, or $\frac{2}{0.000,581,776,482,968,204,352,750}$ (which is the co-efficient of the unknown quantity *t* in the ſecond term of the ſaid equation,) to a mixt number conſiſting of a whole number and a decimal fraction, by dividing it's numerator by it's denominator, and finds it to be equal to the mixt number 3437.746,382,933,985,613,383,297,287. And conſequently by ſubſtituting this number inſtead of $\frac{2}{T}$ in the equation $t^2 + \frac{2}{T} \times t = 1$, he converts the ſaid equation into the equation $t^2 + 3437.746,382,933,985,613,383,297,287 \times t = 1$.

The operation of dividing the numerator 2 by the denominator 0.000,581,776,482,968,204,352,750, is performed by Dr. Mackay in the following manner.

The Quotient.

(3437·746,382,933,985,613,383,297,287

The Divisor.

0.000,581,776,482,968,204,352,75)

The Dividend.

2.000,000,000,000,000,000,000,000,000,00
1.745,329,448,904,613,058,25

·254 670 551 095 386 941 750
232 710 593 187 281 741 100

·21 959 957 908 105 200 650 0
17 453 294 489 046 130 582 5

·4 506 663 419 059 070 067 50
4 072 435 380 777 430 469 25

·434 228 038 281 639 598 250
407 243 538 077 743 046 925

·26 984 500 203 896 551 325 0
23 271 059 318 728 174 110 0

·3 713 440 885 168 377 215 00
3 490 658 897 809 226 116 50

·222 781 987 359 151 098 500
174 532 944 890 461 305 825

·48 249 042 468 689 792 675 0
46 542 118 637 456 348 220 0

·1 706 923 831 233 444 455 0
1 163 552 965 936 408 705 5

·543 370 865 297 035 749 5
523 598 834 671 383 917 5

·19 772 030 625 651 832 0
17 453 294 489 046 130 6

·2 318 736 136 605 701 4
1 745 329 448 904 613 1

·573 406 687 701 088 3
523 598 834 671 383 9

·49 807 853 029 704 4
46 542 118 637 456 3

·3 265 734 392 248 1
2 908 882 414 841 0

·356 851 977 407 1

·356 851 977 407 1
349 065 889 780 9

·7 786 087 626 2
5 817 764 829 7

1 968 322 796 5
1 745 329 498 9

·222 993 347 6
174 532 944 9

·48 460 402 7
46 542 118 6

·1 918 284 1
1 745 329 4

·172 954 7
116 355 3

·56 599 4
52 359 9

·4 239 5
4 072 4

·167 1
116 4

·50 7
46 5

·4 2
4 1

·1

Art. 57. Having thus converted the equation $t^2 + \frac{2}{T} \times t = 1$, into the equation $t^2 + 3437.746,382,933,985,613,383,297,287 \times t = 1$, Dr. Mackay proceeds to resolve this last equation in the following manner.

He begins by observing that the tangent of an arch of 1 minute is greater than the arch itself, but less than half the tangent of an arch of 2 minutes. Now an arch of 1 minute is the 60th part of an arch of 1 degree, or of the 180th part of the semi-circumference of the circle, which when the radius is called 1, (as it is in the present computation,) is $= 3.141,592,653,59$ &c. Therefore an arch of 1 minute will be $= \frac{3.141,592,653,59\ \&c}{60 \times 180}$ $(= \frac{3.141,592,653,59\&c}{10,800})$ $= 0.000,290,888,208,\&c$; and consequently the tangent of the said arch will be greater than $0.000,290,888,208,\&c$. And it will be less than half the tangent of an arch of 2 minutes, which has been shewn to be equal to $0.000,581,776,482,\&c$; and therefore it will be less than $\frac{0.000,581,776,482,\&c}{2}$, or than $0.000,290,888,241,\&c$. Therefore it will be of an intermediate magnitude between $0.000,290,888,241,\&c$, and $0.000,290,888,208,\&c$. And, further, it will be nearer to this lesser limit $0.000,290,888,208,\&c$ than to it's greater limit $0.000,290,888,241,\&c$. Therefore Dr. Mackay conjectures that it will be equal to $0.000,290,888,21$, and then substitutes this number $0.000,290,888,21$, instead of t, in the binomial quantity $t^2 + \frac{2}{T} \times t$, which forms the first, or left-hand, side of the equation $t^2 + \frac{2}{T} \times t = 1$, and compares the result with 1, or the absolute term of the said equation, in order to discover whether the said result will be greater, or will be less, than the said absolute term, and consequently whether the said number $0.000,290,888,21$ (which has been substituted, instead of t, in the binomial quantity $t^2 + \frac{2}{T} \times t$,) will be greater, or will be less, than the true value of t in the said equation $t^2 + \frac{2}{T} \times t = 1$, or than the root of the said equation, which root is equal to the tangent of an arch of 1 minute. This substitution of $0.000,290,888,21$, instead of t, in the said binomial quantity $t^2 + \frac{2}{T} \times t$ is made by Dr. Mackay in the manner following.

$t = 0.000,290,888,21$
$t = 0.000,290,888,21$

0.000 000 058 177 642
26 179 938 90
232 710 568
23 271 056 8
2 327 105 68
58 177 642
2 908 882 1

$t^2 = 0.000,000,084,615,950,717,004,1$

$\frac{2}{T} = 3437.746,382,933,985,613,383,297,287$
$t = 0.000,290,888,21$

0.687 549 276 586 797 122 676 659
309 397 174 464 058 705 204 497
2 750 197 106 347 188 490 709
275 019 710 634 718 849 071
27 501 971 063 471 884 907
687 549 276 586 797 123
34 377 463 829 339 856

$\frac{2}{T} \times t = 0.999,999,891,765,641,623,242,820$

Therefore $t^2 + \frac{2}{T} \times t$ will be $=$

$$\left\{ \begin{array}{l} 0.000,000,084,615,950,717,004,1 \\ + \; 0.999,999,891,765,641,623,242,820 \end{array} \right\}$$

$= 0.999,999,976,381,592,340,246,920$; which is leſs than 1, or the abſolute term of the equation $t^2 + \frac{2}{T} \times t = 1$, or the value of the binomial quantity $t^2 + \frac{2}{T} \times t$ in that equation: and conſequently 0.000,290,888,21, is leſs than the true value of t in the ſaid equation, or than the true value of the tangent of an arch of 1 minute. Q. E. I.

A Process

A Process of Mr. Raphson's Method of Approximation.

Art. 58. Having thus discovered that the number 0.000,290,888,21 is less than the true value of t, or the root of the equation $t^2 + \frac{2}{T} \times t = 1$, Dr. Mackay puts z for their difference; and, in order to obtain a more exact value of t, he substitutes the binomial quantity $0.000,290,888,21 + z$, instead of t, in the equation $t^2 + \frac{2}{T} \times t = 1$. This he does in the following manner.

Since t is $= 0.000,290,888,21 + z$, we shall have

$t^2 = \overline{0.000,290,888,21+z}^2$ $(= \overline{0.000,290,888,21}^2 + 2 \times 0.000,290,888,21 \times z +$ &c

$= \overline{0.000,290,888,21}^2 + 0.000,581,776,42 \times z +$ &c)

$= 0.000,000,084,615,950,717,004,1 + 0.000,581,776,42 \times z +$ &c,

and $\frac{2}{T} \times t = \frac{2}{T} \times \overline{0.000,290,888,21+z}$ $(= \frac{2}{T} \times 0.000,290,888,21 + \frac{2}{T} \times z)$

$= 0.999,999,891,765,641,623,242,820 + 3437.746,382,933,985,613,383,297,287 \times z$; and consequently the binomial quantity $t^2 + \frac{2}{T} \times t$ will be equal to the multinomial quantity

$$\left\{\begin{array}{l} 0.000,000,084,615,950,717,004,1 + 0.000,581,776,42 \times z \text{ \&c} \\ + 0.999,999,891,765,641,623,242,820 + 3437.746,382,933,985,613, \\ \qquad 383,297,287 \times z \end{array}\right\}$$

$= 0.999,999,976,381,592,340,246,920 + 3437.746,964,710,405,613,383,297,287 \times z$. &c

But the binomial quantity $t^2 + \frac{2}{T} \times t = 1$.

Therefore

Therefore the quantity 0.999,999,976,381,592,340,246,920 + 3437.746,964,710,405,613,383,297,287×z will alſo be = 1. And conſequently 3437.746,964,710,405,613,383,297,287 × z will be (= 1 − 0.999,999,976,381,592,340,246,920) = 0.000,000,023,618,407,659,753,080, and z will be = the fraction $\frac{0.000,000,023,618,407,659,753,080}{3437.746,964,710,405,613,383,297,287}$ = 0.000,000,000,006,870,315,908,123. Therefore t, or 0.000,290,888,21 + z, will be = 0.000,290,888,21 + 0.000,000,000,006,870,315,908,123 = 0.000,290,888,216,870,315,908,123; that is, the tangent of an arch of 1 Minute will be very nearly equal to 0.000,290,888,216,870,315,908,123. Q. E. I.

Art. 59. Of this number the firſt twenty-one figures, to wit, 0.000,290,888,216,870,315,908, at leaſt, will certainly be exact, if no miſtake has been made in the calculation; and it is probable that the figure 1 in the twenty-ſecond place of decimal figures (reckoned from the place of units,) is likewiſe exact. But, to remove all doubts upon this matter, Dr. Mackay proceeds to ſubſtitute the ſaid twenty-one figures, to wit, 0.000,290,888,216,870,315,908, inſtead of t, in the binomial quantity $t^2 + \frac{2}{T} \times t$, in order to diſcover whether the value of the ſaid quantity reſulting from ſuch ſubſtitution will be greater, or will be leſs, than 1, or the abſolute term of the equation $t^2 + \frac{2}{T} = 1$; and conſequently whether the ſaid number 0.000,290,888,216,870,315,908 is greater, or is leſs, than the true value of t in that equation. This ſubſtitution he performs in the following manner.

$$t = 0.000,290,888,216,870,315,908$$
$$t = 0.000,290,888,216,870,315,908$$

0.000 000 058 177 643 374 063
26 179 939 518 328
232 710 573 496
23 271 057 350
2 327 105 735
58 177 643
2 908 882
1 745 329
232 711
20 362
87
3
1

$$t^2 = 0.000,000,084,615,954,713,990$$

$$\frac{2}{T} = 3437.746,382,933,985,613,383,297$$
$$t = 0.000,290,888,216,870,315,908$$

0.687 549 276 586 797 122 677
309 397 174 464 058 705 204
2 750 197 106 347 188 491
275 019 710 634 718 849
27 501 971 063 471 885
687 549 276 586 797
34 377 463 829 340
20 626 478 297 604
2 750 197 106 347
240 642 246 805
1 031 323 915
34 377 464
17 188 732
3 093 972
27 502

$$\frac{2}{T} \times t = 0.999,999,915,384,045,285,584$$

$$= 1$$

Therefore

Therefore $t^2 + \frac{2}{T} \times t$ will be $=$

$$\left\{\begin{array}{l} 0.000,000,084,615,954,713,990 \\ + \ 0.999,999,915,384,045,285,584 \end{array}\right\}$$

$= 0.999,999,999,999,999,999,574$; which is lefs than 1, or the abfolute term of the equation $t^2 + \frac{2}{T} \times t = 1$. And therefore the number 0.000,290,888,216,870,315,908 is lefs than the true value of t in that equation, or than the root of that equation. Q. E. I.

Art. 60. But, if this number 0.009,290,888,216,870,315,908, be increafed by only an unit in the laft, or 21ft, place of figures, fo as to be converted into the number 0.000,290,888,216,870,315,909, and the faid new number be fubftituted inftead of t in the binomial quantity $t^2 + \frac{2}{T} \times t$, the value of the faid binomial quantity refulting from fuch fubftitution will be greater than 1, or the abfolute term of the equation $t^2 + \frac{2}{T} \times t = 1$, and confequently the faid new number 0.000,290,888,216,870,315,909, will be greater than the true value of t in that equation. This may be fhewn in the following manner.

Let a be put for the number 0.000,290,888,216,870,315,908, and b for the new, or greater, number 0.000,290,888,216,870,315,909.

Then will b be $(= a + 0.000,000,000,000,000,000,001) = a + \frac{1}{\overline{10}^{21}}$.

Therefore b^2 will be $= \overline{a + \frac{1}{\overline{10}^{21}}}^2$ $(= a^2 + 2a \times \frac{1}{\overline{10}^{21}} + \frac{1}{\overline{10}^{42}} = a^2 +$ $2 \times 0.000,290,\&c \times \frac{1}{\overline{10}^{21}} + \frac{1}{\overline{10}^{42}} = a^2 + 0.000,580,\&c \times \frac{1}{\overline{10}^{21}} + \frac{1}{\overline{10}^{42}}$ $= a^2 + \frac{580\,\&c}{\overline{10}^{6}} \times \frac{1}{\overline{10}^{21}} + \frac{1}{\overline{10}^{42}} = a^2 + \frac{580}{\overline{10}^{27}} + \frac{1}{\overline{10}^{42}} = a^2 + \frac{58\,\&c}{\overline{10}^{26}} +$ $\frac{1}{\overline{10}^{42}}) = a^2 + 0.000,000,000,000,000,000,000,000,58 + \frac{1}{\overline{10}^{42}}$; or (neglecting $\frac{1}{\overline{10}^{42}}$ on account of it's extreme fmallnefs,) b^2 will be $= a^2 + 0.000,000,$

000,000,000,000,000,000,58. And $\frac{2}{T} \times b$ will be $= \frac{2}{T} \times \overline{a + \frac{1}{10|^{21}}} =$ $\frac{2}{T} \times a + \frac{2}{T} \times \frac{1}{10|^{21}} = \frac{2}{T} \times a +$ 3437.&c $\times \frac{1}{10|^{21}} = \frac{2}{T} \times a +$ 3437.&c $\times$ 0.000,000,000,000,000,000,001) $= \frac{2}{T} \times a +$ 0.000,000,000,000,000,003,437,&c. And consequently $b^2 + \frac{2}{T} \times b$ will be $= a^2 +$ 0.000,000,000,000,000,000,000,000,58 $+ \frac{2}{T} \times a +$ 0.000,000,000,000,000,003,437,&c

$$= a^2 + \frac{2}{T} \times a \begin{cases} + \text{0.000,000,000,000,000,000,000,000,58} \\ + \text{0.000,000,000,000,000,003,437,\&c} \end{cases}$$

$$= a^2 + \frac{2}{T} \times a + \text{0.000,000,000,000,000,003,437,000,58}$$

$$= \begin{cases} \text{0.999,999,999,999,999,999,574} \\ + \text{0.000,000,000,000,000,003,437,\&c} \end{cases}$$

$$= \text{1.000,000,000,000,000,003,011,\&c;}$$

which is greater than 1, or the absolute term of the equation $t^2 + \frac{2}{T} \times t = 1$. Therefore b, or 0.000,290,888,216,870,315,909, is greater than the true value of t, or the root of that equation. We may therefore now conclude with certainty that the true value of t in the equation $t^2 + \frac{2}{T} \times t = 1$, or the tangent of an arch of 1 Minute, is greater than 0.000,290,888,216,870,315,908, but less than 0.000,290,888,216,870,315,909, or, in other words, that all the twenty-one figures of the number 0.000,290,888,216,870,315,908, that has been found above for the value of t, or the tangent of 1 Minute, are exact.

Q. E. D.

Art. 61. But this equation $t^2 + \frac{2}{T} \times t = 1$, being a quadratick equation, may also be resolved to any proposed degree of exactness, in a direct manner, or by only one set of arithmetical operations, without having recourse to the repeated processes of Mr. Raphson's Method of Approximation. And Dr. Mackay has furnished me likewise with such a resolution of it; which is as follows.

The

The Bisection of an Arch of 2 *Minutes, performed by resolving in the common way, or by the extraction of the Square-Root, the Quadratick Equation* $t^2 + \frac{2}{T} \times t = 1$, *in which* T *is the Tangent of an Arch of* 2 *Minutes, and is equal to* 0.000,581,776,482,968,204,352,750, (*as it has been found to be above in Art.* 54,) *and* t, *or the Root of the said Equation, is the Tangent of half the said Arch, or of an Arch of* 1 *Minute.*

It has been ſhewn above in Art. 56, that $\frac{2}{T}$ is 3437.746,382,933,985,613, 383,297,287. Therefore, if this number be ſubſtituted, inſtead of $\frac{2}{T}$, in the equation $t^2 + \frac{2}{T} \times t = 1$, the ſaid equation will be converted into the equation $t^2 + 3437.746,382,933,985,613,383,297,287 \times t = 1$. This equation Dr. Mackay reſolves in the following manner.

He divides the co-efficient of t, to wit, the number 3437.746,382,933,985, 613,383,297,287, by 2, and thereby obtains the number 1718.873,191,466, 992,806,691,648,643,5, and he then ſquares this laſt number, and finds it's ſquare to be = 2,954,525.048,343,925,312,842,322,795,310. He then adds this ſquare to both ſides of the equation $t^2 + 3437.746,382,933,985,613,383,297,287 \times t = 1$, and thereby obtains the equation $t^2 + 3437.746,382,933,985,613,383,297,287 \times t + \overline{1718.873,191,466,992,806,691,648,643,5}^2$ (= 1 + 2,954,525.048,343,925,312,842,322,795,310) = 2,954,526.048,343,925, 312,842,322,795,310; and then (by extracting the ſquare-roots of both ſides,) obtains the equation $t + 1718.873,191,466,992,806,691,648,643,5 = \sqrt{1718.}$ 873,482,355,209,677,007,556,766,9. Therefore t will be =

$$\left\{\begin{array}{l} 1718.873,482,355,209,677,007,556,766,9 \\ -\ 1718.873,191,466,992,806,691,648,643,5 \end{array}\right\}$$
$$= \quad 0.000,290,888,216,870,315,908,123,4;$$

which agrees in the first twenty-four figures, 0.000,290,888,216,870,315,908,123, with the value of the same root t that was found above in Art. 58, by means of Mr. Raphson's Method of Approximation. We may therefore now conclude that t, or the tangent of an Arch of 1 Minute, is equal to the decimal fraction 0.000,290,888,216,870,315,908,123. Q. E. I.

The operations of squaring the number 1718.873,191,466,992,806,691,648,643,5, and of extracting the square-root of the number 2,954,526,048,343,925,312,842,322,795,310, are performed by Dr. Mackay in the following manner.

$$\frac{1}{T} = 1718.873,191,466,992,806,691,648,643,5$$
$$\frac{1}{T} = 1718.873,191,466,992,806,691,648,643,5$$

1718,873.191 466 992 806 691 648 643 5
1203 211.234 026 894 964 684 154 050 45
17 188.731 914 669 928 066 916 486 435
13 750.985 531 735 942 453 533 189 148
1 375.098 553 173 594 245 353 318 915
120.321 123 402 689 496 468 415 405
5.156 619 574 400 978 420 074 946
.171 887 319 146 699 280 669 165
.154 698 587 232 029 352 602 248
1 718 873 191 466 992 806 692
687 549 276 586 797 122 677
103 132 391 488 019 568 401
10 313 239 148 801 956 840
1 546 985 872 320 293 526
154 698 587 232 029 353
3 437 746 382 933 986
1 375 098 553 173 594
10 313 239 148 802
1 031 323 914 880
154 698 587 232
1 718 873 191
1 031 323 915
68 754 928
13 750 986
1 031 324
68 755
5 157
859

$$\left|\frac{1}{T}\right|^2 = 2,954,525.048,343,925,312,842,322,795,310.$$

$$\left|\frac{1}{T}\right|^2 = 2,954,525.048,343,925,312,842,322,795,310$$

$$1 = 1.000,000,000,000,000,000,000,000$$

$$\left|1+\frac{1}{T}\right|^2 = 2,954,526.048,343,925,312,842,322,795,310$$

The Square-Root.

(1718.873,482,355,209,677,007,556,766,9.

1

27) 195
189

341) ··645
341

3428) 30426
27424

34368) ·300204
274944

343767) ·2526083
2406369

3437743) ·11971443
10313229

34337464) ·165821492
137509856

343774688) ·2831163653
275019750 4

3437746962) ··8096614912
6875493924

34377469643) 1221120988 84
1031324089 29

343774696465) ·1897968995523
1718873482325

3437746964705) ·1790955131982 2
1718873482352 5

34377469647102) ··7208164962977 9
6875493929420 4

343774696471049) ·3326710335575531 0
3093972268239368 1

·2327380673361629
2062648178826245

·264732494535384
240642287529728

·24090207005656
240642287529 73

···25978252683

···25978252683
240642287 53

·1914023930
1718873482

·195150448
171887348

·23263100
20626482

·2636618
2406423

·230195
206265

·23930
20626

·3304
3094

·210

Art. 62. And thus, by finding by two different methods of reſolving the equation $t^2 + \frac{2}{T} \times t = 1$, or $t^2 + 3437{\cdot}746{,}382{,}933{,}985{,}613{,}383{,}297{,}287 \times t = 1$, that it's root t (which is equal to the tangent of an arch of 1 Minute) is $= 0.000{,}290{,}888{,}216{,}870{,}315{,}908{,}123$, the long and difficult Computation, undertaken by Dr. Mackay, for deriving the length of the Tangent of an Arch of 1 Minute from the Tangent of an Arch of 45 Degrees, by the principles of Plane Geometry, and without the help of the infinite Serieſes invented by Mr. James Gregory for expreſſing the length of a circular arch in powers of the tangent and the radius, and the length of the tangent in powers of the arch and the radius (which Serieſes are obtained by the Method of Fluxions, and the Doctrine of the Reverſion of Serieſes,) is now, at laſt, ſucceſsfully concluded.

End of Dr. Mackay's Computation of the Length of the Tangent of an Arch of 1 Minute of a Degree from the Tangent of an Arch of 45 Degrees by repeated Sections of the said Arch, begun above in page 781.

THE

THE

SECOND COMPUTATION

Mentioned above in page 780, *Art.* 16; *to wit, the Computation of the Length of the Sine of an Arch of* 1 *Minute in a Circle of which the Radius is called* 1, *derived from the Length of the Tangent of the same Arch, which has been found in the foregoing Computation to be* = 0.000,290,888,216,870,315,908, &c.

By Dr. ANDREW MACKAY, LL.D.

Late of Aberdeen in Scotland, and a Fellow of the Royal Society of Edinburgh.

Art. 63. The Secant of any arch in the Quadrant of a Circle is to the Tangent of the ſame arch in the ſame proportion as the radius of the circle is to the ſine of the ſame arch. Therefore the ſecant of an arch of 1 Minute of a Degree is to the tangent of the ſame arch in a circle of which the radius is called 1, in the ſame proportion as 1 is to the ſine of the ſame arch of 1 Minute in the ſame circle; that is, the ſecant of an arch of 1 Minute is to 0.000,290,888,216,870,315,908,&c, in the ſame proportion as 1 is to the ſine of the ſame arch of 1 Minute in the ſame circle. Therefore, if the length of the ſecant of an arch of 1 Minute in the ſaid circle were known, the length of the ſine of the ſame arch of 1 Minute might be found by means of this proportion. But the length of the ſecant of this arch may be found by ſquaring the number 0.000,290,888,216,870,315,908 (which is equal to the tangent of this arch,) and adding to the ſaid ſquare the ſquare of the radius 1, that is, 1×1, or 1, and extracting the ſquare-root of the ſum. Now the ſquare of the tangent 0.000,290,888,216,870,315,908, has been computed by Dr. Mackay above in Art. 59, page 880, and found to = 0.000,000,084,615,954,713,990; to which if we add 1, or 1.000,000,000,000,000,000,000, we ſhall have the number 1.000,000,084,615,954,713,990, for the ſquare of the ſecant of 1 Minute. Therefore the ſecant of 1 Minute will be equal to the ſquare-root of the number 1.000,000,084,615,954,713,990, that is, to the number 1.000,000,042,307,976,462,012; as will appear by the following operation.

1.000,

Square of Secant 1. Secant of 1′.

1.000,000,084,615,954,713,990 (1.000,000,042,307,976,462,012

1
20) ·00
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200) ··00
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2000) ··00
00
20000) ··08
00
200000) ·846
000
2000000) 84615
00000
20000000) 8461595
0000000
200000004) 846159547
800000016
2000000082) ·4615953113
4000000164
20000000843) ·61595294999
60000002529
200000008460) ·159529247000
000000000000
2000000084607) 15952924700000
140000005922 49
200000008461 49) ·1952924107751 00
1800000076153 41
2000000084615 87) ·1529240315975900
1400000059231109
20000000846159 46) ·1292402567447 9100
1200000507695676
200000008461595 24) ··9240251667834 2400
800000033846380 96
2000000084615952 86) 1240251329370430400
1200000050769571716
20000000846159529 22) ··4025127860085868400
4000000169231905844
200000008461595292 40) ··2512769085396255600
00000000000000000000
2000000084615952924 01) 2512769085396255600000
2000000084615952924 01
20000000846159529240 22) ·5127690007803026759900
4000000169231905848044
1127689838571120911856

Therefore the number 1.000,000,042,307,976,462,012 will be to the number 0.000,209,888,216,870,315,908 in the ſame proportion as 1 to the ſine of an arch of 1 minute; and conſequently the ſine of 1 minute will be equal to $\frac{1 \times 0.000,290,888,216,870,315,908}{1.000,000,042,307,976,462,012}$ $(= \frac{0.000,290,888,216,870,315,908}{1.000,000,042,307,976,462,012})$ = 0.000,290,888,204,563,424,596; as will appear from the following operation of diviſion.

The Divisor. *The Quotient.*

1.000,000,042,307,976,462,012) (0.000,290,888,204,563,424,596

The Dividend.

0.000,290,888,216,870,315,908,000,000,000
200 000 008 461 595 292 402 4
·90 888 208 408 720 615 597 60
90 000 003 807 717 881 581 08
··888 204 601 002 734 016 520 0
800 000 003 846 381 169 609 6
·88 204 567 156 352 846 910 40
80 000 003 384 638 116 960 96
·8 204 563 771 714 729 949 440
8 000 000 338 463 811 696 096
·204 563 433 250 918 253 344 0
200 000 008 461 595 292 402 4
··4 563 424 789 322 960 941 600
4 000 000 169 231 905 848 048
·563 424 620 091 055 093 552 0
500 000 021 153 988 231 006 0
·63 424 598 937 066 862 546 00
60 000 002 538 478 587 720 72
·3 424 596 398 588 274 825 280
3 000 000 126 923 929 386 036
·424 596 271 664 345 439 244 0
400 000 016 923 190 584 804 8
·24 596 254 741 154 854 439 20
20 000 000 846 159 529 240 24
·4 596 253 894 995 325 198 960
4 000 000 169 231 905 848 048
·596 253 725 763 419 350 912 0

$$\begin{array}{l}
\cdot 596\ 253\ 725\ 763\ 419\ 350\ 912\ 0 \\
\underline{500\ 000\ 021\ 153\ 988\ 231\ 006\ 0} \\
\cdot 96\ 253\ 704\ 609\ 431\ 119\ 906\ 00 \\
\underline{90\ 000\ 003\ 807\ 717\ 881\ 581\ 08} \\
\cdot 6\ 253\ 700\ 801\ 713\ 238\ 324\ 920 \\
\underline{6\ 000\ 000\ 253\ 847\ 858\ 772\ 072} \\
\cdot 253\ 700\ 547\ 865\ 379\ 552\ 848.
\end{array}$$

Therefore the Sine of an arch of 1 Minute of a Degree, in a circle of which the radius is called 1, will be = 0.000,290,888,204,563,424,596.

Q. E. I.

Art. 64. This value of the Sine of an arch of 1 Minute of a Degree agrees with the value of it found above in Dr. Hutton's Computation of it in page 473, to wit, 0.000,290,888,204,563,5, in the firſt 15 places of figures, reckoned from the place of units, to wit, the figures 0.000,290,888,204,563, and therefore confirms the truth of the ſaid value of the Sine of 1 Minute, obtained by that Computation, in all the ſaid 15 figures. And, therefore, we may now ſafely conclude, from the agreement of the two values of the Sine of an arch of 1 Minute, obtained by theſe two different Computations of it by Dr. Hutton and Dr. Mackay in the firſt fifteen figures, (which Computations have been grounded on the ſimple principles of common Geometry, without any aſſiſtance from the infinite Serieſes of Sir Iſaac Newton and Mr. James Gregory, which are derived from the Method of Fluxions and the Doctrine of the Reverſion of Serieſes,) — I ſay, we may now ſafely conclude, "that the Length of the Sine of an Arch of 1 Minute of a Degree, in a circle of which the radius is called 1, is equal to the decimal fraction 0.000,290,888,204,563, as nearly as it can be expreſſed in 15 places of decimal fractions, reckoned from the place of units;" which was the object of the two preceeding Computations performed by Dr. Mackay.

End of Dr. Mackay's Computation of the Length of the Sine of an Arch of 1 Minute of a Degree by deriving it from the Length of the Tangent of the same Arch.

A Comparison

A Comparison between the foregoing Values of the Tangent and the Sine of an Arch of 1 *Minute of a Degree, in a Circle of which the Radius is called* 1, *with the Values of the same Lines, computed by Means of the infinite Serieses invented by Mr.* JAMES GREGORY *and Sir* ISAAC NEWTON *for expressing the Lengths of the Tangent and the Sine of a Circular Arch that is less than an Arch of* 45 *Degrees, in powers of the Arch and of the Radius.*

BY DR. ANDREW MACKAY, LL.D.
Late of Aberdeen in Scotland, and Fellow of the Royal Society at Edinburgh.

Art. 65. When Dr. Mackay had compleated the two preceeding Computations of the Length of the Tangent of an Arch of 1 Minute of a Degree from the Tangent of an Arch of 45 Degrees, and of the length of the Sine of the ſame Arch of 1 Minute from the value of it's Tangent which he had obtained by his former Computation, he had the curioſity to compute the length of the ſame Tangent and Sine by the help of the two infinite Serieſes, which have been invented by Mr. James Gregory of Aberdeen, and Sir Iſaac Newton, for expreſſing the lengths of the Tangent of any given arch of a Quadrant, that is leſs than 45 Degrees, and of the Sine of the ſame Arch, in terms involving the powers of the arch and of the radius of the Circle. The former of theſe two Serieſes, (which was invented by Mr. James Gregory, about the month of December, 1670,) is as follows.

If the radius of a circle be called r, and any arch of it that is leſs than 45 Degrees, be called a, and it's Tangent be called t, the Tangent t will be $=$ the Series $a + \frac{a^3}{3r^2} + \frac{2a^5}{15r^4} + \frac{17a^7}{315r^6} + \frac{62a^9}{2835r^8} + \frac{1382a^{11}}{155,925r^{10}}$ + &c. This Series may, perhaps, be true alſo when the arch a is greater than an arch of 45 Degrees, but leſs than the radius r; but it converges ſo ſlowly when the arch a is greater than 45°, and even when it is equal, or nearly equal, to 45°, as to be of little, or no, uſe. But, when the arch is small, it converges with very great ſwiftneſs, and is exceedingly uſeful. The Inveſtigation of this Series has been given in the 3d volume of this Collection of Tracts, called *Scriptores Logarithmici*, in pages 419, 420, 421, &c, - - - 431, and again in pages 432, 433, 434, &c, - - - 437, of the ſame volume. If the radius r be equal to 1, (as it is in the preceeding computations of the Tangent and Sine of an arch of 1 Minute of a Degree,) this Series will become $=$ the Series $a + \frac{a^3}{3} + \frac{2a^5}{15} + \frac{17a^7}{315} + \frac{62a^9}{2835} + \frac{1382a^{11}}{155,925}$ + &c.

Art. 66. Now the length of an arch of 1 Minute of a Degree in a Circle of which the Radius is called 1, and of which the ſemi-circumference is conſequently $=$ 3.141,592,653,589,793,238,462,6,&c, is $= \frac{3.141,592,653,589,793,238,462,6\ \&c}{60 \times 180}$ $\left(= \frac{3.141,592,653,589,793,238,462,6\ \&c}{10,800}\right) =$ 0.000,290,888,208,665,721,596,153, &c; or, if we carry the figures only to 21 places below the place of units, it is $=$ 0.000,290,888,208,665,721,596. Dr. Mackay therefore takes $a =$ 0.000,290,888,208,665,721,596, and then computes the four following powers of a in the following manner.

a^3 or

a, or Length of an Arch of 1′ = 0.000,290,888,208,665,721,596

0.000,290,888,208,665,721,596

0.000 000 058 177 641 733 144 3
26 179 938 779 914 9
232 710 566 932 6
23 271 056 693 3
2 327 105 669 3
58 177 641 7
2 327 105 7
174 532 9
17 453 3
1 454 4
203 6
5 8
3
1

a^2 = 0.000,000,084,615,949,940,752,2

a = 0.000,290,888,208,665,721,596

0.000 000 000 023 271 056 693 3
1 163 552 834 7
174 532 925 2
2 908 882 1
1 454 441 0
261 799 4
116 355
26 180
261 8
116
2

a^3 = 0.000,000,000,024,613,782,102,8

a = 0.000,290,888,208,665,721,596,

0.000 000 000 000 005 817 764 2
1 163 552 8
174 532 9
2 908 9
872 7
203 6
23 3
6

a^4 = 0.000,000,000,000,007,159,859,0

a^5 =

$$a^4 = 0.000,000,000,000,007,159,859,0$$
$$a = 0.000,290,888,208,665,721,596$$
$$0.000,000,000,000,000,002,036,2$$
$$29\ 1$$
$$4\ 5$$
$$2\ 6$$
$$2$$
$$a^5 = 0.000,000,000,000,000,002,082,6$$
$$2$$
$$2a^5 = 0.000,000,000,000,000,004,165,2$$
$$\frac{2a^5}{15} = 0.000,000,000,000,000,000,277,7$$

But a^3 is $= 0.000,000,000,024,613,782,102,8.$

Therefore $\frac{a^3}{3}$ is $\left(= \frac{0.000,000,000,024,613,782,102,8}{3}\right)$
$= 0.000,000,000,008,204,594,034,3,$

and $a + \frac{a^3}{3} + \frac{2a^5}{15}$ + &c will be $=$

$$\left\{\begin{array}{l} 0.000,290,888,208,665,721,596 \\ + 0.000,000,000,008,204,594,034,3 \\ + 0.000,000,000,000,000,000,277,7 \end{array}\right\}$$

$= 0.000,290,888,216,870,315,908,0$; that is, t, or the tangent of an arch of 1 Minute, will be $= 0.000,290,888,216,870,315,908$; which agrees with the values of the ſaid tangent found above in Art. 58 and 61, in all it's 21 figures.

Art. 67. The ſecond of the two infinite Serieſes above-mentioned, which was invented by Sir Iſaac Newton, and expreſſes the length of the Sine of a given arch (that is not greater than the arch of a Quadrant of a circle,) in terms involving the powers of the arch and of the radius of the circle, is as follows. If the radius of the circle be denoted by r, the arch by a, and it's Sine by s, the Sine s will be equal to the infinite Series $a -$

$$\frac{a^3}{2.3.r^2} + \frac{a^5}{2.3.4.5.r^4} - \frac{a^7}{2.3.4.5.6.7.r^6} + \frac{a^9}{2.3.4.5.6.7.8.9.r^8} - \frac{a^{11}}{2.3.4.5.6.7.8.9.10.11.r^{10}}$$

+ &c, or $a - \frac{a^3}{6r^2} + \frac{a^5}{120r^4} - \frac{a^7}{5040r^6} + \frac{a^9}{362,880r^8} - \frac{a^{11}}{39,916,800r^{10}} +$

&c. Therefore, when r is $= 1$, or the radius of the circle is denoted by 1,

(as

(as it is in the foregoing Computations of the Tangent and Sine of an arch of 1 Minute,) the Sine s will be equal to the Series $a - \frac{a^3}{6} + \frac{a^5}{120} - \frac{a^7}{5040} + \frac{a^9}{362,880} - \frac{a^{11}}{39,916,800} + \&c.$

Now, when the arch a is an arch of 1 Minute of a Degree, it appears from the foregoing article that it's length will be = 0.000,290,888,208,665,721,596, and that it's cube, or a^3, will be = 0.000,000,000,024,613,782,102,8, and that it's fifth power, or a^5, will be = 0.000,000,000,000,000,002,082,6. Therefore $\frac{a^3}{6}$ will be $(= \frac{0.000,000,000,024,613,782,102,8}{6}) = 0.000,000,000,004,102,297,017,1$, and $\frac{a^5}{120}$ will be $= \frac{0.000,000,000,000,000,002,082,6}{120} = 0.000,000,000,000,000,000,017,3$. Therefore $a - \frac{a^3}{6} + \frac{a^5}{120} - \&c$ will be =

$$\left\{\begin{array}{l} 0.000,290,888,208,665,721,596 \\ -\ 0.000,000,000,004,102,297,017,1 \\ +\ 0.000,000,000,000,000,000,017,3 \\ -\ \&c \end{array}\right\}$$

$$= \left\{\begin{array}{l} 0.000,290,888,208,665,721,613,3 \\ -\ 0.000,000,000,004,102,297,017,1 \\ -\ \&c \end{array}\right\}$$

= 0.000,290,888,204,563,424,596,2, — &c. Therefore s, or the Sine of 1 Minute of a Degree, in a circle of which the radius is called 1, will be = 0.000,290,888,204,563,424,596,2; which agrees in the first twenty-one figures, reckoned from the place of units, with the value of the said Sine found above in Art. 63, to wit, 0.000,290,888,204,563,424,596. And consequently this number 0.000,290,888,204,563,424,596 may be safely concluded to be the true value of the Sine of an arch of 1 Minute of a Degree as exactly as it can be expressed in twenty-one places of decimal fractions.

End of the Comparison of the Values of the Tangent and Sine of an Arch of 1 *Minute of a Degree, obtained by the Principles of Common Geometry, with the Values of the same Lines, obtained by means of the Infinite Serieses invented by Mr. James Gregory and Sir Isaac Newton.*

OBSER-

Observations on the foregoing Computation of the Length of the Tangent of an Arch of 1 *Minute of a Degree, by repeated Sections of an Arch of* 45 *Degrees.*

By Dr. ANDREW MACKAY, LL.D.

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Art. 68. The foregoing Computation of the length of the Tangent of an arch of 1 Minute of a Degree in a circle of which the radius is called 1, by repeated Sections of an Arch of 45 Degrees; to wit, 1ſt, A Quinquiſection of the ſaid arch, by which we obtain the length of the tangent of an arch of 9 Degrees; and 2ndly, A Triſection of an arch of 9 Degrees, by which we obtain the length of the tangent of on arch of 3 Degrees; and, 3dly, a Triſection of an arch of 3 Degrees, by which we obtain the length of the tangent of an arch of 1 Degree, and, 4thly, a Quinquiſection of an arch of 1 Degree, or 60 Minutes, by which we obtain the length of the tangent of an arch of 12 Minutes; and, 5thly, a Triſection of an arch of 12 Minutes, by which we obtain the length of the tangent of an arch of 4 Minutes; and, 6thly, a Biſection of an arch of 4 Minutes, by which we obtain the length of the tangent of an arch of 2 Minutes; and, 7thly, a Biſection of an arch of 2 Minutes, by which we obtain the length of the tangent of an arch of 1 Minute;—I ſay, this method of deriving the length of the tangent of an arch of 1 Minute from the tangent of an arch of 45 Degrees (which is known to be equal to the radius of the Circle, or to 1,) is the moſt obvious and direct method that could be taken for the purpoſe. But it is alſo very laborious, as we have ſeen. And part of this

this labour might have been avoided by taking another method of diſcovering the length of the tangent of an arch of 3 degrees, which would have required much leſs calculation than was employed for that purpoſe in the foregoing Quinquiſection of an arch of 45 Degrees, and Triſection of an arch of 9 Degrees. This method is as follows.

Art. 69. An arch of 3 Degrees is the difference of an arch of 15 Degrees and an arch of 18 Degrees. Now it is ſhewn in books of Trigonometry, that, if there be two unequal arches in the quadrant of a circle, each of which is leſs than the arch of the whole quadrant, and the leſſer of theſe two arches is ſubtracted from the greater, and we draw the tangents and ſecants of the ſaid two arches, and likewiſe the tangent and ſecant of the arch which is the difference of the two former arches, the tangent of this third arch (which is equal to the difference of the two former arches,) will be to the difference of the tangents of the two former arches in the ſame proportion as the ſquare of the radius of the circle is to the ſum of the ſaid ſquare of the radius and the rectangle contained under the tangents of the two former arches; or, if r be put for the radius of the circle, and a for the tangent of the greater of the two former arches, and b for the tangent of the leſſer of them, and c for the tangent of their difference, the tangent c will be to $a-b$ (the difference of the two former tangents,) as rr is to $rr+ab$. And conſequently, when r is $=1$, or the radius of the circle is called 1, we ſhall have c to $a-b$ as 1×1 is to $1\times1+ab$, or c to $a-b$ as 1 is to $1+ab$; and therefore c will be $\frac{a-b}{1+ab}$. Therefore, if the two tangents a and b of the two former arches are known, the tangent c of the third arch (which is equal to the difference of the two former arches,) may be derived from them by computing the expreſſion $\frac{a-b}{1+ab}$. And conſequently, if the two former arches were arches of 18 Degrees and 15 Degrees, and we knew the lengths of their tangents a and b, the tangent of their difference, which is an arch of 3 Degrees, would be $(=\frac{a-b}{1+ab}) = \frac{\text{tang. } 18^\circ - \text{tang. } 15^\circ}{1+\text{tang. } 18^\circ \times \text{tang. } 15^\circ}$, and might be obtained by computing the value of the ſaid fraction. Now the values of the tangents of two arches of 18° and 15° may be obtained with only a moderate quantity of calculation, by the following methods.

A PROBLEM.

To find the Length of the Tangent of an Arch of 18 *Degrees, in a Circle of which the Radius is called* 1.

SOLUTION.

Art. 70. An arch of 18 Degrees is the fifth part of an arch of 90 Degrees, or of a whole quadrant of a circle. Now, if the tangent of any circular arch, that is less than the arch of a quadrant, be denoted by the capital letter T, and the tangent of the fifth part of such arch, be denoted by the small letter t, and the radius of the circle be called 1, the relation between the tangents T and t will be expressed by the quinquinomial equation $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5 = T$. Therefore, if T denote the tangent of any arch, how great soever, that is less than 90 Degrees, (as, for example, the tangent of an arch of 89°, 59′, 59″), and t denote the tangent of the fifth part of the said arch, it will still be true that the quinquinomial quantity $5t + 10T \times t^2 - 10t^3 - 5T \times t^4 + t^5$ is equal to T, or to $T \times 1$, and consequently that the quinquinomial quantity $\frac{5t}{T} + \frac{10T \times t^2}{T} - \frac{10t^3}{T} - \frac{5T \times t^4}{T} + \frac{t^5}{T}$ is $= \frac{T \times 1}{T}$, or that the quinquinomial quantity $\frac{5t}{T} + 10t^2 - \frac{10t^3}{T} - 5t^4 + \frac{t^5}{T}$ is $= 1$. Now in the case of such a near approach of the greater arch (of which T is the Tangent,) to an arch of 90 Degrees, the tangent T will become immensely great in comparison of the tangent t, which (being the tangent of the fifth part of the greater arch, which is less than 90 degrees,) can never be so great as the tangent of the fifth part of 90 Degrees, or the tangent of 18 Degrees; and consequently the three terms $\frac{5t}{T}$, $\frac{10t^3}{T}$, and $\frac{t^5}{T}$ must be excessively small in comparison of the two terms $10t^2$ and $5t^4$, so as to be only the millionth part, or the millionth part of the millionth part, of the said

ſaid two terms, or to be leſs than the ſaid two terms in any other and greater proportion of minority whatſoever. And hence it follows that, when the greater arch (of which T is the tangent,) is quite equal to an arch of 90 Degrees, or the arch of the whole quadrant, and the ſaid tangent T becomes abſolutely infinite, the ſaid three quantities $\frac{5t}{T}$, $\frac{10t^3}{T}$, and $\frac{t^5}{T}$ will have no magnitude at all, and conſequently the ſaid equation $\frac{5t}{T} + 10t^2 - \frac{10t^3}{T} - 5t^4 + \frac{t^5}{T}$ $= 1$ will become $0 + 10t^2 - 0 - 5t^4 + 0 = 1$, or $10t^2 - 5t^4 = 1$. Therefore (dividing all the terms by 5,) we ſhall have $2t^2 - t^4 = \frac{1}{5}$, and (ſubtracting both ſides from 1,) we ſhall have $1 - 2t^2 + t^4 \left(= 1 - \frac{1}{5}\right) = \frac{4}{5}$, and (extracting the ſquare-roots of both ſides,) $1 - t^2 = \frac{2}{\sqrt{5}}$, and conſequently $1 = \frac{2}{\sqrt{5}} + t^2$, and $t^2 = 1 - \frac{2}{\sqrt{5}}$, and $t = \sqrt{1 - \frac{2}{\sqrt{5}}} = \sqrt{1 - \frac{2\sqrt{5}}{5}}$; that is, t, or the tangent of an arch of 18 Degrees, will be $= \sqrt{1 - \frac{2\sqrt{5}}{5}}$, or $\sqrt{1 - \frac{4\sqrt{5}}{10}}$, or $\sqrt{1 - 0.4 \times \sqrt{5}}$. Q. E. I.

Another demonſtration that the tangent of an arch of 18 Degrees is equal to $\sqrt{1 - \frac{2}{\sqrt{5}}}$, or $\sqrt{1 - \frac{2\sqrt{5}}{5}}$ has been given in the third volume of this Collection of Mathematical Tracts, called *Scriptores Logarithmici*, pages 148 and 149; to which I refer the reader.

Art. 71. The computation of this expreſſion $\sqrt{1 - 0.4 \times \sqrt{5}}$ requires four arithmetical operations, to wit, the extraction of the ſquare-root of the number 5, the multiplication of this ſquare-root into the decimal fraction 0.4, the ſubtraction of the product of this multiplication from 1, and the extraction of the ſquare-root of the remainder of this ſubtraction. Neither of theſe four operations is very difficult, and the ſecond and third are remarkably eaſy. The extraction of the ſquare-root of 5 has been made to a great number of figures by Mr. Abraham Sharp, the great Engliſh Calculator of the latter part

of the 17th century; and he has found the ſaid ſquare-root to be = 2.236,067,977,499,789,696,409,173,668,731,276,235,440,618,357,375,457,7 &c, as may be ſeen in page 149 of this Collection of Tracts, called *Scriptores Logarithmici*, lines 7th and 6th from the bottom.

The next operation, of multiplying the ſaid ſquare into the decimal fraction 0.4, will be as follows.

$$\begin{array}{rl} \sqrt{5} = & 2.236,067,977,499,789,696,409,173,668,731,276 \\ & \underline{0.4} \\ 0.4\times\sqrt{5} = & 0.894,427,190,999,915,878,563,669,467,492,710,4 \end{array}$$

The third operation, of ſubtracting $0.4\times\sqrt{5}$ from 1, will be as follows.

$$\begin{array}{rl} & 1.000,000,000,000,000,000,000,000,000,000,000,0 \\ & \underline{0.894,427,190,999,915,878,563,669,467,492,710,4} \\ 1-0.4\times\sqrt{5} = & 0.105,572,809,000,084,121,436,330,532,507,289,6 \end{array}$$

And the ſquare-root of this laſt number, which is equal to $1-0.4\times\sqrt{5}$, has been found by Mr. Abraham Sharp to be = 0.324,919,696,232,906,326,155,871,412,215,134,4 &c, as may be ſeen in the third volume of theſe *Scriptores Logarithmici*, page 146.

Therefore $\sqrt{1-0.4\times\sqrt{5}}$, or the tangent of an arch of 18 Degrees, in a circle of which the radius is called 1, is = 0.324,919,696,232,906,326,155,871,412,215,134,4 &c. Q. E. I.

PRO-

PROBLEM II.

To find the Length of the Tangent of an Arch of 15 *Degrees, in a Circle of which the Radius is called* 1.

SOLUTION.

Art. 72. The ſecant of any arch in the quadrant of a circle, that is leſs than 90 Degrees, or the arch of the whole quadrant, is a third proportional to the co-ſine of the ſame arch and to the radius of the circle. Therefore the ſecant of an arch of 60 Degrees will be a third proportional to the co-ſine of an arch of 60 Degrees and to the radius. But the co-ſine of an arch of 60 Degrees is the ſine of an arch of 30 Degrees, which is equal to half the radius of the circle, or, when the radius is called 1, to the fraction $\frac{1}{2}$. Therefore the ſecant of an arch of 60 Degrees will be a third proportional to the fraction $\frac{1}{2}$ and 1, and conſequently will be equal to 2. Therefore the ſquare of the ſecant of an arch of 60 Degrees will be $(= 2 \times 2) = 4$, and the exceſs of the ſquare of this ſecant above the ſquare of the radius, or 1×1, will be $(= 4 - 1 \times 1 = 4 - 1) = 3$. But the exceſs of the ſquare of the ſecant of any arch above the ſquare of the radius is equal to the ſquare of the tangent of the ſame arch. Therefore the ſquare of the tangent of an arch of 60 Degrees, in a circle of which the radius is called 1, will be equal to 3; and conſequently the ſaid tangent itſelf will be equal to $\surd 3$, or the ſquare-root of 3.

But it is ſhewn in books of Trigonometry that the difference between the ſecant and the tangent of any arch in the quadrant of a circle is equal to the tangent

tangent of half the complement of the ſaid arch to an arch of 90 Degrees, or the arch of the whole Quadrant. Therefore the difference between the ſecant and the tangent of an arch of 60 Degrees will be equal to the tangent of half the complement of an arch of 60 Degrees to an arch of 90 Degrees, or to the tangent of an arch of half of 30 Degrees, or to the tangent of an arch of 15 Degrees. But, ſince the ſecant of an arch of 60 Degrees is equal to 2, and the tangent of the ſame arch is equal to $\surd 3$, the difference of the ſecant and the tangent of an arch of 60 Degrees, will be equal to $2-\surd 3$. Therefore $2-\surd 3$ will be equal to the tangent of an arch of 15 Degrees, and conſequently the length of the tangent of an arch of 15 Degrees may be obtained by firſt extracting the ſquare-root of 3, and then ſubtracting it from 2.

Q. E. I.

Now it appears from Mr. Sharp's Quadrature of the Circle from the tangent of an arch of 30 Degrees, ſet-down in page 142 of the 3d volume of theſe *Scriptores Logarithmici*, that $\surd 12$, or $2\times\surd 3$, is $=$ 3.464,101,615,137,754,587,054,892,683,011,744,&c. Therefore $\surd 3$ will be equal to half this number, that is, to 1.732,050,807,568,877,293,527,446,341,505,872,&c, and $2-\surd 3$ will be $=$

$$\left\{\begin{array}{l} 2.000,000,000,000,000,000,000,000,000,000,000 \\ -\ 1.732,050,807,568,877,293,527,446,341,505,872,\&c \end{array}\right\}$$

$=$ 0.267,949,192,431,122,706,472,553,658,494,128; that is, the tangent of an arch of 15 Degrees will be equal to the decimal fraction 0.267,949,192,431,122,706,472,553,658,494,128. Q. E. I.

Art. 73. Having thus found the tangent of an arch of 18 Degrees to be $=$ 0.324,919,696,232,906,326,155,871,412,215,134, and the tangent of an arch of 15 Degrees to be $=$ 0.267,949,192,431,122,706,472,553,658,494,128, we muſt ſubtract the latter of theſe tangents from the former, and their difference will be $=$

$$\left\{\begin{array}{l} 0.324,919,696,232,906,326,155,871,412,215,134 \\ -\ 0.267,949,192,431,122,706,472,553,658,494,128 \end{array}\right\}$$

$=$ 0.056,970,503,801,783,619,683,317,753,721,006; or, if the tangent of 18 Degrees be denoted by a, and the tangent of 15 Degrees be denoted by

by b, we ſhall have $a-b$, (or the numerator of the fraction $\frac{a-b}{1+ab}$,) = 0.056, 970,503,801,783,619,683,317,753,721,006. We muſt then multiply the tangent a, or 0.324,919,696,232,906,326,155,871,412,&c into the tangent b, or 0·267,949,192,431,122,706,472,553,658,&c, or (increaſing the laſt figure 8 by an unit, on account of the following figures 494,128,&c which are omitted,) 0·267,949,192,431,122,706,472,553,659; and we ſhall find the product ab to be = 0.087,061,970,210,572,952,731,324,783; to which if we add 1, we ſhall have $1+ab$ (or the denominator of the fraction $\frac{a-b}{1+ab}$,) = 1.087,061,970,210,572,952,731,324,783. And laſtly, we muſt divide the number 0.056,970,503,801,783,619,683,317,753,721, or the numerator $a-b$ of the ſaid fraction, by the number 1.087,061,970,210,572,952,731,324,783, or it's denominator; and the quotient of this diviſion, which will be 0.052, 407,779,283,041,204,038,805, will be equal to the fraction $\frac{a-b}{1+ab}$, or $\frac{\text{tangent of } 18^\circ - \text{tangent of } 15^\circ}{\text{square of radius } 1 + \text{tang. } 18^\circ \times \text{tang. } 15^\circ}$, and conſequently will be equal to the tangent of an arch of 3 Degrees. Q. E. I.

Art. 74. The operations of multiplying the tangent of an arch of 18 Degrees into the tangent of an arch of 15 Degrees, and of dividing the difference of thoſe tangents by the ſum of 1 (the ſquare of the radius of the circle,) and the product of the ſaid multiplication, are performed by Dr. Mackay in the following manner.

Tangent of 18° = 0.324,919,696,232,906,326,155,871,412
Tangent of 15° = 0.267,949,192,431,122,706,472,553,659

0.064 983 939 246 581 265 231 174 282
19 495 181 773 974 379 569 352 285
2 274 437 873 630 344 283 091 100
292 427 726 609 615 693 540 284
12 996 787 849 316 253 046 235
2 924 277 266 096 156 935 403
32 491 969 623 290 632 616
29 242 772 660 961 569 354
649 839 392 465 812 652
129 967 878 493 162 530
9 747 590 886 987 190
324 919 696 232 906
32 491 969 623 291
6 498 393 924 658
649 839 392 466
227 443 787 363
1 949 518 177
129 967 878
22 744 379
649 839
162 460
16 246
975
195
16
3

Tang. 18° × Tang. 15° = 0.087,061,970,210,572,952,731,324,783
Square of Radius = 1.

1.087,061,970,210,572,952,731,324,783

The Divisor. *The Quotient.*

1.087,061,970,210,572,952,731,324,783) (0.052,407,779,283,041,204,038,805

The Dividend.

0.056,970,503,801,783,619,683,317,753
54 353 098 510 528 647 636 566 239
· 2 617 405 291 254 972 046 751 514
2 174 123 940 421 145 905 462 650
· 443 281 350 833 826 141 288 864
434 824 788 084 229 181 092 530
·· 8 456 562 749 596 960 196 334
7 609 433 791 474 010 669 119
· 847 128 958 122 949 527 215
760 943 379 147 401 066 912
· 86 185 578 975 548 460 303
76 094 337 914 740 106 691
10 091 241 060 808 353 612
9 783 557 731 895 156 575
· 307 683 328 913 197 037
217 412 394 042 114 591
· 90 270 934 871 082 446
86 964 957 616 845 836
· 3 305 977 254 236 610
3 261 185 910 631 719
·· 44 791 343 694 891
43 482 478 808 423
· 1 308 864 796 468
1 087 061 970 211
: 221 802 826 257
217 412 394 042
·· 4 390 432 215
4 348 247 881
·· 42 184 334
32 611 859
· 9 572 475
8 696 496
· 875 979
869 650
·· 6 329
5 435
· 894

The value of the tangent of an arch of 3 Degrees obtained by this diviſion, to wit, the quotient 0.052,407,779,283,041,204,038,805, agrees with the value found for this tangent in the foregoing computation in Art. 32, page 817, in all it's twenty-four figures.

Art. 75. By theſe calculations ſuggeſted and performed by Dr. Mackay, (which have taken-up only 9 pages,) we have obtained the length of the tangent of an arch of 3 Degrees to the ſame degree of exactneſs as it was obtained above, by the quinquiſection of an arch of 45 Degrees, and the triſection of an arch of 9 Degrees, in the 39 pages from page 780 to page 820.

End of Dr. Mackay's Observations on the foregoing Computation of the Length of the Tangent of an Arch of 1 Minute of a Degree by repeated Sections of an Arch of 45 Degrees.

Art. 76. But there is ſtill another method of proceeding to facilitate the Computation of the Tangent of an Arch of 1 Minute of 1 Degree, in the former part of ſuch computation, and untill we have obtained the value of the tangent of an arch of 1 Degree; which has been communicated to me by my very learned and ingenious friend, Mr. James Ivory, of the Royal Military College at Great Marlow in Buckinghamſhire, in conſequence of my having mentioned to him the foregoing laborious Computation of it by the Quinquiſection of an arch of 45 Degrees, and the following two Triſections, the ſecond Quinquiſection, the third Triſection, and the two following Biſections, of arches, which had been performed by Dr. Mackay. And, as I am confident that this Method of Mr. Ivory will give great ſatisfaction to the lovers of theſe ſubjects, I will now proceed to ſtate it under a ſeparate title, as the work of its ingenious Inventor.

A SKETCH

A

SKETCH OF A METHOD

OF

COMPUTING THE LENGTH OF THE TANGENT OF AN ARCH OF 1 DEGREE, IN A CIRCLE, OF WHICH THE RADIUS IS CALLED 1.

By JAMES IVORY, M. A.

OF THE ROYAL MILITARY COLLEGE AT GREAT MARLOW IN BUCKINGHAMSHIRE.

PROBLEM I.

Art. 77. To find the Length of the Tangent of an Arch of 9 Degrees.

SOLUTION.

Let the tangent of twice this arch, or of an arch of 18 Degrees, be denoted by the ſmall letter t, the radius of the circle being called 1.

Then will the tangent of four times this arch, or of an arch of 72 Degrees, be equal to the fraction $\frac{4t-4t^3}{1-6t^2+t^4}$; as has been ſhewn above in pages 770, 771, 772, and 773.

But an arch of 18 Degrees is the complement of an arch of 72 Degrees to an arch of 90 Degrees, or the arch of a quadrant. Therefore the rectangle contained under the tangent of 18 Degrees, and the tangent of an arch of 72 Degrees will be equal to the ſquare of the radius; that is, $t \times$ the fraction $\frac{4t-4t^3}{1-6t^2+t^4}$ will be equal to 1×1, or 1; or $\frac{4t^2-4t^4}{1-6t^2+t^4}$ will be $= 1$. Therefore $4t^2-4t^4$ will be $(= \overline{1-6t^2+t^4} \times 1) = 1-6t^2+t^4$; and (adding $6t^2$ to both ſides,) $10t^2-4t^4$ will be $= 1+t^4$, and (ſubtracting t^4 from both ſides,) $10t^2-5t^4$ will be $= 1$, and (dividing all the terms by 5,) $2t^2-t^4$ will be $= \frac{1}{5}$. Therefore, if we ſubtract both ſides from 1, we ſhall have $1 - \overline{2t^2-t^4}$ $(= 1 - \frac{1}{5} = \frac{5}{5} - \frac{1}{5}) = \frac{4}{5}$, or $1-2t^2+t^4 = \frac{4}{5}$, or $\overline{1-t^2}^2 = \frac{4}{5}$. Therefore $1-t^2$ will be $= \sqrt{\frac{4}{5}}$ $(= \frac{\sqrt{4}}{\sqrt{5}} = \frac{2}{\sqrt{5}}) = \frac{2\sqrt{5}}{5}$, and 1 will be $= \frac{2\sqrt{5}}{5} + t^2$, and t^2 will be $= \sqrt{1 - \frac{2\sqrt{5}}{5}}$, and t will be $= \sqrt{1 - \frac{2\sqrt{5}}{5}}$ $= \sqrt{\frac{5-2\sqrt{5}}{5}}$; that is, the tangent of an arch of 18 Degrees will be equal to $\sqrt{\frac{5-2\sqrt{5}}{5}}$.

Art. 78. Having thus found the tangent of an arch of 18 Degrees to be $= \sqrt{\frac{5-2\sqrt{5}}{5}}$, let this tangent be denoted by the capital letter T; and let the ſmall letter t denote the tangent of an arch of 9 Degrees, or of half the arch of which the tangent is $\sqrt{\frac{5-2\sqrt{5}}{5}}$, or T. And then the relation between the tangents T and t will be expreſſed by the quadratick equation $t^2 + \frac{2}{T} \times t = 1$; as has been ſhewn above in page 773. Now this equation may be reſolved as follows.

Add $\frac{1}{T^2}$ to both ſides. And we ſhall have $t^2 + \frac{2}{T} \times t + \frac{1}{T^2} = 1 + \frac{1}{T^2}$, that is, $\overline{t + \frac{1}{T}}^2$ will be $= 1 + \frac{1}{T^2}$. Therefore $t + \frac{1}{T}$ will be $= \sqrt{1 +}$

$\sqrt{\left(1+\frac{1}{T^2}\right)}$, and consequently t will be $=\sqrt{\left(1+\frac{1}{T^2}\right)}-\frac{1}{T}$; that is, the tangent of 9 Degrees will be equal to the binomial quantity $\sqrt{\left(1+\frac{1}{T^2}\right)}-\frac{1}{T}$.

Q. E. I.

Art. 79. This binomial quantity $\sqrt{\left(1+\frac{1}{T^2}\right)}-\frac{1}{T}$ is equal to the trinomial quantity $1+\sqrt{5}-\sqrt{(5+2\sqrt{5})}$; as may be ſhewn in the following manner.

It has been ſhewn above in Art. 77, that the tangent of an arch of 18 Degrees is $=\sqrt{\frac{5-2\sqrt{5}}{5}}$. But, in our preſent notation, T is equal to the tangent of an arch of 18 Degrees. Therefore T is $=\sqrt{\frac{5-2\sqrt{5}}{5}}$, and conſequently T^2 is $=\frac{5-2\sqrt{5}}{5}$. Therefore $T^2\times\overline{5+2\sqrt{5}}$ will be $=\frac{5-2\sqrt{5}}{5}\times\overline{5+2\sqrt{5}}$ $\left(=\frac{\overline{5-2\sqrt{5}}\times\overline{5+2\sqrt{5}}}{5}=\frac{\overline{5}^2-\overline{2\sqrt{5}}^2}{5}=\frac{25-4\times5}{5}=\frac{25-20}{5}=\frac{5}{5}\right)=1$. Therefore $\frac{1}{T^2}$ will be $=5+2\sqrt{5}$, and $1+\frac{1}{T^2}$ will be $=1+5+2\sqrt{5}$ $=\overline{1+\sqrt{5}}^2$, and conſequently $\sqrt{\left(1+\frac{1}{T^2}\right)}$ will be $=1+\sqrt{5}$. And, becauſe $\frac{1}{T^2}$ is $=5+2\sqrt{5}$, we ſhall have $\frac{1}{T}=\sqrt{(5+2\sqrt{5})}$. Therefore $\sqrt{\left(1+\frac{1}{T^2}\right)}-\frac{1}{T}$ will be $=1+\sqrt{5}-\sqrt{(5+2\sqrt{5})}$.

Q. E. D.

Therefore the tangent of an arch of 9 Degrees will be equal to the trinomial quantity $1+\sqrt{5}-\sqrt{(5+2\sqrt{5})}$; the value of which may be computed by two extractions of the ſquare-root, to wit, that of the ſquare-root of 5, and that of the ſquare-root of the binomial quantity $5+2\sqrt{5}$; which operations would require much leſs labour than the other method of obtaining the value of the ſame tangent by reſolving the quinquinomial equation $5t+10t^2-10t^3-5t^4+t^5=1$ by repeated proceſſes of Mr. Raphſon's Method of Approximation.

PRO-

PROBLEM II.

Art. 80. *To find the Length of the Tangent of an Arch of* 10 *Degrees, when the Radius of the Circle is called* 1.

SOLUTION.

The ſine of an arch of 30 Degrees is equal to half the radius of the circle, and therefore is $= \frac{1}{2}$, when the radius is called 1. Therefore the ſquare of this ſine is $= \frac{1}{4}$, and conſequently the ſquare of the co-ſine of this arch (which is equal to the exceſs of the ſquare of the radius above the ſquare of the ſine,) will be $(= 1 \times 1 - \frac{1}{4} = 1 - \frac{1}{4}) = \frac{3}{4}$, and the co-ſine itſelf will be $= \frac{\sqrt{3}}{2}$. But the co-ſine of every arch in a circle that is leſs than 90 Degrees, or the arch of a quadrant, is to the ſine of the ſame arch as the radius of the circle is to the tangent of the ſame arch. Therefore $\frac{\sqrt{3}}{2}$ is to $\frac{1}{2}$ as 1 is to the tangent of an arch of 30 Degrees; and conſequently the ſaid tangent will be equal to $\frac{1}{\sqrt{3}}$.

Now it is known, by the Doctrine of Angular Sections, that, if any arch of the quadrant of the circle that is leſs than 90 Degrees, or the arch of the whole quadrant, is divided into three equal parts, and the tangent of the whole arch be denoted by the capital letter T, and the tangent of the leſſer arch, or the third part of the greater arch, be denoted by the ſmall letter *t*,

(the

(the radius of the circle being called 1,) the relation between the two tangents T and t will be expreſſed by the cubick equation $3t+3T\times t^2-t^3=T$; which has two roots, (that is, real and affirmative roots,) of which the leſſer will be the tangent of the leſſer arch, or of the third part of the greater arch of which T is the tangent. Therefore, when the greater of the two arches is an arch of 30 Degrees, and the leſſer of them is an arch of 10 Degrees, and the tangent, T, of the greater arch is conſequently $=\frac{1}{\sqrt{3}}$, the equation $3t+3T\times t^2-t^3=T$ will become $3t+3\times\frac{1}{\sqrt{3}}\times t^2-t^3=\frac{1}{\sqrt{3}}$, or $3t+\frac{3}{\sqrt{3}}\times t^2-t^3=\frac{1}{\sqrt{3}}$, or $3t+\sqrt{3}\times t^2-t^3=\frac{1}{\sqrt{3}}$; and t, or the leſſer root of this equation, will be equal to the tangent of an arch of 10 Degrees. We muſt therefore endeavour to reſolve this equation $3t+\sqrt{3}\times t^2-t^3=\frac{1}{\sqrt{3}}$.

Art. 81. To ſimplify this equation, let us ſuppoſe x to be $=\frac{t}{\sqrt{3}}$, and conſequently $x\times\sqrt{3}$ to be $=t$.

Then will t^2 be $=3x^2$, and t^3 will be $=3\sqrt{3}\times x^3$, and $3t$ will be $=3\times\sqrt{3}\times x$, and $\sqrt{3}\times t^2$ will be $(=\sqrt{3}\times 3x^2)=3\sqrt{3}\times x^2$, and $3t+\sqrt{3}\times t^2-t^3$ will be $=3\sqrt{3}\times x+3\sqrt{3}\times x^2-3\sqrt{3}\times x^3$.

But $3t+\sqrt{3}\times t^2-t^3$ is $=\frac{1}{\sqrt{3}}$.

Therefore $3\sqrt{3}\times x+3\sqrt{3}\times x^2-3\sqrt{3}\times x^3$ will alſo be $=\frac{1}{\sqrt{3}}$; and conſequently $3\times\sqrt{3}\times\sqrt{3}\times x+3\times\sqrt{3}\times\sqrt{3}\times x^2-3\times\sqrt{3}\times\sqrt{3}\times x^3$ will be $=\frac{\sqrt{3}}{\sqrt{3}}$, that is, $3\times 3\times x+3\times 3\times x^2-3\times 3\times x^3$ will be $=1$, or $9x+9x^2-9x^3$ will be $=1$; and conſequently $x+x^2-x^3$ will be $=\frac{1}{9}=0.111$, 111,111,111,111,111,111,111,&c; which is an equation of a very ſimple and manageable form, and may be reſolved with exactneſs to 20, or more, places of figures, by four ſucceſſive proceſſes of Mr. Raphſon's Method of Approximation, taking the fraction $\frac{1}{10}$, or 0.1, for the firſt near value of x, from

which

which we begin the approximation. And, when we ſhall have found the value of x to the intended degree of exactneſs, we muſt multiply it into $\sqrt{3}$, or the mixt number 1.732,050,807,568,877,293,527,446,341,&c; and the product will be $= t$, or the tangent of an arch of 10 Degrees.

Q. E. I.

PROBLEM III.

Art. 82. *To find the Tangent of an Arch of 1 Degree.*

SOLUTION.

Let a denote the tangent of an arch of 10 Degrees, and b the tangent of an arch of 9 Degrees, and y the tangent of an arch of 1 Degree, which is the difference of the two former arches of 10 Degrees and 9 Degrees, the tangents of which are a and b. Then, by the Doctrine of Angular Sections, the tangent y of the difference of the ſaid two arches of 10 and 9 Degrees, will be to $a-b$, the difference of the tangents of thoſe arches, as the ſquare of the radius 1 is to the ſum of the ſaid ſquare and the rectangle contained under the two tangents a and b; that is, y will be to $a-b$ as 1×1 is to $1\times 1+a\times b$, or as 1 is to $1+ab$; and conſequently y will be $\frac{\overline{a-b}\times 1}{1+ab}$, or $\frac{a-b}{1+ab}$; that is, the tangent of an arch of 1 Degree will be $= \frac{a-b}{1+ab}$. We muſt therefore, firſt, ſubtract the tangent of an arch of 9 Degrees from the tangent of an arch of 10 Degrees, in order to obtain the value of $a-b$, the numerator of the ſaid fraction $\frac{a-b}{1+ab}$; and

and we muſt then multiply the former tangent into the latter tangent, and add the product of the ſaid multiplication to 1, in order to obtain the value of $1+ab$, the denominator of the ſaid fraction; and, laſtly, we muſt divide the ſaid value of $a-b$ by the ſaid value of $1+ab$; and the quotient of this diviſion will be equal to the tangent of an arch of 1 Degree. Q. E. I.

Art. 83. The length of the tangent of an arch of 1 Degree may, I think, be computed by the method above-deſcribed with much leſs trouble than by the method purſued by Dr. Mackay. The length of the computation of the tangent of an arch of 1 Degree by the one method would, I apprehend, be conſiderably leſs than half the length of the computation of the ſame tangent by the other method. The two methods appear to be equally direct and elementary.

But, when we have diſcovered the length of the tangent of an arch of 1 Degree, and are deſirous of deriving from the known length of that tangent the length of the tangent of an arch of 1 Minute of a Degree, there appears to be no method better than that purſued by Dr. Mackay, to wit, by repeated ſections of the arch of 1 Degree.

July 28, 1805. JAMES IVORY.

Art. 84. Dr. Mackay has alſo furniſhed me with a computation of the length of the tangent of an arch of 1 Degree, in a circle of which the radius is called 1, according to the method here recommended by Mr. Ivory. This Computation is as follows.

A COMPUTATION

OF THE

Length of the Tangent of an Arch of 1 *Degree, in a Circle of which the Radius is called* 1, *in the Method recommended by Mr.* IVORY.

By Dr. ANDREW MACKAY, LL.D.

Late of Aberdeen in Scotland, and Fellow of the Royal Society of Edinburgh.

The firſt part of this computation conſiſts in finding the value of the trinomial quantity $1+\sqrt{5}-\sqrt{5+2\sqrt{5}}$, which is equal to the tangent of an arch of 9 Degrees. And for this purpoſe it is neceſſary to extract the ſquare-root of the number 5, and then to double the ſaid ſquare-root in order to obtain the value of $2\sqrt{5}$, and to add the ſaid value to 5, and extract the ſquare-root of the ſum $5+2\sqrt{5}$, and, laſtly, to ſubtract this ſquare-root from $1+\sqrt{5}$, or the ſum of 1 and the ſquare-root of 5. Theſe operations Dr. Mackay performs in the following manner.

5.000,000,000,000,000,000,000,000 (2.236,067,977,499,789,696,409,
[173,668,9

4
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84
443) ·1600
1329
4466) ·27100
26796
44720) ··30400
00000
447206) 3040000
2683236
4472127) ·35676400
31304889
44721349) ·437151100
402492141
447213587) ·3465895900
3130495109
4472135947) ·33540079100
31304951629
44721359544) ·2235127471 00
1788854381 76
447213595489) ·44627308924 00
402492235940 1
447213595498 9) ·43780853299900
402492235940 1
4472135954998 7) ·353162970499900
313049516849909
44721359549998 8) ·401134536499 9100
357770876399 9584
·433636600999516

·433636600999516
402492235949953
·311443650495 63
268328157299 97
·431154931956 6
40249223595 00
·286626960066
26832815730 0
·18298802766
1788543820
··410258946
402492236
··7766710
4472136
3294574
3130495
·164079
134164
·29915
26833
·3082
2683
·399
358
·41

$$\sqrt{5} = 2.236,067,977,499,789,696,409,173,668,9\overline{2}$$

$$2\sqrt{5} = 4.472,135,954,999,579,392,818,347,337,8\overline{5.000,000,000,000,000,000,000,000,0}$$

$$5+2\sqrt{5} = 9.472,135,954,999,579,392,818,347,337,8$$

9.472,135,954,999,579,392,818,347,337,8 (3.077,683,537,175,253,402,
9 [570,290,576

60) ·47
00

607) 4721
4249

6147) ·47235
43029

61546) ·420695
369276

615528) ·5141949
4924224

6155363) ·21772599
18466089

61553665) ·330651057
307768325

615536703) ·2288273293
1846610109

6155367067) ·44166318492
43087569469

61553670741) ·10787902381
6155367074 1

615536707427) ·4632123164083
4308756951989

61553670743 45) ·3233662120944 7
307768353717 25

615536707435045) ·3287124228522978
307768353717 5225

·2094406913477 53
1846610122305 13

·2477967911724 0
2462146829740 2

··1582108198 38
1231073414 87

·3510347835 1

·3510347835 1
307768353 72

·4326642979
4308756952

··17886027
12310734

·5575293
5539830

··35463
30777

·4686
4309

·377
369

··8

Therefore

Therefore $\sqrt{5+2\sqrt{5}}$ is $= 3.077,683,537,175,253,402,570,290,576$, and consequently the trinomial quantity $1+2\sqrt{5}-\sqrt{5+2\sqrt{5}}$ will be $=$

$$\left\{\begin{array}{ll} & 1.000,000,000,000,000,000,000,000,000 \\ + & 2.236,067,977,499,789,696,409,173,668 \\ - & 3.077,683,537,175,253,402,570,290,576 \end{array}\right\} =$$

$$\left\{\begin{array}{ll} & 3.236,067,977,499,789,696,409,173,668 \\ - & 3.077,683,537,175,253,402,570,290,576 \end{array}\right\}$$

$= 0.158,384,440,324,536,293,838,883,092.$ Therefore the tangent of an arch of 9 Degrees is equal to 0.158,384,440,324,536,293,838,883,092. Q. E. I.

This value of the tangent of an arch of 9 Degrees agrees in it's twenty-five first figures, to wit, 0.158,384,440,324,536,293,838,883,0 with 0.158,384,440,324,536,293,838,883,06, or the value of the same tangent found above in Art. 24, by the quinquisection of an arch of 45 Degrees by resolving the quinquinomial equation $5t+10t^2-10t^3-5t^4+t^5 = 1$, of which the said tangent is the root. And the preceeding calculations by which it has been obtained, (which consist chiefly of the extractions of the square-roots of the number 5 and the binomial quantity $5+2\sqrt{5}$,) are evidently much fewer and less laborious than the calculations employed above in the resolution of the quinquinomial equation $5t+10t^2-10t^3-5t^4+t^5 = 1$, which extend from page 781 to page 799, or through 18 pages.

Art. 85. The second part of this computation of the length of the tangent of an arch of 1 Degree, in the Method recommended by Mr. Ivory, consists in finding the length of the tangent of an arch of 10 Degrees. And this is to be performed by resolving the cubick equation $3t+3T\times t^2-t^3 = T$, in which T is the tangent of an arch of 30 Degrees and is $= \frac{1}{\sqrt{3}}$, that is, by resolving the cubick equation $3t+3\times\frac{1}{\sqrt{3}}\times t^2-t^3 = \frac{1}{\sqrt{3}}$, or, (if we put $x = \frac{t}{\sqrt{3}}$, or $x\times\sqrt{3} = t$,) by first resolving the cubick equation $x+x^2-x^3 = \frac{1}{9}$, or $x+x^2-x^3 = 0.111,111,111,111,111,111,111,111,111$,&c, and then multiplying the value found for x by such resolution into the value of $\sqrt{3}$,

or

or the mixt number 1.732,050,807,568,877,293,527,446,341,&c, in order to obtain the value of t, or the tangent of an arch of 10 Degrees. Theſe operations are performed by Dr. Mackay in the following manner.

The Reſolution of the Cubick Equation $x+x^2-x^3 =$ 0.111,111,111,111,111,111,111,111,111,&c *by Mr. Raphson's Method of Approximation.*

Art. 86. Since the trinomial quantity $x+x^2-x^3 = \frac{1}{9}$, it is evident that x muſt be leſs than 1. For, if x were $= 1$, x^2 would be $= 1$, and x^3 would alſo be $= 1$, and $x+x^2-x^3$ would be $(= 1+1-1) = 1$, which is equal to nine times the abſolute term $\frac{1}{9}$, or 9 times the true value of the trinomial quantity $x+x^2-x^3$ in the propoſed equation $x+x^2-x^3 = \frac{1}{9}$. Therefore the true value of x in that equation will be much leſs than 1, and probably, not very different from $\frac{1}{9}$. We will therefore conjecture that it is $= \frac{1}{10}$, or 0.1, and will ſubſtitute 0.1 in it's ſtead in the trinomial quantity $x+x^2-x^3$, in order to diſcover whether the value of the ſaid quantity reſulting from ſuch ſubſtitution will be greater, or will be leſs, than the abſolute term $\frac{1}{9}$, or 0.111,111,111,&c, and conſequently whether the ſaid conjectural value of x, to wit, 0.1, will be greater, or will be leſs, than the true value of x in the propoſed equation $x+x^2+x^3 =$ 0.111,111,111,&c.

Now, if x be $=$ 0.1, we ſhall have $x^2 =$ 0.01, and $x^3 =$ 0.001, and conſequently $x+x^2-x^3$ $(= 0.1 + 0.010 - 0.001 = 0.110 - 0.001) = 0.109$; which is leſs than 0.111,111,111,&c, or the abſolute term of the propoſed equation. Therefore 0.1 muſt be leſs than the true value of x in that equation.

A Process

A Process of Mr. Raphson's Method of Approximation.

Art. 87. We will therefore put z for the difference been 0.01 and the true value of x in the propoſed equation, and ſubſtitute the binomial quantity $0.1 + z$ inſtead of x in the ſaid equation. Now, if x is $= 0.1 + z$, we ſhall have $x^2 = \overline{0.1+z}^2$ $(= \overline{0.1}^2 + 2\times 0.1 \times z + \&c) = 0.01 + 0.2\times z + \&c$, and $x^3 = \overline{0.1+z}^3$ $(= \overline{0.1}^3 + 3\overline{\times 0.1}^2 \times z + \&c = 0.001 + 3\times 0.01 \times z + \&c) =$ $0.001 + 0.03\times z + \&c$, and conſequently $x + x^2 - x^3$ will be $=$

$$\left\{\begin{array}{rrl} & 0.1 & + z \\ + & 0.01 & + 0.20 \times z + \&c \\ - & 0.001 & - 0.03 \times z - \&c \end{array}\right\} = \left\{\begin{array}{l} 0.110 + 1.20\times z + \&c \\ - 0.001 + 0.03\times z - \&c \end{array}\right\} =$$

$0.109 + 1.17\times z$.

But $x + x^2 - x^3$ is $= 0.111,111,111,\&c$.

Therefore $0.109 + 1.17\times z$ will alſo be $= 0.111,111,111,\&c$; and conſequently $1.17\times z$ will be $(= 0.111,111,111,\&c - 0.109) = 0.002,111,111,$ $\&c$, and z will be $= \frac{0.002,111,111,\&c}{1.17} = 0.001,8$.

Therefore x, or $0.1 + z$, will be $= 0.1 + 0.001,8 = 0.101,8$, that is, the ſecond near value of x will be 0.101,8, or (neglecting the laſt figure 8, as perhaps not exact,) will be 0.101. Q. E. I.

Art. 88. Dr. Mackay then ſubſtitutes this ſecond near value of x, to wit, 0.101, inſtead of x in the trinomial quantity $x + x^2 - x^3$, in order to diſcover whether the value of the ſaid quantity reſulting from this ſubſtitution will be greater, or will be leſs, than 0.111,111,111,&c, or the abſolute term of the propoſed equation $x + x^2 - x^3 = 0.111,111,111,\&c$, and conſequently whether 0.101, will be greater, or will be leſs, than the true value of x in that equation. This ſubſtitution may be made as follows.

$x =$

$$
\begin{array}{rl}
x = & 0.101 \\
x = & 0.101 \\
\hline
& 101 \\
& 1010 \\
\hline
x^2 = & 0.010,201 \\
& 0.101 \\
\hline
& 10201 \\
& 1020\,10 \\
\hline
x^3 = & 0.001,030,301
\end{array}
$$

$$
\begin{array}{rl}
x = & 0.101 \\
x^2 = & 0.010,201 \\
\hline
x+x^2 = & 0.111,201,000 \\
x^3 = & 0.001,030,301 \\
\hline
x+x^2-x^3 = & 0.110,170,699
\end{array}
$$

Therefore the value of the trinomial quantity $x+x^2-x^3$ reſulting from this ſubſtitution is 0.110,170,699; which is leſs than 0.111,111,111,&c, or the abſolute term of the propoſed equation $x+x^2-x^3 = 0.111,111,111,\&c$. And conſequently the number 0.101 is leſs than the root of the ſaid equation.

Q. E. I.

A Second Process of Mr. Raphson's Method of Approximation.

Art. 89. Having thus diſcovered that 0.101 is leſs than the true value of x in the equation $x+x^2-x^3 = 0.111,111,111,111,111,\&c$, Dr. Mackay puts z for their difference, and then ſubſtitutes the binomial quantity $0.101+z$, inſtead of x, in the ſaid equation. This he does in the following manner.

Since x is $= 0101+z$, we ſhall have:

$$x^2 = \overline{0.101+z}^2 \; (= \overline{0.101}^2 + 2\times 0.101\times z+\&c = \overline{0.101}^2 + 0.202\times z+\&c) = 0.010,201 + 0.202\times z+\&c,$$

$$\text{and } x^3 = \overline{0.101+z}^3 \; (= \overline{0.101}^3 + 3\times\overline{0.101}^2\times z+\&c = \overline{0.101}^3 + 3\times 0.010,201\times z+\&c) = 0.001,030,301 + 0.030,603\times z+\&c.$$

Therefore

Therefore the trinomial quantity $x+x^2-x^3$ will be equal to the multinomial quantity

$$\left\{\begin{array}{lcl} 0.101 & + & z \\ + \ 0.010,201 & + & 0.202 \times z + \&c \\ - \ 0.001,030,301 & - & 0.030,603 \times z - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{lcl} 0.111,201 & + & 1.202 \times z + \&c \\ - \ 0.001,030,301 & - & 0.030,603 \times z + \&c \end{array}\right\}$$

$$= 0.110,170,699 \quad + \quad 1.171,397 \times z - \&c.$$

But the trinomial quantity $x+x^2-x^3$ is $=$ 0.111,111,111,111,&c.

Therefore the quantity 0.110,170,699 + 1.171,397 z + &c will alſo be $=$ 0.111,111,111,111,&c. And conſequently 1.171,397 $\times z$ + &c will be $(=$ 0.111,111,111,111,&c $-$ 0.110,170,699$) =$ 0.000,940,412,111,&c, and z will be $= \frac{0.000,940,412,111,\&c}{1.171,397} =$ 0.000,802,8. Therefore x, or 0.101 + z, will be $=$ 0.101 + 0.101,802,8 $=$ 0.101,802,8; that is 0.101,802,8 will be a third near value of the root x of the propoſed equation $x+x^2-x^3 =$ 0.111,111,111,111,&c. Q. E. I.

Art. 90. Of this third near value of x it is probable that the ſix firſt figures 0.101,802 are exact. And Dr. Mackay therefore ſubſtitutes thoſe ſix figures, inſtead of x, in the trinomial quantity $x+x^2-x^3$, in order to diſcover whether the value of the ſaid quantity reſulting from ſuch ſubſtitution will be greater, or will be leſs, than the abſolute term 0.111,111,111,111,111,111,&c, of the propoſed equation, and conſequently whether the ſaid number 0.101,802 will be greater, or will be leſs, than the true value of it's root x.

Now, if x is $=$ 0.101,802, we ſhall have x^2 $(= \overline{0.101,802}^2)$ 0.010,363,647,204, and x^3 $(= \overline{0.101,802}^3) =$ 0.001,055,040,012,661,608, as will appear from the following operations.

$x = 0.101,802$
$x = 0.101,802$
203 604
81 441 60
101 802
10 180 20
$x^2 = 0.010,363,647,204$
$x = 0.101,802$
20 727 294 408
8 290 917 763 20
10 363 647 204
1 036 364 720 40
$x^3 = 0.001,055,040,012,661,608$

$x = 0.101,802$
$x^2 = 0.010,363,647,204$
$x + x^2 = 0.112,165,647,204$
From $0.112,165,647,204$
Take $0.001,055,040,012,661,608$
$0.111,110,607,191,338,392$
$= x + x^2 - x^3$.

Therefore $x + x^2$ will be $= 0.112,165,647,204$, and $x + x^2 - x^3$ will be $(= 0.112,165,647,204 - 0.001,055,040,012,661,608) = 0.111,110,607,191,338,392$; which is less than $0.111,111,111,111$, &c, or the absolute term of the equation $x + x^2 - x^3 = 0.111,111,111,111$, &c. And consequently $0.101,802$ is less than the true value of x, or the root of the said equation.

A Third Process of Mr. Raphson's Method of Approximation.

Art. 91. Dr. Mackay then puts z for the difference of $0.101,802$ and x, and substitutes the binomial quantity $0.101,802 + z$, instead of x, in the said equation, in order to obtain a fourth and more exact value of it's root x. This he does in the following manner.

Since x is $= 0.101,802 + z$, we shall have

$x^2 = \overline{0.101,802 + z}^2$ $(= \overline{0.101,802}^2 + 2 \times 0.101,802 \times z + \&c =$
$\overline{0.101,802}^2 + 0.203,604 \times z + \&c) = 0.010,363,647,204$
$+ 0.203,604 \times z + \&c$;
and $x^3 = \overline{0.010,802 + z}^3$ $(= \overline{0.101,802}^3 + 3 \times \overline{0.101,802}^2 \times z + \&c$
$= \overline{0.101,802}^3 + 3 \times 0.010,363,647,204 \times z + \&c$
$= \overline{0.101,802}^3 + 0.031,090,941,612 \times z + \&c)$
$= 0.001,055,040,012,661,608 + 0.031,090,941,612 \times z + \&c.$

Therefore

Therefore the trinomial quantity $x+x^2-x^3$ will be equal to the multinomial quantity

$$\left\{\begin{array}{lll} & 0.101,802 & +z \\ + & 0.010,363,647,204 & +\ 0.203,604\times z + \&c \\ - & 0.001,055,040,012,661,608 & -\ 0.031,090,941,612\times z - \&c \end{array}\right\} =$$

$$\left\{\begin{array}{lll} & 0.112,165,647,204 & +\ 1.203,604\times z + \&c \\ - & 0.001,055,040,012,661,608 & -\ 0.031,090,941,612\times z + \&c \end{array}\right\}$$

$$= 0.111,110,607,191,338,392 + 1.172,513,058,388\times z \ \&c.$$

But the trinomial quantity $x+x^2-x^3$ is $= 0.111,111,111,111,111,111,\&c.$

Therefore the compound quantity 0.111,110,607,191,338,392 + 1.172,513,058,388 × *z* &c will alſo be = 0.111,111,111,111,111,111,&c. And conſequently 1.172,513,058,388 × *z* will be (= 0.111,111,111,111,111,111, &c − 0.111,110,607,191,338,392) = 0.000,000,503,919,772,719, and *z* will be $= \frac{0.000,000,503,919,772,719}{1.172,513,058,388} = 0.000,000,429,777$. Therefore *x*, or 0.101,802+*z*, will be (= 0.101,802 + 0.000,000,429,777) = 0.101,802,429,777, and conſequently 0.101,802,429,777 will be a fourth near value of the root of the equation $x+x^2-x^3 = 0.111,111,111,111,111,111,111,\&c.$

Q. E. I.

Art. 92. Dr. Maekay then proceeds to ſubſtitute this fourth near value of the root of the equation $x+x^2-x^3 = 0.111,111,111,111,111,111,111,\&c$, to wit, 0.101,802,429,777, (which, it is probable, is exact in all it's figures,) in the trinomial quantity $x+x^2-x^3$, in order to diſcover whether the value of the ſaid qnantity reſulting from ſuch ſubſtitution will be greater, or will be leſs, than 0.111,111,111,111,111,111,111,&c, or the abſolute term of the propoſed equation $x+x^2-x^3 = 0.111,111,111,111,111,111,111,\&c$, and conſequently whether the ſaid fourth near value of *x* is greater, or is leſs, than the true value of *x* in that equation. And for this purpoſe he computes the ſquare and the cube of this fourth near value of *x* in the following manner.

x = 0.101,802,429,777
x = 0.101,802,429,777

712 617 008 439
7 126 170 084 39
71 261 700 843 9
916 221 867 993
2 036 048 595 54
40 720 971 910 8
203 604 859 554
81 441 943 821 60
101 802 429 777
10 180 242 977 70

x^2 = 0.010,363,734,708,501,016,269,729
x = 0.101,802,429,777

0.001 036 373 470 850 101 626 973
10 363 734 708 501 016 270
8 290 987 766 800 813 016
20 727 469 417 002 033
4 145 493 883 400 406
207 274 694 170 020
93 273 612 376 509
7 254 614 295 951
725 461 429 595
72 546 142 959

x^3 = 0.001,055,053,374,889,632,273,732

x = 0.101,802,429,777
x^2 = 0.010,363,734,708,501,016,269,729

$x+x^2$ = 0.112,166,164,485,501,016,269,729
x^3 = 0.001,055,053,374,889,632,273,732

$x+x^2-x^3$ = 0.111,111,111,110,611,383,995,997

Therefore $x+x^2-x^3$ is = 0.111,111,111,110,611,383,995,997; which is leſs than the abſolute term 0.111,111,111,111,111,111,111,111,&c of the propoſed equation. And conſequently the number 0.101,802,429,777 is leſs than the true value of x in that equation. Q. E. I.

A Fourth

A Fourth Process of Mr. Raphson's Method of Approximation.

Art. 93. Having thus found that the number 0.101,802,429,777 is leſs than the true value of x, Dr. Mackay puts z for their difference, and then has recourſe to a fourth proceſs of Mr. Raphſon's Method of Approximation, to obtain another near value of x that ſhall be much nearer than the number 0.101,802,429,777 to it's true value. And for this purpoſe he ſubſtitutes the binomial quantity $0.101,802,429,777+z$, inſtead of x, in the terms of the equation $x+x^2-x^3 = 0.111,111,111,111,111,111,111,111,111,$&c. This he does in the following manner.

Since x is $= 0.101,802,429,777+z$, we ſhall have

$x^2 = \overline{0.101,802,429,777+z}^2$ $(= \overline{0.101,802,429,777}^2 + 2\times 0.101,802,429,777\times z + \&c = \overline{0.101,802,429,777}^2 + 0.203,604,859,554\times z + \&c) = 0.010,363,734,708,501,016,269,729 + 0.203,604,859,554\times z + \&c,$

and $x^3 = \overline{0.101,802,429,777+z}^3$ $(= \overline{0.101,802,429,777}^3 + 3\times \overline{0.101,802,429,777}^2 \times z + \&c = \overline{0.101,802,429,777}^3 + 3\times 0.010,363,734,708,501,016,269,729\times z + \&c = \overline{0.101,802,429,777}^3 + 0.031,091,204,125,503,048,809,187\times z + \&c) = 0.001,055,053,374,889,632,273,732 + 0.031,091,204,125,503,048,809,187\times z + \&c.$

Therefore the trinomial quantity $x+x^2-x^3$ will be equal to the multinomial quantity

$$\left\{\begin{array}{rrrrr} & 0.101,802,429,777 & & + & z \\ + & 0.010,363,734,708,501,016,269,729 & + & 0.203,604,859,554\times z+\&c & \\ - & 0.001,055,053,374,889,632,273,732 & - & 0.031,091,204,125,503,048, & \end{array}\right\}$$
$$809,187\times z - \&c$$
$$= 0.112,$$

$$= \left\{ \begin{array}{r} 0.112,166,164,485,501,016,269,729 + 1.203,604,859,554 \times z \\ + \&c \\ - 0.001,055,053,374,889,632,273,732 + 0.031,091,204,125,503, \\ 048,809,187 \times z - \&c \end{array} \right\}$$

$$= 0.111,111,111,110,611,383,995,997 + 1.172,513,655,428,496, 951,190,813 \times z + \&c.$$

But the trinomial quantity $x + x^2 - x^3$ is $= 0.111,111,111,111,111,111, 111,111,111,\&c.$

Therefore the compound quantity $0.111,111,111,110,611,383,995,997 + 1.172,513,655,428,496,951,190,813 \times z$ will alſo be $= 0.111,111,111,111, 111,111,111,111,111,\&c.$ And conſequently $1.172,513,655,428,496,951,190, 813 \times z$ will be $=$

$$\left\{ \begin{array}{r} 0.111,111,111,111,111,111,111,111,111,\&c \\ - 0.111,111,111,110,611,383,995,997, \end{array} \right\}$$

$= 0.000,000,000,000,499,727,115,114,111,\&c$, and z will be

$= \frac{0.000,000,000,000,499,727,115,114,111,\&c}{1.172,513,655,428,496,951,190,813} = 0.000,000,000,000,426,201,531, 044.$ Therefore x, or $0.101,802,429,777 + z$, will be $= 0.101,802,429,777 + 0.000,000,000,000,426,201,531,044 = 0.101,802,429,777,426,201,531, 044$; that is, the root of the propoſed equation $x + x^2 - x^3 = \frac{1}{9}$, or $x + x^2 + x^3$ $0.111,111,111,111,111,111,111,111,111,\&c$, will be $= 0.101,802,429, 777,426,201,531,044.$ Q. E. I.

The Diviſion of $0.000,000,000,000,499,727,115,114,111,\&c$ by $1.172,513, 655,428,\&c$ is as follows.

The

The Divisor. *The Quotient.*

1.172,513,655,428) (0.000,000,000,000,426,201,531,044

The Dividend.

0.000,000,000,000,499,727,115,114,111,111,111,111,&c

469 005 462 171 2

·30 721 652 942 91

23 450 273 108 56

·7 271 379 834 351

7 035 081 932 568

·236 297 901 783 1

234 502 731 085 6

··1 795 170 697 511

1 172 513 655 428

·622 657 042 083 1

586 256 827 7140

·36 400 214 369 11

35 175 409 662 84

·1 224 804 706 271

1 172 513 655 428

··52 291 050 843 11

46 900 546 217 12

·5 390 504 625 991

4 690 054 621 712

·700 450 004 289

Art. 94. This laſt, or fifth, near value of x, or the root of the equation $x+x^2-x^3=\frac{1}{9}$, or $x+x^2-x^3=$ 0.111,111,111,111,111,111,111,111,111, &c, to wit, the number, 0.101,802,429,777,426,201,531,044, is probably exact in all it's twenty-four figures. But that the degree of it's exactneſs may be aſcertained, Dr. Mackay has ſubſtituted it inſtead of x in the trinomial quantity $x+x^2-x^3$, and has found that the value of that quantity which reſults from it's ſubſtitution co-incides exactly with the abſolute term of the equation $x+x^2-x^3$ = 0.111,111,111,111,111,111,111,111,111,&c in the firſt twenty-four figures of the ſaid abſolute term, or is = 0.111,111,111,111,111,111,111,111. And for this purpoſe he has computed the ſquare and the cube of the ſaid number 0.101,802,429,777,426,201,531,044 to twenty-four places of figures, in the following manner.

* =

x = 0.101,802,429,777,426,201,531,044
x = 0.101,802,429,777,426,201,531,044

0.010 180 242 977 742 620 153 104 4
101 802 429 777 426 201 531 0
81 441 943 821 940 961 224 8
203 604 859 554 852 403 1
40 720 971 910 970 480 6
2 036 048 595 548 524 0
916 221 867 996 835 8
71 261 700 844 198 3
7 126 170 084 419 8
712 617 008 442 0
40 720 971 911 0
2 036 048 595 5
610 814 578 7
20 360 486 0
101 802 4
50 901 2
3 054 1
101 8
4 1
4

x^2 = 0.010,363,734,708,587,792,972,599,0
x = 0.101,802,429,777,426,201,531,044

0.001 036 373 470 858 779 297 259 9
10 363 734 708 587 792 972 6
8 290 987 766 870 234 378 1
20 727 469 417 175 585 9
4 145 493 883 435 117 2
207 274 694 171 755 9
93 273 612 377 290 1
7 254 614 296 011 5
725 461 429 601 1
72 546 142 960 1
4 145 493 883 4
207 274 694 2
62 182 408 3
2 072 746 9
10 363 7
5 181 9
310 9
104
4

x^3 = 0.001,055,053,374,902,883,392,532,5

Therefore

Therefore $x+x^2-x^3$ will be equal to

$$\left.\begin{array}{l} 0.101,802,429,777,426,201,531,044 \\ +\ 0.010,363,734,708,587,792,972,599 \\ -\ 0.001,055,053,374,902,883,392,532 \end{array}\right\}$$

$$= \left.\begin{array}{l} 0.112,166,164,486,013,994,503,643 \\ -\ 0.001,055,053,374,902,883,392,532 \end{array}\right\}$$

$= 0.111,111,111,111,111,111,111,111$; which are the firſt twenty-four firſt figures of the infinite ſeries 0.111,111,111,111,111,111,111,111, 111,111,&c, which is equal to $\frac{1}{9}$, or the abſolute term of the propoſed equation $x+x^2-x^3 = \frac{1}{9}$. Therefore the ſaid number 0.101,802,429,777, 426,201,531,044 is the true value of x, or the root of that equation.

Art. 95. This number 0.101,802,429,777,426,201,531,044 is leſs than the true value of x in the equation $x+x^2-x^3 = \frac{1}{9}$, or $x+x^2-x^3 = 0.111$, 111,111,111,111,111,111,111,111,111,&c; becauſe, when it is ſubſtituted inſtead of x in the trinomial quantity $x+x^2-x^3$, it makes the ſaid trinomial quantity be equal to only the firſt twenty-four figures of the infinite ſeries which is equal to $\frac{1}{9}$. But, if this number were to be increaſed by only an unit in the laſt, or twenty-fourth, place of decimal fractions, the new number thence ariſing, to wit, the number 0.101,802,429,777,426,201,531,045, would be greater than the true value of x in the ſaid equation $x+x^2-x^3 = 0.111,111$, 111,111,111,111,111,111,111,111,111,&c, or, if it were ſubſtituted, inſtead of x in the trinomial quantity $x+x^2-x^3$, would make that quantity be greater than the abſolute term $\frac{1}{9}$, or 0.111,111,111,111,111,111,111,111,111,111,&c of the ſaid equation. This may be ſhewn in the following manner.

Let a be put for the number 0.101,802,429,777,426,201,531,044. Then will the new number 0.101,802,429,777,426,201,531,045 be $= a+0.000$, 000,000,000,000,000,000,001, or $= a + \frac{1}{\overline{10}^{24}}$. Therefore the ſquare of the ſaid new number will be $= \overline{a + \frac{1}{\overline{10}^{24}}}^2 = a^2 + 2a \times \frac{1}{\overline{10}^{24}} + \frac{1}{\overline{10}^{48}}$, and

the cube of the ſaid new number will be $= \overline{a + \frac{1}{\overline{10}|^{24}}}^{3} = a^3 + 3a^2 \times \frac{1}{\overline{10}|^{24}}$ $+ 3a \times \frac{1}{\overline{10}|^{48}} + \frac{1}{\overline{10}|^{72}}$. Therefore, if this new number were to be ſubſtituted, inſtead of x, in the trinomial quantity $x+x^2-x^3$, the ſaid trinomial quantity would be $= a + \frac{1}{\overline{10}|^{24}} + a^2 + 2a \times \frac{1}{\overline{10}|^{24}} + \frac{1}{\overline{10}|^{48}} - a^3 - 3a^2 \times \frac{1}{\overline{10}|^{24}}$ $- 3a \times \frac{1}{\overline{10}|^{48}} - \frac{1}{\overline{10}|^{72}}$; or (neglecting the terms $\frac{1}{\overline{10}|^{48}}$, $3a \times \frac{1}{\overline{10}|^{48}}$, and $\frac{1}{\overline{10}|^{72}}$ on account of their extreme ſmallneſs in compariſon of the other terms,) the ſaid trinomial quantity would be $= a + \frac{1}{\overline{10}|^{24}} + a^2 + 2a \times \frac{1}{\overline{10}|^{24}} - a^3 -$ $3a^2 \times \frac{1}{\overline{10}|^{24}}$, or $= a + a^2 - a^3 + \frac{1}{\overline{10}|^{24}} + 2a \times \frac{1}{\overline{10}|^{24}} - 3a^2 \times \frac{1}{\overline{10}|^{24}}$, which exceeds the quantity $a+a^2-a^3$ by the quantity $\frac{1}{\overline{10}|^{24}} + 2a \times \frac{1}{\overline{10}|^{24}} - 3a^2 \times$ $\frac{1}{\overline{10}|^{24}}$. Now, becauſe a is nearly equal to 0.1, and a^2 to 0.01, it follows that $2a$ will be nearly $= 0.2$, and $3a^2$ will be nearly $= 0.03$, which is much leſs than 0.2, or $2a$; and conſequently $2a \times \frac{1}{\overline{10}|^{24}}$ will be greater than $3a^2 \times \frac{1}{\overline{10}|^{24}}$, and the quantity $\frac{1}{\overline{10}|^{24}} + 2a \times \frac{1}{\overline{10}|^{24}} - 3a^2 \times \frac{1}{\overline{10}|^{24}}$ will be greater than $\frac{1}{\overline{10}|^{24}}$; that is, the exceſs of the value of the trinomial quantity $x+x^2-x^3$ reſulting from the ſubſtitution of $a + \frac{1}{\overline{10}|^{24}}$ in it's terms inſtead of x, above the value of it found above by ſubſtituting a, or 0.101,802,429,777,426,201,531, 044, in it's terms inſtead of x, will be greater than $\frac{1}{\overline{10}|^{24}}$, or than 0.000,000, 000,000,000,000,000,001. And conſequently the value of the ſaid trinomial quantity reſulting from the ſubſtitution of the new number $a + \frac{1}{\overline{10}|^{24}}$, or 0.101,802,429,777,426,201,531,045, in it's terms inſtead of x, will be greater than 0.111,111,111,111,111,111,111,111 + 0.000,000,000,000,000,000, 000,001; that is, than 0.111,111,111,111,111,111,111,112, and conſequently than the infinite ſeries 0.111,111,111,111,111,111,111,111,111,111,&c, which

is

is leſs than 0.111,111,111,111,111,111,111,112. It appears therefore that the number a, or 0.101,802,429,777,426,201,531,044 is leſs than the true value of x in the propoſed equation $x+x^2-x^3 = \frac{1}{9}$, and that the number 0.101,802,429,777,426,201,531,045 is greater than the ſaid true value, and conſequently that the former of theſe numbers, to wit, the number 0.101,802,429,777,426,201,531,044, is exact in all it's twenty-four figures. Q. E. D.

Art. 96. Having thus found the value of x, or the root of the equation $x+x^2-x^3 = \frac{1}{9}$, to be 0.101,802,429,777,426,201,531,044, Dr. Mackay proceeds to multiply it into the number 1.732,050,807,568,877,293,527,446,3, which is equal to the ſquare-root of 3, in order to obtain the value of $x \times \sqrt{3}$, which is equal to t, or the root of the equation $3t + 3 \times \frac{1}{\sqrt{3}} \times t^2 - t^3 = \frac{1}{\sqrt{3}}$, or to tangent of an arch of 10 Degrees. This multiplication is performed in the following manner.

$\sqrt{3}$ = 1.732,050,807,568,877,293,527,446,3
x = 0.101,802,429,777,426,201,531,044

0.173 205 080 756 887 729 352 744 6
1 732 050 807 568 877 293 527 4
1 385 640 646 055 101 834 822 0
3 464 101 615 137 754 587 1
692 820 323 027 550 917 4
34 641 016 151 377 545 9
15 588 457 268 119 895 6
1 212 435 565 298 214 1
121 243 556 529 821 4
12 124 355 652 982 1
692 820 323 027 6
34 641 016 151 4
10 392 304 845 4
346 410 161 5
1 732 050 8
866 025 4
51 961 5
1 732 1
693
6 9

Tangent of 10 Degrees = 0.176,326,980,708,464,973,471,089,5

And

And thus we have compleated the ſecond part of the Computation of the Length of the Tangent of an Arch of 1 Degree in the Method recommended by Mr. Ivory, by finding the length of the tangent of an arch of 10 Degrees.

Art. 97. The third and laſt part of this Computation conſiſts in deriving from the lengths of the tangents of arches of 9 and 10 Degrees, (which have now been diſcovered by the foregoing calculations,) the tangent of the difference of thoſe arches, or of an arch of 1 Degree, by computing the value of the fraction $\frac{a-b}{1+ab}$, in which a ſtands for the tangent of an arch of 10 Degrees, and b ſtands for the tangent of an arch of 9 Degrees, and conſequently a is $= 0.176,326,980,708,464,973,471,089,5$, and b is $= 0.158,384,440,324,536,293,838,883,092$.

$$\begin{array}{rl} a = & 0.176,326,980,708,464,973,471,089,5 \\ b = & 0.158,384,440,324,536,293,838,883,1 \\ \hline a-b = & 0.017,942,540,383,928,679,632,206,4 \end{array}$$

The multiplication of a into b, in order to find the value of ab in the denominator $1+ab$ of the fraction $\frac{a-b}{1+ab}$, is performed by Dr. Mackay in the following manner.

a = 0.176,326,980,708,464,973,471,089,5
b = 0.158,384,440,324,536,293,838,883,1

0.017 632 698 070 846 497 347 108 95
8 816 349 035 423 248 673 554 47
1 410 615 845 667 719 787 768 72
52 898 094 212 539 492 041 33
14 106 158 456 677 197 877 69
705 307 922 833 859 893 88
70 530 792 283 385 989 39
7 053 079 228 338 598 94
52 898 094 212 539 49
3 526 539 614 169 30
705 307 922 833 86
88 163 490 354 23
5 289 809 421 25
1 057 961 884 25
35 265 396 14
15 869 428 26
528 980 94
158 694 28
5 289 81
1 410 62
141 06
14 11
53
2

---

ab = 0.027,927,450,153,625,532,923,391,52
1.

$1+ab$ = 1.027,927,450,153,625,532,923,391,52

The

The Division of $a-b$ *by* $1+ab$.

The Divisor. *The Quotient.*

1.027,927,450,153,625,532,923,391,52) (0.017,455,064,928,217,585,765

The Dividend.

0.017,942,540,383,928,679,632,206,4

10 279 274 501 536 255 329 233 9

· 7 663 265 882 392 424 302 972 5

7 195 492 151 075 378 730 463 7

· 467 773 731 317 045 572 508 8

411 170 980 061 450 213 169 4

· 56 602 751 255 595 359 339 4

51 396 372 507 681 276 646 2

· 5 206 378 747 914 082 693 2

5 139 637 250 768 127 664 6

·· 66 741 497 145 955 028 6

61 675 647 009 217 532 0

· 5 065 850 136 737 496 6

4 111 709 800 614 502 1

954 140 336 122 994 5

925 134 705 138 263 0

· 29 005 630 984 731 5

20 558 549 003 072 5

· 8 447 081 981 659 0

8 223 419 601 229 0

· 223 662 380 430 0

205 585 490 030 7

· 18 076 890 399 3

10 279 274 501 5

· 7 797 615 897 8

7 195 492 151 1

· 602 123 746 7

513 963 725 1

· 88 160 021 6

82 234 196 0

· 5 925 825 6

5 139 637 2

· 786 188 4

719 549 2

· 66 639 2

61 675 6

· 4 963 6

Therefore $\frac{a-b}{1+ab}$ is = 0.017,455,064,928,217,585,765, which is true to the neareſt unit in the twenty-firſt, or laſt, place of decimal fractions; and conſequently the tangent of an arch of 1 Degree, in a circle of which the radius is called 1, is now found to be equal to the ſaid number 0.017,455,064,928,217,585,765. Q. E. I.

This number 0.017,455,064,928,217,585,765 agrees in all it's figures with the firſt twenty-one figures of the number 0.017,455,064,928,217,585,765,130,659, which was found above in Art. 39, page 831, for the value of the tangent of an arch of 1 Degree.

This Computation of the Length of the Tangent of an arch of 1 Degree by the Method recommended by Mr. Ivory begun in page 914, and therefore has taken-up only 21 pages: but in the former Computation of the ſame Tangent, (by, firſt, quinquiſecting an arch of 45 Degrees, and then triſecting an arch of 9 Degrees, and afterwards triſecting an arch of 3 Degrees,) no fewer than 51 pages were employed, to wit, the pages from page 780 to page 832. So that Mr. Ivory's Method muſt be acknowledged to be much ſhorter and leſs laborious than the other, and yet equally clear and ſatisfactory.

End of Dr. Mackay's Computation of the Length of the Tangent of an Arch of 1 Degree, according to the Method recommended by Mr. Ivory.

Another

ANOTHER

METHOD

OF

COMPUTING THE VALUE OF THE TANGENT OF AN ARCH OF 9 DEGREES,

OR

OF QUINQUISECTING AN ARCH OF 45 DEGREES,

By Means of the Quinquinomial Equation $5r^4t + 10r^3t^2 - 10r^2t^3 - 5rt^4 + t^5 = r^5$, *which expresses the Relation between the Tangent of an Arch of* 45 *Degrees, (which is equal to* r, *or the Radius of the Circle,) and* t, *or the Tangent of the fifth Part of the said Arch of* 45 *Degrees, or of an Arch of* 9 *Degrees.*

Art. 98. This equation $5r^4t + 10r^3t^2 - 10r^2 - 10r^2t^3 - 5rt^4 + t^5 = r^5$ has two other roots, (that is, real and affirmative roots,) beſides the root t, or the tangent of an arch of 9 Degrees; to wit, firſt, r, the radius of the circle, or the tangent of the arch of 45 Degrees itſelf; and, ſecondly, the co-tangent of an arch of 9 Degrees, or the tangent of an arch of 81 Degrees, which is the complement of an arch of 9 Degrees to an arch of 90 Degrees, or the arch of the whole quadrant; which analogy between theſe tangents, when I firſt diſcovered it, was, I confeſs, matter of ſome ſurprize to me. That theſe two quantities will be roots of the ſaid equation as well as t, or the tangent of an arch of 9 Degrees, may be ſhewn in the following manner.

Art. 99. In the firſt place, if the radius r be ſubſtituted inſtead of t in the quinquinomial quantity $5r^4t + 10r^3t^2 - 10r^2t^3 - 5rt^4 + t^5$, which forms the firſt,

or

or left-hand, ſide of the equation $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5 = r^5$, the ſaid quinquinomial quantity will be thereby converted into the quinquinomial quantity $5r^4\times r+10r^3\times r^2-10r^2\times r^3-5r\times r^4+r^5$, or $5r^5+10r^5-10r^5-5r^5+r^5$, which is equal to r^5, or the abſolute term of the ſaid quinquinomial equation $5r^4t+10r^4t^2-10r^2t^3-5rt^4+t^5 = r^5$. Therefore r is a root of the ſaid equation, as well as t, or the tangent of an arch of 9 Degrees.

Q. E. D.

Art. 100. Secondly, let b be put for the tangent of an arch of 81 Degrees, or the complement of an arch of 9 Degrees to an arch of 90 Degrees, or the arch of the whole quadrant. And let b be ſubſtituted inſtead of t in the quinquinomial quantity $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5$, whereby the ſaid quantity will be converted into the quinquinomial quantity $5r^4b+10r^3b^2-10r^2b^3-5rb^4+b^5$. And it may be ſhewn that this new quinquinomial quantity $5r^4b+10r^3b^2-10r^2b^3-5rb^4+b^5$ will be equal to r^5, (the abſolute term of the propoſed equation $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5 = r^5$,) as well as the quinquinomial quantity $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5$ and the quinquinomial quantity $5r^4\times r+10r^3\times r^2-10r^2\times r^3-5r\times r^4+r^5$, and therefore that b is a root of the quinquinomial equation $5r^4t+10r^3t^2-10r^2t^3-1rt^4+t^5 = r^5$, as well as t, the tangent of an arch of 9 Degrees, and r, the radius of the circle. This may be ſhewn in the following manner.

Since b is the tangent of the complement of an arch of 9 Degrees to an arch of 90 Degrees, or the arch of the whole quadrant, the rectangle under the tangent b and the tangent t will be equal to the ſquare of the radius r. Therefore t will be $= \frac{rr}{b}$, and conſequently t^2 will be $(= \overline{\frac{rr}{b}}^2) = \frac{r^4}{b^2}$, and t^3 will be $(= \overline{\frac{rr}{b}}^3) = \frac{r^6}{b^3}$, and t^4 will be $(= \overline{\frac{rr}{b}}^4) = \frac{r^8}{b^4}$, and t^5 will be $(= \overline{\frac{rr}{b}}^5) = \frac{r^{10}}{b^5}$. Therefore the quinquinomial quantity $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5$ will be $(= 5r^4\times\frac{rr}{b}+10r^3\times\frac{r^4}{b^2}-10r^2\times\frac{r^6}{b^3}-5r\times\frac{r^8}{b^4}+\frac{r^{10}}{b^5}) = \frac{5r^6}{b}+\frac{10r^7}{b^3}-\frac{10r^8}{b^3}-\frac{5r^9}{b^4}+\frac{r^{10}}{b^5}$.

But $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5$ is equal to r^5.

Therefore $\frac{5r^6}{b}+\frac{10r^7}{b^2}-\frac{10r^8}{b^3}-\frac{5r^9}{b^4}+\frac{r^{10}}{b^5}$ will alſo be equal to r^5. Therefore (by multiplying all the terms of this equation into the fraction $\frac{b^5}{r^5}$,) we ſhall have $5rb^4+10r^2b^3-10r^3b^2-5r^4b+r^5=b^5$; and (by adding $10r^2b^3+5r^4b$ to both ſides,) we ſhall have $5rb^4+10r^2b^3+r^5=b^5+10r^3b^2+5r^4b$, and (by ſubtracting $5rb^4+10r^2b^3$ from both ſides,) we ſhall have $r^5=b^5+10r^3b^2+5r^4b-5rb^4-10r^2b^3$, or $5r^4b+10r^3b^2-10r^2b^3-5rb^4+b^5=r^5$. Q. E. D.

Therefore r and b, or $\frac{rr}{t}$, that is, the radius of the circle, and the tangent of an arch of 81 Degrees, or the co-tangent of an arch of 9 Degrees, are roots of the quinquinomial equation $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5=r^5$, as well as t, or the tangent of an arch of 9 Degrees.

Art. 101. Now, ſince the quinquinomial quantity $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5$ is equal to r^5, and the quinquinomial quantity $5r^4\times r+10r^3\times r^2-10r^2\times r^3-5r\times r^4+r^5$ is alſo equal to r^5, it follows that the quinquinomial quantity $5r^4t+10r^3t^2-10r^2t^3-5rt^4+t^5$ will be equal to the quinquinomial quantity $5r^4\times r+10r^3\times r^2-10r^2\times r^3-5r\times r^4+r^5$. Therefore (adding $10r^2\times r^3+5r\times r^4$ to both ſides,) we ſhall have $5r^4t+10r^3t^2+10r^2\times r^3+5r\times r^4-10r^2t^3-5rt^4+t^5=5r^4\times r+10r^3\times r^2+r^5$; and (ſubtracting $5r^4t+10r^3t^2+t^5$ from both ſides,) we ſhall have $10r^2\times r^3+5r\times r^4-10r^2\times t^3-5rt^4=5r^4\times r+10r^3\times r^2+r^5-5r^4t-10r^3t^2-t^5$, or $10r^2\times r^3-10r^2\times t^3+5r\times r^4-5r\times t^4=5r^4\times r-5r^4\times t+10r^3\times r^2-10r^3\times t^2+r^5-t^5$, or $10r^2\times\overline{r^3-t^3}+5r\times\overline{r^4-t^4}=5r^4\times\overline{r-t}+10r^3\times\overline{r^2-t^2}+r^5-t^5$, and (dividing both ſides by the binomial quantity $r-t$,) we ſhall have $10r^2\times\frac{r^3-t^3}{r-t}+5r\times\frac{r^4-t^4}{r-t}=5r^4\times\frac{r-t}{r-t}+10r^3\times\frac{r^2-t^2}{r-t}+\frac{r^5-t^5}{r-t}$ or $10r^2\times\overline{r^2+rt+t^2}+5r\times\overline{r^3+r^2t+rt^2+t^3}=5r^4\times 1+10r^3\times\overline{r+t}+r^4+r^3t+r^2t^2+rt^3+t^4$,

or

$$\text{or } \left\{\begin{array}{rrr} & 10r^4 & + 5r^4 \\ + & 10r^3t & + 5r^3t \\ + & 10r^2t^2 & + 5r^2t^2 \\ & & + 5rt^3 \end{array}\right\} = \left\{\begin{array}{rrr} & & 5r^4 \\ + & 10r^4 & + r^4 \\ + & 10r^3t & + r^3t \\ & & + r^2t^2 \\ & & + rt^3 \\ & & + t^4 \end{array}\right\} \text{ or}$$

$15r^4+15r^3t+15r^2t^2+5rt^3 = 16r^4+11r^3t+r^2t^2+rt^3+t^4$, and (subtracting $15r^4+11r^3t$ from both sides) $4r^3t+15r^2t^2+5rt^3 = r^4+r^2t^2+rt^3+t^4$, and (subtracting $r^2t^2+rt^3$ from both sides,) $4r^3t+14r^2t^2+4rt^3 = r^4+t^4$, and, lastly, (subtracting t^4 from both sides,) $4r^3t+14r^2t^2+4rt^3-t^4 = r^4$; which equation has two roots, of which t, or the tangent of an arch of 9 Degrees, is the leſſer. And therefore the tangent of an arch of 9 Degrees might have been obtained by reſolving the quadrinomial equation $4r^3t+14r^2t^2+4rt^3-t^4 = r^4$, or, if the radius r be called 1, by reſolving the quadrinomial equation $4t+14t^2+4t^3-t^4 = 1$, as well as by reſolving the quinquinomial equation $5t+10t^2-10t^3-5t^4+t^5 = 1$; and with much leſs labour of calculation. This reſolution may be performed as follows.

The Reſolution of the Quadrinomial Equation $4t+14t^2+4t^3-t^4 = 1$, *of which the leſſer Root is equal to* t, *or the Tangent of an Arch of* 9 *Degrees in a Circle of which the Radius is called* 1.

Art. 102. A circular arch of 9 Degrees is the 20th part of the ſemi-circumference of the circle, and therefore is $= \frac{3.141,592,653,\&c}{20} = 0.157,079,\&c.$ But the tangent of a circular arch is greater than the arch itſelf. And therefore the tangent of an arch of 9 Degrees muſt be greater than 0.157,079. We will therefore ſuppoſe it to be equal to 0.158, and will try the effect of this ſuppoſition by raiſing the powers of 0.158, and ſubſtituting 0.158, and $\overline{0.158}^2$, $\overline{0.158}^3$, and $\overline{0.158}^4$, inſtead of t, t^2, t^3, and t^4, in the quadrinomial quantity $4t+14t^2+4t^3-t^4$, (which forms the firſt, or left-hand, ſide of the propoſed equation $4t+14t^2+4t^3-t^4 = 1$,) in order to diſcover whether the value of the

ſaid quadrinomial quantity ariſing from ſuch ſubſtitution will be greater, or will be leſs, than the abſolute term, 1, of the ſaid equation, and conſequently whether the ſaid number 0.158 will be greater, or will be leſs, than the true value of t in the ſaid equation. This may be done in the following manner.

$t = 0.158$
$t = 0.158$
1 264
790
158
$t^2 = 0.024,964$
14
99 856
249 64
$14t^2 = 0.349,496$

0.024,964
0.158
199 712
1 248 20
2 496 4
$t^3 = 0.003,944,312$
4
$4t^3 = 0.015,777,248$

0.003,944,312
0.158
31 554 496
197 215 60
394 431 2
$t^4 = 0.000,623,201,296$

0.158
4
$4t = 0.632$

$$\begin{aligned} 0.632,000,000,000 &= 4t \\ + \ 0.349,496,000,000 &= 14t^2 \\ + \ 0.015,777,248,000 &= 4t^3 \\ \hline 0.997,273,248,000 &= 4t + 14t^2 + 4t^3 \\ - \ 0.000,623,201,296 &= t^4 \\ \hline 0.996,650,046,704 &= 4t + 14t^2 + 4t^3 - t^4. \end{aligned}$$

Therefore the value of the quadrinomial quantity $4t + 14t^2 + 4t^3 - t^4$ reſulting from the ſuppoſition that t is equal to 0.158, is the decimal fraction 0.996,650,046,704; which is leſs than 1, or the abſolute term of the propoſed equation $4t + 14t^2 + 4t^3 - t^4 = 1$. Therefore the number 0.158 is leſs than the true value of t in that equation. Q. E. I.

A Process of Mr. Raphson's Method of Approximation.

Art. 103. Let z be put for the exceſs of the true value of t in the equation $4t + 14t^2 + 4t^3 - t^4 = 1$ above 0.158, and let the binomial quantity $0.158 + z$, be ſubſtituted inſtead of t in the ſaid equation.

Then

Then will t^2 be $= \overline{0.158+z}^2$ $(= \overline{0.158}^2 + 2\times 0.158\times z + \&c = \overline{0.158}^2 + 0.316\times z + \&c) = 0.024{,}964 + 0.316\times z$;

and t^3 will be $= \overline{0.158+z}^3$ $(= \overline{0.158}^3 + 3\times\overline{0.158}^2\times z + \&c = \overline{0.158}^3 + 3\times 0.024{,}964\times z + \&c = \overline{0.158}^3 + 0.074{,}892\times z + \&c) = 0.003{,}944{,}312 + 0.074{,}892\times z + \&c$;

and t^4 will be $= \overline{0.158+z}^4$ $(= \overline{0.158}^4 + 4\times\overline{0.158}^3\times z + \&c = \overline{0.158}^4 + 4\times 0.003{,}944{,}312\times z + \&c = \overline{0.158}^4 + 0.015{,}777{,}248\times z + \&c) = 0.000{,}623{,}201{,}296 + 0.015{,}777{,}248\times z + \&c$.

Therefore $4t$ will be $= 4\times\overline{0158+z}$ $(= 4\times 0.158 + 4\times z) = 0.632 + 4z$;

and $14t^2$ will be $= 14\times\overline{0.024{,}964 + 0.316\times z + \&c}$ $(= 14\times 0.024{,}964 + 14\times 0.316\times z + \&c = 14\times 0.024{,}964 + 4.424\times z + \&c) = 0.349{,}496 + 4.424\times z + \&c$;

and $4t^3$ will be $= 4\times\overline{0.003{,}944{,}312 + 0.074{,}892\times z + \&c}$ $(= 4\times 0.003{,}944{,}312 + 4\times 0.074{,}892\times z + \&c = 4\times 0.003{,}944{,}312 + 0.299{,}568\times z + \&c) = 0.015{,}777{,}248 + 0.299{,}568\times z + \&c$.

Therefore the whole quadrinomial quantity $4t + 14t^2 + 4t^3 - t^4$ will be equal to the multinomial quantity

$$\left.\begin{array}{rrrl} & 0.632{,}000 & + & 4z \\ + & 0.349{,}496 & + & 4.424\times z + \&c \\ + & 0.015{,}777{,}248 & + & 0.299{,}568\times z + \&c \\ - & 0.000{,}623{,}201{,}296 & - & 0.015{,}777{,}248\times z + \&c \end{array}\right\}$$

$$= \left\{\begin{array}{rrrl} & 0.997{,}273{,}248{,}000 & + & 8.723{,}568\times z + \&c \\ - & 0.000{,}623{,}201{,}296 & - & 0.015{,}777{,}248\times z + \&c \end{array}\right\}$$

$$= 0.996{,}650{,}046{,}704 + 8.707{,}790{,}752\times z - \&c.$$

But the quadrinomial quantity $4t + 14t^2 + 4t^3 - t^4$ is $= 1$.

Therefore

Therefore the quantity 0.996,650,046,704 + 8.707,790,752×z &c will alſo be = 1. And conſequently 8.707,790,752 × z + &c will be (= 1 — 0.996,650,046,704 = 0.003,349,953,296, and z will be = $\frac{0.003,349,953,296}{8.707,790,752}$ = 0.000,384. Therefore *t*, or 0.158+z, will be = 0.158 + 0.000,384 = 0.158,384; that is, the tangent of an arch of 9 Degrees will be very nearly equal to 0.158,384. Q. E. I.

Art. 104. Having thus obtained the number 0.158,384, for a ſecond near value of *t*, or the leſſer root of the quadrinomial equation $4t + 14t^2 + 4t^3 - t^4 = 1$, which is probably exact in all it's figures, let us ſubſtitute it inſtead of *t* in the quadrinomial quantity $4t + 14t^2 + t^3 - t^4$, in order to diſcover whether the value of the ſaid quantity, reſulting from ſuch ſubſtitution, will be greater, or will be leſs than 1, or the abſolute term of the ſaid equation, and conſequently whether the ſaid ſecond near value of *t* is greater, or is leſs, than it's true value in that equation.

Now, if *t* is = 0.158,384, we ſhall have

$t^2 = (\overline{0.158,384}^2) = 0.025,085,491,456,$

and $t^3 = (\overline{0.158,384}^3) = 0.003,973,140,478,767,104,$

and $t^4 = (\overline{0.158,384}^4) = 0.000,629,281,881,589,048,999,936$; as will appear from the following operations of multiplication.

$$
\begin{array}{r}
t = 0.158,384 \\
t = 0.158,384 \\
\hline
633\,536 \\
12\,670\,72 \\
47\,515\,2 \\
1\,267\,072 \\
7\,919\,20 \\
15\,838\,4 \\
\hline
t^2 = 0.025,085,491,456
\end{array}
$$

$t^3 =$

$t^2 = 0.025,085,491,456$
$t = 0.158,384$

100 341 965 824
2 006 839 316 48
7 525 647 436 8
200 683 931 648
1 254 274 572 80
2 508 549 145 6

$t^3 = 0.003,973,140,478,767,104$
$t = 0.158,384$

15 892 561 915 068 416
317 851 238 301 368 32
1 191 942 143 630 131 2
31 785 123 830 136 832
198 657 023 938 355 20
397 314 047 876 710 4

$t^4 = 0.000,629,281,881,589,048,999,936$

Therefore $4t$ will be $(= 4 \times 0.158,384) = 0.633,536$;
and $14t^2$ will be $(= 14 \times 0.025,085,491,456) = 0.351,196,880,384$;
and $4t^3$ will be $(= 4 \times 0.003,973,140,478,767,104) = 0.015,892,561,915,068,416$.

Therefore $4t + 14t^2 + 4t^3$ will be $=$

$$\left\{\begin{array}{l} 0.633,536 \\ + \ 0.351,196,880,384 \\ + \ 0.015,892,561,915,068,416 \end{array}\right\}$$

$= 1.000,625,442,299,068,416$, and the whole quadrinomial quantity $4t + 14t^2 + 4t^3 - t^4$ will be $=$

$$\left\{\begin{array}{l} 1.000,625,442,299,068,416,000,000 \\ - \ 0.000,629,281,881,589,048,999,936 \end{array}\right\}$$

$= 0.999,996,160,417,479,367,000,064$; which is less than 1, or the absolute term of the equation $4t + 14t^2 + 4t^3 - t^4 = 1$, and consequently the said number 0.158,384, or the second near value of the lesser root, t, of the said equation, will be less than it's true value. Q. E. I.

A Second

A Second Process of Mr. Raphson's Method of Approximation.

Art. 105. In order therefore to obtain a third near value of t, or the leſſer root of this equation, let z be put for the exceſs of the ſaid root above it's ſecond near value 0.158,384, and let the binomial quantity $0.158,384+z$ be ſubſtituted, inſtead of t, in the ſaid equation $4t+14t^2+4t^3-t^4=1$.

Then will t^2 be $= \overline{0.158,384+z}^2$ $(= \overline{0.158,384}^2 + 2\times 0.158,384\times z + \&c = \overline{0.158,384}^2 + 0.316,768\times z + \&c) =$ $0.025,085,491,456 + 0.316,768\times z + \&c$;

and t^3 will be $= \overline{0.158,384+z}^3$ $(= \overline{0.158,384}^3 + 3\times\overline{0.158,384}^2 \times z + \&c = \overline{0.158,384}^3 + 3\times 0.025,085,491,456 \times z + \&c = \overline{0.158,384}^3 + 0.075,256,474,368\times z + \&c) = 0.003,973,140,478,767,104 + 0.075,256,474,368\times z + \&c$;

and t^4 will be $= \overline{0.158,384+z}^4$ $(= \overline{0.158,384}^4 + 4\times\overline{0.158,384}^3 \times z + \&c = \overline{0.158,384}^4 + 4\times 0.003,973,140,478,767,104\times z + \&c = \overline{0.158,384}^4 + 0.015,892,561,915,068,416\times z + \&c) = 0.000,629,281,881,589,048,999,936 + 0.015,892,561,915,068,416\times z + \&c.$

Therefore $4t$ will be $= 4\times\overline{0.158,384+z}$ $(= 4\times 0.158,384+4\times z) = 0.633,536\times 4z$;

and $14t^2$ will be $= 14\times\overline{0.025,085,491,456 + 0.316,768\times z + \&c}$ $(= 14\times 0.025,085,491,456 + 14\times 0.316,768\times z + \&c = 14\times 0.025,085,491,456 + 4.434,752\times z + \&c) = 0.351,196,880,384 + 4.434,752\times z + \&c$;

and

and $4t^3$ will be $= 4 \times 0.003,973,140,478,767,104 + 0.075,256,474,368 \times z + \&c$ (= $4 \times 0.003,973,140,478,767,104 + 4 \times 0.075,256,474,368 \times z + \&c = 4 \times 0.003,973,140,478,767,104 + 0.301,025,897,472 \times z + \&c) = 0.015,892,561,915,068,416 + 0.301,025,897,472 \times z + \&c.$

Therefore the whole quadrinomial quantity $4t + 14t^2 + 4t^3 - t^4$ will be equal to the multinomial quantity

$$\left\{\begin{array}{ll} 0.633,536 & + 4z \\ + 0.351,196,880,384 & + 4.434,752 \times z + \&c \\ + 0.015,892,561,915,068,416 & + 0.301,025,897,472 \times z + \&c \\ - 0.000,629,281,881,589,048,999,936 & - 0.015,892,561,915,068,416 \times z - \&c \end{array}\right\}$$

$$= \left\{\begin{array}{ll} 1.000,625,442,299,068,416 & + 8.735,777,897,472 \times z + \&c \\ - 0.000,629,281,881,589,048,999,936 & - 0.015,892,561,915,068,416 \times z - \&c \end{array}\right\}$$

$$= 0.999,996,160,417,479,367,000,064 + 8.719,885,335,556,931,584 \times z + \&c.$$

But the quadrinomial quantity $4t + 14t^2 + 4t^3 - t^4$ is equal to 1.

Therefore the quantity $0.999,996,160,417,479,367,000,064 + 8.719,885,335,556,931,584 \times z$ &c will alſo be $= 1$. And conſequently $8.719,885,335,556,931,584 \times z$ will be $(= 1 - 0.999,996,160,417,479,367,000,064) = 0.000,003,839,582,520,632,999,936$, and z will be $=$ the fraction $\frac{0.000,003,839,582,520,632,999,936}{8.719,885,335,556,931,584} = 0.000,000,440,324$. Therefore t, or $0.158,384 + z$, will be $(= 0.158,384 + 0.000,000,440,324) = 0.158,384,440,324$, that is, the tangent of an arch of 9 Degrees will be very nearly equal to $0.158,384,440,324$. Q. E. I.

This third near value of t, or the tangent of an arch of 9 Degrees, is exact in all it's twelve figures, the more accurate value of it being $0.158,384,440,324,536,293,838,883$, as has been ſhewn above in Art. 25, page 803.

A Third Process of Mr. Raphson's Method of Approximation.

Art. 106. Having thus found the number 0.158,384,440,324, conſiſting of 12 figures, for a third near value of t, or the leſſer root of the equation $4t + 14t^2 + 4t^3 - t^4 = 1$, we will now proceed to make this number the foundation of another Proceſs of Mr. Raphſon's Method of Approximation, by which we may obtain a ſtill nearer value of t, which will be exact to 24, or, at leaſt, to 23, places of figures. And, for this purpoſe, we muſt, in the firſt place, ſubſtitute the ſaid number 0.158,384,440,324, inſtead of t, in the quadrinomial quantity $4t + 14t^2 + 4t^3 - t^4$, (which forms the firſt, or left-hand, ſide of the equation $4t + 14t^2 + 4t^3 - t^4 = 1$,) in order to diſcover whether the value of the ſaid quantity reſulting from ſuch ſubſtitution will be greater, or will be leſs, than 1, or the abſolute term of the ſaid equation, and conſequently whether the ſaid number 0.158,384,440,324 will be greater, or will be leſs, than the true value of t in that equation. Now this ſubſtitution may be made in the following manner.

If t is = 0.158,384,440,324, t^2 will be = 0.025,085,630,936,746,717,224,976, and t^3 will be = 0.003,973,173,616,091,048,653,022, and t^4 will be = 0 000,629,288,879,494,663,981,535, as will appear from the following operations of multiplication.

t = 0.158,384,440,324
t = 0.158,384,440,324

633 537 761 296
3 167 688 806 48
47 515 332 097 2
6 335 377 612 960
63 353 776 129 6
633 537 761 296
12 670 755 225 92
47 515 332 097 2
1 267 075 522 592
7 919 222 016 20
1 583 844 403 24

t^2 = 0.025,085,630,936,746,717,224,976

t =

$t^2 = 0.025,085,630,936,746,717,224,976$
$t = 0.158,384,440,324$

0.002 508 563 093 674 671 722 497 6
1 254 281 546 837 335 861 248 80
200 685 047 493 973 737 799 808
7 525 689 281 024 015 167 492
2 006 850 474 939 737 377 998
100 342 523 746 986 868 899
10 034 252 374 698 686 889
1 003 425 237 469 868 688
7 525 689 281 024 015
501 712 618 734 934
100 342 523 746 086

$t^3 = 0.003,973,173,616,091,048,653,022,109$

$t^3 = 0.003,973,173,616,091,048,653,022$
$t = 0.158,384,440,324$

0.000 397 317 361 609 104 865 302 2
198 658 680 804 552 432 651 10
31 785 388 928 728 389 224 176
1 191 952 084 827 314 595 906
317 853 889 287 283 892 241
15 892 694 464 364 194 612
1 589 269 446 436 419 461
158 926 944 643 641 946
1 191 952 084 827 314
79 463 472 321 820
15 892 694 464 364

$t^4 = 0.000,629,288,879,494,663,981,535,140$

Therefore $4t$ will be $(= 4 \times 0.158,384,440,324) = 0.633,537,761,296$;
and $14t^2$ will be $(= 14 \times 0.025,085,630,936,746,717,224,976) = 0.351,198,833,114,454,041,149,664$;
and $4t^3$ will be $(= 4 \times 0.003,973,173,616,091,048,653,022) = 0.015,892,694,464,364,194,612,088$; and consequently $4t + 14t^2 + 4t^3$ will be equal to

$$\left\{\begin{array}{l} 0.633,537,761,296 \\ + \ 0.351,198,833,114,454,041,149,664 \\ + \ 0.015,892,694,464,364,194,612,088 \end{array}\right\}$$

$= 1.000,629,288,874,818,235,761,752$, and the quadrinomial quantity $4t + 14t^2 + 4t^3 - t^4$ will be equal to

$$= \left\{ \begin{array}{l} 1.000,629,288,874,818,235,761,752 \\ -\ 0.000,629,288,879,494,663,981,535 \end{array} \right\}$$

$= 0\ 999,999,999,995,323,571,780,217$; which is less than 1, or the absolute term of the equation $4t + 14t^2 + 4t^3 - t^4 = 1$, and consequently the number 0.158,384,440,324 is less than the true value of t in the said equation. Q. E. I.

Art. 107. Having thus found that 0.158,384,440,324 is less than t, let z be put for their difference; and, in order to avoid the frequent repetition of the long number 0.158,384,440,324, let the letter a be used instead of it.

Then, since t is $= 0.158,384,440,324 + z$, or $a + z$, we shall have

$$t^2\ (= \overline{a+z}^2) = a^2 + 2az + \&c,$$
$$\text{and } t^3\ (= \overline{a+z}^3) = a^3 + 3a^2z + \&c,$$
$$\text{and } t^4\ (= \overline{a+z}^4) = a^4 + 4a^3z + \&c,$$
$$\text{and } 4t\ (= 4 \times \overline{a+z}) = 4a + 4z,$$
$$\text{and } 14t^2\ (= 14 \times \overline{a^2 + 2az + \&c}) = 14a^2 + 28az + \&c,$$
$$\text{and } 4t^3\ (= 4 \times \overline{a^3 + 3a^2z + \&c}) = 4a^3 + 12a^2z + \&c.$$

Therefore the quadrinomial quantity $4t + 14t^2 + 4t^3 - t^4$ will be equal to the multinomial quantity

$$\begin{array}{llll} & 4a & + & 4z \\ + & 14a^2 & + & 28az + \&c \\ + & 4a^3 & + & 12a^2z + \&c \\ - & a^4 & - & 4a^3z - \&c. \end{array}$$

But the quadrinomial quantity $4t + 14t^2 + 4t^3 - t^4$ is $= 1$.

Therefore the multinomial quantity $4a + 14a^2 + 4a^3 - a^4 + 4z + 28az + 12a^2z + \&c - 4a^3z - \&c$ will also be $= 1$. And consequently $4z + 28az + 12a^2z + \&c - 4a^3z - \&c$ will be $= 1 - \overline{4a + 14a^2 + 4a^3 - a^4}$, and z will be $= \frac{1 - \overline{4a + 14a^2 + 4a^3 - a^4}}{4 + 28a + 12a^2 - 4a^3\ \&c}$.

But $4a + 14a^2 + 4a^3 - a^4$ has been shewn to be $= 0.999,999,999,995,323,571,$

571,780,217. Therefore $1-\sqrt{4a+14a^2+4a^3-a^4}$ is (= 1 — 0.999,999,999,995,323,571,780,217) = 0.000,000,000,004,676,428,219,783. Therefore z is $= \frac{0.000,000,000,004,676,428,219,783}{4+28a+12a^2-4a^3}$.

But $28a$ is (= 28×0.158,384,440,324) 4.434,764,329,072,
and $12a^2$ is (= 12×0.025,085,630,936,746,717,224,976) = 0.301,027,571,240,960,606,699,712,
and $4a^3$ is (= 4×0.003,973,173,616,091,048,653,022) = 0.015,892,694,464,364,194,612,088.

Therefore $4+28a+12a^2$ will be =

$$\left\{\begin{array}{l} 4. \\ +\ 4.434,764,329,072 \\ +\ 0.301,027,571,240,960,606,699,712 \end{array}\right\}$$

= 8.735,791,900,312,960,606,699,712, and the whole quadrinomial quantity $4+28a+12a^2-4a^3$ will be =

$$\left\{\begin{array}{l} 8.735,791,900,312,960,606,699,712 \\ -\ 0.015,892,694,464,364,194,612,088 \end{array}\right\}$$

= 8.719,899,205,848,596,412,087,624.

Therefore z, or the fraction $\frac{1-\sqrt{4a+14a^2+4a^3-a^4}}{4+28a+12a^2-4a^3}$ will be = the numeral fraction $\frac{0.000,000,000,004,676,428,219,783}{8.719,899,205,848,596,412,087,624}$ = 0.000,000,000,000,536,293,838,883. Therefore t, or $a+z$, or 0.158,384,440,324+z, will be (= 0.158,384,440,324,536,293,838,883) = 0.158,384,440,324,536,293,838,883; that is, the tangent of an arch of 9 Degrees will be very nearly equal to the number 0.158,384,440,324,536,293,838,883.

Q. E. I.

This number 0.158,384,440,324,536,293,838,883, here obtained for the value of the tangent of an arch of 9 Degrees in a circle of which the radius is called 1, by resolving the quadrinomial equation $4t+14t^2+4t^3-t^4 = 1$, agrees with the value of the same tangent found above in Art. 25, page 803, by resolving the quinquinomial equation $5t+10t^2-10t^3-5t^4+t^5 = 1$, in all it's 24 figures. And the labour of resolving this quadrinomial equation has been

been conſiderably leſs than that of reſolving the aforeſaid quinquinomial equation, that reſolution having taken-up 23 pages, to wit, from page 780 to page 804, and this laſt reſolution having taken-up only 11 pages, to wit, from page 938 to page 950. The thought of inveſtigating the value of this tangent by reſolving the quadrinomial equation $4t+14t^2+4t^3-t^4=1$ was ſuggeſted to me by the ingenious Mr. William Frend, M. A. and Fellow of Jeſus College, Cambridge.

End of the Computation of the Length of the Tangent of an Arch of 9 Degrees, in a Circle of which the Radius is called 1, *by reſolving the Quadrinomial Equation* $4t+14t^2+4t^3-t^4=1$.

Theſe various computations of the lengths of the tangents of circular arches having extended this volume of the *Scriptores Logarithmici* to an uncommon ſize, I ſhall here put an end to it.

FRANCIS MASERES.

Inner Temple, April 24, 1806.

End of the Sixth Volume of the present Collection of Mathematical Tracts, intitled, SCRIPTORES LOGARITHMICI.

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ERRATA.

IN THE PREFACE.

IN page xli, line 2, inſtead of " 23 pages" read " 24 pages."
And in the ſame page xli, line 3, inſtead of " page 473," read " page 474."
In page lxii, line 1, inſtead of 0.000,163, read 0.001,163.
And in the ſame page lxii, line 18, inſtead of "If we divide by," read "If we divide 2 by."
In page lxxxiii, line 1, inſtead of " to both ſides," read " from both ſides."

IN THE BODY OF THE WORK.

In page 5, line 5 from the bottom, inſtead of $\frac{50,521\ m^8\ d^3}{241,290}$ read $\frac{50,521\ m^8\ d^3}{241,920}$.

And in the ſame page 5, in the bottom line, inſtead of $\frac{41,581 \times 99 m^8\ d^5}{967,680}$ read $\frac{41,581 \times 13\ m^8\ d^5}{967,680}$.

In page 6, line 6, inſtead of $\frac{41,581 \times 99\ m^5\ d^8}{967,680}$, read $\frac{41,581 \times 13\ m^5\ d^8}{967,680}$.

In page 7, line 17, inſtead of " + c," read " + & c."

In page 9, line 11, inſtead of a^3 read d^3.

In page 10, line 8, after $\frac{C}{16}$ inſert a comma.

And in ſame page 10, line 12, inſtead of $\frac{5 \times 0.069,114}{8} = \frac{0.345,570}{8}$ read $\frac{5 \times 0.069,114}{128} = \frac{0.345,570}{128}$.

In page 11, line 1, inſtead of −0.000,196, read −0.000,194; and inſtead of −0.000,198, read 0.000,196.

N. B. This ſmall error runs through the remainder of the calculation.

And in the ſame page 11, line 8, inſtead of $-0.002,074x^8$, read $-0.002,704x^8$.

In page 21, line 16, inſtead of .55737, read .557,355.

In page 35, line 5, instead of 12,625 a^6, read 12,625 a^{10}.

In page 47, line 5 from the bottom, instead of $\frac{-D^1 x^5}{\sqrt{1+x}^5}$ read $\frac{-D^1 v x^5}{\sqrt{1+x}^5}$.

In page 120, line 15, instead of $x^2 + 2xe$ + &c, read $\overline{x^2 + 2xe}$ &c.

In page 134, line 15, instead of $\frac{m-1}{2} \times {}^{m-2}xe^2$ read $\frac{m-1}{2} \times y^{m-2} \times e^2$.

In page 136, line 6 from the bottom, instead of "quantita," read "quantitate."

In page 138, line 2, instead of $-x^{1+}$ read $-x^{1+3}$.

In page 139, in the second line from the bottom, instead of $-z^2$ read $-z^3$.

In page 142, line 8, instead of $\frac{\sqrt{pp+4} - p}{2}$, read $\frac{\sqrt{pp+4q} - p}{2}$.

And in the same page 142, line 9, instead of $\frac{p + \sqrt{pp+4q}}{2}$, read $\frac{\sqrt{pp+4q} - p}{2}$.

In page 143, line 12, instead of $2py$, read $2pye$.

And in the same page 143, line 20, instead of $\frac{\sqrt{pp+3q} - p}{3}$, read $\frac{\sqrt{pp+3q} - p}{3}$.

And again in the same page 143, line 22, instead of $\frac{\sqrt{pp+3q} - p}{3}$, read $\frac{\sqrt{pp+3q} - p}{3}$.

In page 146, line 2 from the bottom, instead of "$+ x$" read "$\times x$."

In page 148, line 15, instead of $y + e^4$, read $\overline{y + e}^4$.

In page 149, line 8, instead of "$-\frac{q}{2} + x$," read "$-\frac{q}{2} \times x$."

In page 152, line 12 from the bottom, instead of $Bx^2 Cx^3$, read $Bx^2 + Cx^3$.

In page 155, line 2 from the bottom, instead of d read *sed*.

In page 159, line 11, instead of $3x\dot{x}^2 + 3x\dot{x}^2$, read $3x^2\dot{x} + 3x\dot{x}^2$.

In page 164, line 14, instead of "$+ x$", read "$\times x$."

And in the same page 164, line 4 from the bottom, instead of *major*, in two places, read *minor*.

In page 165, line 3, instead of $P\,\overline{n}$, read $P\,\frac{1}{n}$.

In page 173, line 10, instead of qx, read qx^2.

In page 175, line 9, instead of $x + \dot{x}^2$, read $\overline{x + \dot{x}}^2$.

In page 226, line 10 from the bottom, instead of x^4, read x^3.

In page 229, line 7, instead of $+ mmx$, read $-mmx$.

In page 231, line 6, instead of x^1, read x^3.

In

In page 245, line 3 from the bottom, instead of $\surd^4(ln^3 + p$, read $\surd^4(ln^3 + p^4$.
In page 247, line 13, instead of n^3, read n^3x.
And in the same page 247, line 15, instead of *quàm m*, read *quàm n*.
In page 258, line 5, instead of $\surd\left(\frac{1}{4}ll + mm + nn + pp\right.$, read $\surd\left(\frac{1}{4}ll + mm + nn + pp - zz\right.$.
In page 262, line 10 from the bottom, instead of $\surd\left(\frac{1}{2}ll + mm\right.$, read $\surd\left(\frac{1}{4}ll + mm\right.$.
And in the same page 262, line 8 from the bottom, instead of $\frac{1}{2}l + \surd\left(\frac{1}{4}ll + mm\right.$, read $-\frac{1}{2}l + \surd\left(\frac{1}{4}ll + mm\right.$.
In page 263, line 5, instead of $\surd\left(\frac{1}{4}ll + mm - zz\right.$, read $\surd\left(\frac{1}{4}ll + mm - zz\right.$.
And in the same page 263, line 8, instead of $-l$, read $-\frac{1}{2}l$.
In page 266, line 3 from the bottom, instead of *radix unca*, read *radix unica*.
In page 268, line 5, instead of *l*, read *n*.
In page 276, line 5, instead of *minor*, read *major*.
In page 278, line 3 from the bottom, instead of $\surd^3\left(\frac{p^4}{n^3}\right.$, read $\surd^3\left(\frac{p^4}{l}\right.$.
In page 284, line 7, instead of $\surd^3(l^3 + m^3 + n^3$, read $\surd^3(l^3 + m^3 + n^3$.
In page 296, line 1, instead of "x esse $= 45$," read "x esse $= 4.5$."
In page 297, at the beginning of line 3, dele z.
In page 298, line 12 from the bottom, instead of $x^4 + lx^3$, read $x^3 + lxx$.
In page 299, line 2 from the bottom, instead of 551,207, read 551,204.

N. B. This error runs through the rest of the calculation in the next page.

In page 304, line 8, instead of $\surd dd + xx$, read $\surd(dd + xx$
In page 305, line 7 from the bottom, instead of 43,507,212, read 43,507,216.
And in the same page 305, line 2 from the bottom, instead of xxx, read xx.
In page 311, lines 17, 19, 22, 23, 24, 26, 27, and 29, instead of 141,773,228x, read 141,873,228x. This number 141,873,228 is the product of the multiplication of the two numbers 321,708 and 441.

And in the ſame page 311, lines 10 and 12 from the bottom, inſtead of "10,404*xx*," read 10,404x^4.

In page 312, line 4, inſtead of *ſexti*, read *decimi*.

In page 331, line 7, in the Parentheſis, inſtead of 4,228,931.085,087,352, read 4,228,931.085,087,852.

In page 333, line 6, inſtead of 30,931,573,941.090,563,610,4, read 30,931,573, 941.090,565,610,4.

In page 335, line 2, inſtead of 4,228,913.085, in two places, read 4,228, 931.085.

And in the ſame page 335, and the ſame line 2, inſtead of 2,114,456,542.500, read 2,114,465,542.500.

And in the ſame page 335, line 3, inſtead of 16,180,483,150.000, read 8,090,241,575.000.

And again in the ſame page 335, line 5, inſtead of 2,114,456,542.500 + 16,180,483,150.000, read 2,114,465,542.500 + 8,090,241,575.000.

And again in the ſame page 335, lines 6 and 7, inſtead of 18,294,939,692.500, read 10,204,707,117.500.

In page 338, line 9, inſtead of 56,249,104,561, read 56,249,134,561.

In page 342, line 10 from the bottom, inſtead of $4x^3$, read $40x^3$.

And in the ſame page 342, line 3 from the bottom, inſtead of CD, read CB.

In page 346, line 9, after the laſt mark of &c, and before $\overbrace{11.5}^4$, inſert the mark of equality, $=$.

In page 357, line 11 from the bottom, in the Denominator of the fraction, inſtead of 5288.6875, read 5287.6875.

In page 361, line 4, inſtead of 35*z*, read 34*z*.

In page 366, lines 8, 9, and 10 from the bottom, inſtead of $-2x^2$, read $+2x^2$.

In page 368, line 8, inſtead of $+ \frac{z+2}{\sqrt{2z}}$, read $\times \frac{z+2}{\sqrt{2z}}$.

And in the ſame page 368, the laſt line, inſtead of $\frac{\sqrt{\ }}{\sqrt{z-3}}$, read $\frac{\sqrt{2}}{\sqrt{z-z^3}}$.

In page 383, line 10, inſtead of 43.24, $\times$ *zz*, read 34.24 $\times$ *zz*.

In page 385, line 8, inſtead of 0.9299, read 0.9269.

In page 393, lines 9 and 13 from the bottom, inſtead of 1.033,908,514,968, 522,139,793,133,755, read 1.033,908,514,968,522,139,743,133,755.

In

In page 403, line 6, at the end, dele the words "*will be*," and insert them in line 7, after 1.242,639, &c. × *z*.

In page 410, line 7 from the bottom, instead of "5," read "0·5."

In page 415, line 9, after x^4 insert =.

In page 420, line 15, instead of 14245,709,068,198,771, read 14,245·709,068,198,771.

In page 423, line 13, instead of 246·170,297,206, read 246·170,166,297,206.

And in the same page 423, line 8 from the bottom, instead of 14,245·709,968,198,771, read 14,245·709,068,198,771.

In page 424, line 2 from the bottom, instead of x^3, read $80x^3$.

In page 431, line 4 from the bottom, instead of $\frac{w^4}{2586.\&c}$, read $\frac{w^4}{5286.\&c}$.

In page 438, line 8 from the bottom, instead of "480 ×," read, "80 ×."

And in the same page 438, line 6 from the bottom, instead of "— will be," read "$-x^4$ will be."

In page 447, line 16, immediately after the number 1970·051,130,81, insert *x*.

In page 460, line 14, instead of ·000,369,321,603,595,3, read ·000,399,321,603,595,3.

And in the same page 460, line 11 from the bottom, instead of "$\frac{c+g^3-3g}{3\times 1-g^2}$" read $\frac{c+g^3-3g}{3\times\sqrt{1-g^2}}$.

In page 467, line 12, instead of ·000,369,321,603,595,3, read ·000,399,321,603,595,3.

And in the same page 467, line 15, instead of 33657769, read 33657759.

In page 468, line 14, instead of ·999,817,236,416,885, read 4·999,817,236,416,885.

In page 470, line 10, instead of +·000,000,008,307,061, read +·000,000,008,307,070.

In page 473, line 2 from the bottom, instead of ·000,295,888,204,563,5 read ·000,290,888,204,563,5.

In page 485, line 13, instead of 200,000,000,000,000, read 50,000,000,000,000.

In the same page 485, line 18, after 7,071,068, instead of a full stop "." insert a Comma ",".

In

In page 486, line 7, after the word *in* insert the small Greek Letter δ.
And in the same page 486, line 14, instead of *Kw*, read *κw*.
In page 630, line 15, instead of *CG*, read *CF*.
In page 636, line 3, instead of 200,000,000,000,000, read 50,000,000,000,000.
And in the same page 636, the last line, instead of *HK*, read *HI*.
In page 638, line 10, instead of 0.000,000,000,000,01, read 0.000,000, 100,000,01.
In page 647, line 13, after the word "quadrant," instead of *CAP*, read *CAB*.
In page 660, line 2, between *a* and $\frac{a^3}{r^2}$ insert "$\frac{aa}{r}$,"
In page 664, line 7, instead of "Isacc" read Isaac.
In page 671, line 2 from the bottom, instead of 0.000,000,111, read 0.000, 000,333.
And in the same page 671, in the last line, instead of 1.000,500,111, read 1.000,500,333.
In page 672, line 1, instead of 1.000,500,111, read 1.000,500,333.
And in the same page 672, lines 2, 3, and 4, instead of .001,11, read .003,33.
In page 673, line 9, instead of 9,989,068, read 9,989,968.
In page 674, line 4, instead of $\frac{0.000,001,000}{3}$, read $\frac{0.000,100,000}{3}$; and instead of $\frac{0.000,000,010,}{4}$, read $\frac{0.000,001,000}{4}$; and instead of 0.000,000,333, read 0.000, 033,333.
And in the same page 674, line 5, instead of 0.000,000,002, read 0.000, 000,250; and, instead, of 1.005,000,335, read 1.005,033,583.
And in the same page 674, line 6, instead of 1.005,000,335, read 1.005,033,583.
And in the same page 674, line 7, instead of 100,500.0335, read 100,503.3583.
And in the same page 674, line 8, instead of 100,500.0335, read 100,503. 3583.
And in the same page 674, line 8, instead of 100,500.0335, read 100,503.3583.
And in the same page 674, line 9, instead of .0335, read .3583; and in line 10, instead of 100,500, read 100,503; and in line 18, instead of 100,500, read 100,503.
In page 675, line 1, and likewise line last, instead of 100,500, read 100,503.
In page 677, line 2, instead of 1,053,605,154, read 1.053,605,154.
In page 682, line 5 from the bottom, instead of $\checkmark \overline{1 - \frac{1}{r}}^{n}$, read $\checkmark^{n} \overline{1 - \frac{1}{r}}^{n}$.

In

In page 685, line 5, instead of $\frac{1}{5^{r^5}}$, read $\frac{1}{5^{r^4}}$.

In page 690, line 6, instead of 100,500.0335, read 100,503.3583.

And in the same page 690, line 4 from the bottom, instead of 55,504.174, 301,25, read 55,504.174,801,25.

In page 692, line 7 from the bottom, between the fractions $\frac{x^5}{2.3.4.5.r}$ and $\frac{x}{6r}$ insert the mark of multiplication ×.

And in the same page 692, line 6 from the bottom, instead of "$\frac{x}{r}$ 2,222.1 =", read $\frac{x}{r} = 2{,}222.1$.

In page 694, line 11, instead of 10^6, read $\overline{10}^6$.

In page 712, line 4, instead of 40,317,433, read 81,425,681.

In page 714, line 9 from the bottom, instead of 177,433, read 2909.

And again in the same page, line 8 from the bottom, in two places, instead of 40,317,433, read 81,425,681.

In page 769, line 13, instead of $\frac{rr \times a+b}{rr-ab}$, read $\frac{rr \times \overline{a+b}}{rr-ab}$.

And in the same page 769, line 18, instead of $\frac{rr \times A+a}{rr-Aa}$, read $\frac{rr \times \overline{A+a}}{rr-Aa}$.

In page 783, line 5 from the bottom, instead of $-10t^4$, read $-5t^4$.

In page 785, line 12 from the bottom, instead of $-10t^4$, read $-5t^4$.

In page 786, line 2 from the top, and line 2 from the bottom, instead of $-10t^4$, read $-5t^4$.

In page 804, line 14 from the bottom, instead of 0.05246, read 0.05279

And in the same page 804, line 4 from the bottom, instead of 0.0542, read 0.0524.

In page 806, line 10, instead of 0.01524, read 0.0524.

And in the same page 806, line 11, immediately after the parenthesis) insert the mark =.

And again in the same page 806, line 16, instead of 0.00524, read 0.0524; and in line 18, instead of 0.002,745,745,76, read 0.002,745,76.

In page 839, the bottom line, instead of t^2 read t^4.

In page 849, line 2 from the bottom, instead of $3T$, read $3t$.

And in the same page 849, in the bottom line, instead of the number 0.003, 490,672,881,596,250,318,516, read 0.003,490,672,681,596,250,318,516.

In

In page 854, line 17, at the beginning of the line, insert t^3 ; and at the beginning of line 18, insert t.

In page 881, line 8, instead of 0.009,290,888,216,870,315,908, read 0.000,290,888,216,870,315,908.

In page 893, the bottom line, instead of a^3, read a^4.

In page 929, line 8, dele the word "*first*."

In page 938, line 4, instead of $= b$, read $= b^5$.

In page 944, line 6 from the bottom, instead of $0.633,536 \times 4z$, read $0.633,536 + 4z$.

In page 949, immediately after the parenthesis) insert the mark of equality =.

END OF THE ERRATA.



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Maseres, Francis, 1731-1824

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